uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

DIFFERENTIAL CALCULUS
EXTRAORDINARY EXAM - SOLUTIONS
Degree in Applied Mathematics and Computation

Time: 3 hours

Problem 1 (**1 + 1 + 1 = 3 points**)

a) Calculate the domain of the function $f(x) = \arcsin\left(\dfrac{x}{x^2+1}\right)$,

b) Find the minimum value k such that f is injective on $[k,\infty)$ and obtain the inverse function on that interval.

c) Prove that

$$\arctan\left(\frac{1+x}{1-x}\right) - \arctan(x) = \frac{\pi}{4}, \qquad x < 1.$$

SOLUTION:

a) The domain of $\arcsin(x)$ is $[-1,1]$, so we need $g(x) = \dfrac{x}{x^2+1} \in [-1,1]$. We observe that

$$g'(x) = \frac{1-x^2}{(x^2+1)^2} = 0 \quad \Longrightarrow \quad x = \pm 1.$$

These are the critical points, g' is positive on $(-1,1)$ and negative on $(-\infty,-1)\cup(1,\infty)$, so 1 is the global maximum and -1 the global minimum. Since $g(1) = \frac{1}{2}$ and $g(-1) = -\frac{1}{2}$ we obtain that $g(x) \in [-\frac{1}{2},\frac{1}{2}] \in [-1,1]$ for $x \in \mathbb{R}$. This means that $\mathrm{Dom}(f) = \mathbb{R}$.

b) We can study the derivative:

$$f'(x) = \frac{1}{\sqrt{1-\left(\dfrac{x}{x^2+1}\right)^2}}\frac{1-x^2}{(x^2+1)^2} = \frac{1-x^2}{\sqrt{(x^2+1)^2 - x^2}(x^2+1)} = 0 \quad \Longrightarrow \quad x = \pm 1.$$

Thus f is injective for $x \in [1,\infty)$ because it is monotonous there. That is, $k = 1$. For the inverse:

$$y = \arcsin\left(\frac{x}{x^2+1}\right) \quad \Longrightarrow \quad \sin(y) = \frac{x}{x^2+1} \quad \Longrightarrow \quad \sin(y)x^2 - x + \sin(y) = 0,$$

$$\Longrightarrow \quad x = \frac{1 \pm \sqrt{1-4\sin^2(y)}}{2\sin(y)} = \frac{1}{2\sin(y)} \pm \sqrt{\frac{1}{4\sin^2(y)} - 1}.$$

The inverse is defined only when $x \geq 1$, so we choose the positive sign:

$$f^{-1}(y) = \frac{1}{2\sin(y)} + \sqrt{\frac{1}{4\sin^2(y)} - 1}.$$

c) First, we define the function:

$$h(x) = \arctan\left(\frac{1+x}{1-x}\right) - \arctan(x).$$

The domain is $\mathbb{R}\setminus\{1\}$, because $\arctan(x)$ is defined on the whole real line. The derivative is:

$$h'(x) = \frac{1}{1+\left(\dfrac{1+x}{1-x}\right)^2}\,\frac{1-x+(1+x)}{(1-x)^2} - \frac{1}{1+x^2} = 0,$$

hence the function is constant on each interval of its domain, in particular it is constant on $(-\infty, 1)$ and observe that: $h(0) = \arctan(1) - \arctan(0) = \dfrac{\pi}{4}$.

Problem 2 (**2 points**)

Plot the function $f(x) = x\sqrt{|x^2-4|}$, with all the calculations.

SOLUTION:
$\mathrm{Dom}(f) = \mathbb{R}$, and $f(-x) = -f(x)$, so the function is odd.

$$\lim_{x\to\infty} x\sqrt{|x^2-4|} = \infty, \qquad \lim_{x\to-\infty} x\sqrt{|x^2-4|} = -\infty,$$

and then there are no horizontal asymptotes. Also:

$$\lim_{x\to\pm\infty} \sqrt{|x^2-4|} = \infty,$$

thus, there are no oblique asymptotes either. There are no vertical asymptotes because the domain is $\mathbb{R}$. To obtain the derivative we first decompose f:

$$f(x) = \begin{cases} x\sqrt{4-x^2}, & x\in[-2,2], \\ x\sqrt{x^2-4}, & x\in(-\infty,-2)\cup(2,\infty). \end{cases} \implies f'(x) = \begin{cases} \dfrac{4-2x^2}{\sqrt{4-x^2}}, & x\in(-2,2), \\ \dfrac{2x^2-4}{\sqrt{x^2-4}}, & x\in(-\infty,-2)\cup(2,\infty). \end{cases}$$

So, f is not differentiable at $x=-2$ nor at $x=2$. The critical points are $x=\pm\sqrt{2}$, both in $(-2,2)$.

$$\begin{array}{lll} f'>0 & \text{if} \quad x\in(-\infty,-2)\cup(-\sqrt{2},\sqrt{2})\cup(2,\infty) & \implies f \text{ increases here} \\ f'<0 & \text{if} \quad x\in,\ (-2,-\sqrt{2})\cup(\sqrt{2},2), & \implies f \text{ decreases here} \end{array}$$

Hence, $x=-2$ and $x=\sqrt{2}$ are local maxima while $x=-\sqrt{2}$ and $x=2$ are local minima. The values are:

$$f(2) = f(-2) = 0, \quad f(\sqrt{2}) = 2, \quad f(-\sqrt{2}) = -2.$$

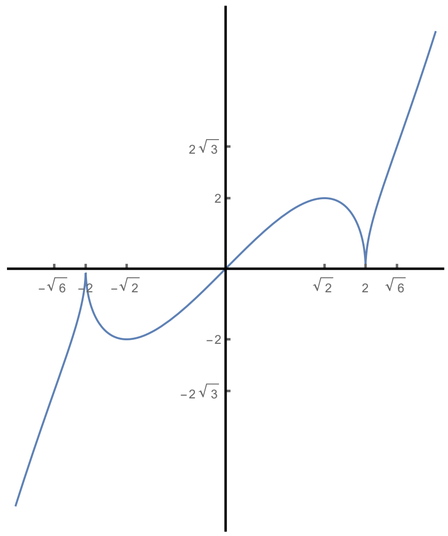
The second derivative is:

$$f''(x) = \begin{cases} \dfrac{2x(x^2-6)}{(4-x^2)^{3/2}}, & x\in(-2,2), \\ \dfrac{2x(x^2-6)}{(x^2-4)^{3/2}}, & x\in(-\infty,-2)\cup(2,\infty). \end{cases}$$

$$\begin{array}{lll} f''>0 & \text{if} \quad x\in(-\sqrt{6},-2)\cup(-2,0)\cup(\sqrt{6},\infty) & \implies f \text{ convex here} \\ f''<0 & \text{if} \quad x\in,\ (-\infty,-\sqrt{6})\cup(0,2)\cup(2,\sqrt{6}) & \implies f \text{ concave here} \end{array}$$

Inflection points are $x=-\sqrt{6}$, $x=0$ and $x=\sqrt{6}$, with values:

$$f(0) = 0, \quad f(\sqrt{6}) = 2\sqrt{3}, \quad f(-\sqrt{6}) = -2\sqrt{3}.$$



Problem 3 (**1 + 1 = 2 points**)

a) Compute the limit: $\displaystyle \lim_{n\to\infty} \frac{\sum_{k=1}^{n} \sin\left(\frac{\pi}{k}\right)}{\log n}$.

b) Study the convergence of the sequence defined by: $a_0 = 1/2, \quad a_{n+1} = 2 + \dfrac{4}{a_n}$.

Hint: Observe that $a_n > 2$ for $n \geq 1$.

SOLUTION:

a) The denominator is increasing to ∞, so we can apply the Stolz criterion:

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \sin\left(\frac{\pi}{k}\right)}{\log n} = \lim_{n\to\infty} \frac{\sin\left(\frac{\pi}{n}\right)}{\log\left(\frac{n}{n-1}\right)} = \lim_{n\to\infty} \frac{\pi \sin\left(\frac{\pi}{n}\right)}{\frac{\pi}{n} \log\left(\frac{n}{n-1}\right)^n} = \pi.$$

b) We consider the function $f(x) = 2 + \dfrac{4}{x}$. The possible limit satisfies:

$$L = f(L) \quad \Longrightarrow \quad L = 2 + \frac{4}{L} \quad \Longrightarrow \quad L = 1 \pm \sqrt{5}.$$

Since $L > 0$ the possible limit is $L = 1 + \sqrt{5}$. Now we study if the sequence is convergent using the fixed point theorem: $f'(x) = -\dfrac{4}{x^2} < 0$, so the sequence is alternating. Also $f'(x)| = \dfrac{4}{x^2} < 1$ for $x > 2$, which is the case of our sequence, so it is convergent and converges to $L = 1 + \sqrt{5}$.

To prove the hint, observe that:

$$a_0 = \frac{1}{2} > 0 \quad \Longrightarrow \quad a_n > 0 \quad \forall n \quad \Longrightarrow \quad a_{n+1} = 2 + \frac{4}{a_n} > 2 \quad \forall n \geq 1.$$

Problem 4 (**1 + 1 + 1 = 3 points**)

a) Study the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{n(1+a)^n}{e^{an}}$ for different values of $a > -1$.

b) Sum and obtain the interval of convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^n(2n+1)!}$.

c) Obtain the Taylor series and the interval of convergence of $f(x) = \log\left(\dfrac{1}{1+2x}\right) + 2x$.

SOLUTION:

a) $\displaystyle\lim_{n\to\infty} \sqrt[n]{a_n} = \frac{1+a}{e^a} < 1$ for all $a > -1$, $a \neq 0 \Rightarrow C$; if $a = 0$ the series is $\displaystyle\sum_{n=1}^{\infty} n = \infty$

b)

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^n(2n+1)!} = \frac{\sqrt{2}}{x} \sum_{n=1}^{\infty} \frac{(-1)^n \left(\frac{x}{\sqrt{2}}\right)^{2n+1}}{(2n+1)!} = \frac{\sqrt{2}}{x}\left(\sin\left(\frac{x}{\sqrt{2}}\right) - \frac{x}{\sqrt{2}}\right) = \frac{\sqrt{2}}{x}\sin\left(\frac{x}{\sqrt{2}}\right) - 1.$$

The series converges on $\mathbb{R}$ because the series of the sine converges on $\mathbb{R}$.

c) Observe that $f(x) = -\log(1+2x) + 2x$, so we use the series of the logarithm:

$$-\log(1+2x) + 2x = -\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2x)^n}{n} + 2x = 2x + \sum_{n=1}^{\infty} \frac{(-2x)^n}{n} = \sum_{n=2}^{\infty} \frac{(-2x)^n}{n}.$$

This series is convergent for $2x \in (-1, 1]$, that is, it converges for $x \in (-\frac{1}{2}, \frac{1}{2}]$.

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