uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

DIFFERENTIAL CALCULUS
FINAL EXAM - SOLUTIONS
Degree in Applied Mathematics and Computation

Time: 3 hours

Problem 1 (**1,5 points**)

Minimize the function $f(x) = \dfrac{x^p}{p} - bx$ for $x \in (0, \infty)$ and prove the following inequality:

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}, \qquad \text{where} \quad a, b > 0, \quad p, q > 1, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

SOLUTION:
$f'(x) = x^{p-1} - b = 0 \Longrightarrow x = b^{1/(p-1)}$, f' is positive for $x > b^{1/(p-1)}$ and negative for $0 < x < b^{1/(p-1)}$, so $x = b^{1/(p-1)}$ is an absolute minimum in $(0, \infty)$, so for any $0 < a$:

$$f(a) = \frac{a^p}{p} - ba \ge \frac{b^{p/(p-1)}}{p} - bb^{1/(p-1)} = -\frac{1}{q}b^q \Longrightarrow \frac{a^p}{p} + \frac{b^q}{q} \ge ab.$$

Problem 2 (**2 + 1 = 3 points**)

a) Plot the graph of this function, studying the derivative at the left of 0, but without f''.

$$f(x) = \frac{e^{1/x}}{1+x}, \qquad x \ne 0; \qquad f(0) = 0,$$

b) Study in a reasoned way how many solutions the equation $\dfrac{e^{1/x}}{1+x} = x^3$ has in $\mathbb{R}$.

SOLUTION:

a) $\mathrm{Dom}(f) = \mathbb{R} \setminus \{-1\}$, and there are no symmetries. We look for asymptotes:

$$\lim_{x\to\infty} \frac{e^{1/x}}{1+x} = 0^+, \qquad \lim_{x\to-\infty} \frac{e^{1/x}}{1+x} = 0^-,$$

thus, $y = 0$ is a horizontal asymptote for $x \to \pm\infty$. Also:

$$\lim_{x\to 0^+} \frac{e^{1/x}}{1+x} = \infty, \qquad \lim_{x\to 0^-} \frac{e^{1/x}}{1+x} = 0 = f(0),$$

$$\lim_{x\to -1^+} \frac{e^{1/x}}{1+x} = \infty, \qquad \lim_{x\to -1^-} \frac{e^{1/x}}{1+x} = -\infty.$$

Then, there is a vertical asymptote at $x = -1$ from the two sides and another vertical asymptote at $x = 0$ from the right hand side.

The derivative satisfies:

$$f'(x) = \frac{e^{1/x}[\frac{(-1)}{x^2}(1+x) - 1]}{(1+x)^2} = \frac{-e^{1/x}(1+x+x^2)}{x^2(1+x)^2} = 0 \Longrightarrow x = \frac{-1 \pm \sqrt{-3}}{2},$$

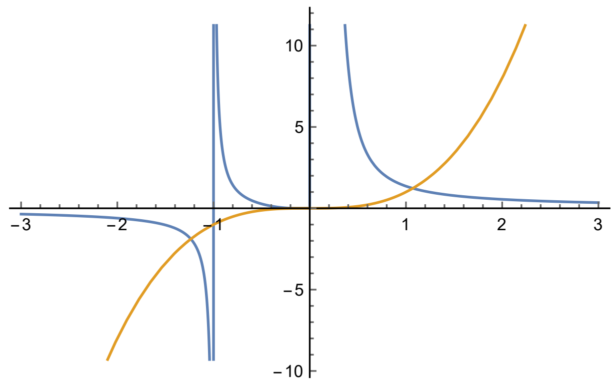
so there are no critical points. $f' < 0$ on $\mathbb{R} \setminus \{0, -1\}$, and thus f is decreasing on each interval $(-\infty, -1) \cup (-1, 0) \cup (0, \infty)$.

At the left of zero the derivative is:

$$\lim_{x \to 0^-} \frac{-e^{1/x}(1 + x + x^2)}{x^2(1+x)^2} = \lim_{x \to 0^-} \frac{-e^{1/x}}{x^2} = \lim_{t \to -\infty} (-t^2 e^t) = \lim_{t \to -\infty} \frac{(-t^2)}{e^{-t}}$$
$$= \lim_{t \to -\infty} \frac{2t}{e^{-t}} = \lim_{t \to -\infty} \frac{2}{-e^{-t}} = 0^-.$$

with the change: $1/x = t$, $x \to 0^- \implies t \to -\infty$, and the use of L'Hôpital twice.

With these data, the function will be convex on $(-1, 0) \cup (0, \infty)$ and concave on $(-\infty, -1)$.



b) There are three solutions of the equation: one is $x = 0$. Let us consider the function $g(x) = f(x) - x^3$, that is continuous on $(-\infty, -1) \bigcup (-1, 0) \bigcup (0, \infty)$. On $(-\infty, -1)$, observe that

$$\lim_{x \to -\infty} \left(\frac{e^{1/x}}{1+x} - x^3 \right) = \infty \quad \text{and} \quad \lim_{x \to -1^-} \left(\frac{e^{1/x}}{1+x} - x^3 \right) = -\infty,$$

so, there exist two points $x_1 < x_2 \in (-\infty, -1)$ such that $g(x_1) > 0 > g(x_2)$, and by the Bolzano theorem we know that there exists at least a root on $(-\infty, -1)$. Even more, since both $\frac{e^{1/x}}{1+x}$ and $-x^3$ increase on that interval, then $g(x)$ increases, the root is unique.

The third root is in $(0, \infty)$:

$$\lim_{x \to 0^+} \left(\frac{e^{1/x}}{1+x} - x^3 \right) = \infty, \quad \text{and} \quad \lim_{x \to \infty} \left(\frac{e^{1/x}}{1+x} - x^3 \right) = -\infty,$$

hence, using the same argument as before, by Bolzano's theorem we know that there is at least one root here and it is unique because also here $g(x)$ is decreasing.

There are no roots on $(-1, 0)$, because $f(x) > 0$ and $x^3 < 0$ there. In the picture you see $f(x)$ together with x^3.

Problem 3 (1 point)
Calculate the Taylor polynomial of degree 3 at the origin of $f(x) = \sin(2x) - e^{2x}$ and a bound of the error when we approximate at $x = 1/2$ the function by the polynomial.

SOLUTION:
We use the Taylor polynomials of degree 3 at the origin of the two functions:

$$P_{3,0} f(x) = 2x - \frac{(2x)^3}{3!} - 1 - 2x - \frac{(2x)^2}{2} - \frac{(2x)^3}{3!} = -1 - 2x^2 - \frac{8x^3}{3}.$$

To estimate the error at $x = 1/2$ we use Lagrange's formula for the remainder of the Taylor polynomial:

$$R_{3,0}f\left(\frac{1}{2}\right) = \frac{f^{IV)}(t)}{4!}\left(\frac{1}{2}\right)^4 = \frac{16(\sin(2t) - \mathrm{e}^{2t})}{4!}\left(\frac{1}{2}\right)^4 = \frac{\sin(2t) - \mathrm{e}^{2t}}{4!},$$

where $t \in (0, 1/2)$, so we have the bound:

$$\left|R_{3,0}f\left(\frac{1}{2}\right)\right| \leq \frac{1+\mathrm{e}}{4!}.$$

Problem 4 (**1 + 1,5 = 2.5 points**)

a) Compute the limit: $\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k^2}{n^2}\sin\frac{1}{k}$.

b) Study the convergence of the sequence defined by: $\displaystyle a_{n+1} = \frac{a_n^3+5}{6}, \quad a_0 = 1/2.$

SOLUTION:

a) We write this as a quotient and use Stolz criterion. We can since the denominator n^2, is an increasing sequence with limit ∞:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k^2}{n^2}\sin\frac{1}{k} = \lim_{n\to\infty}\frac{\sum_{k=1}^{n}k^2\sin\frac{1}{k}}{n^2} = \lim_{n\to\infty}\frac{n^2\sin(1/n)}{n^2-(n-1)^2} = \frac{1}{2}.$$

b) We have $a_{n+1} = f(a_n)$ with $f(x) = \dfrac{x^3+5}{6}$. The derivative is $f'(x) = \dfrac{x^2}{2} > 0$, so the sequence is monotonous. Besides:

$$a_0 = \frac{1}{2}, \qquad a_1 = \frac{41}{48} > \frac{1}{2},$$

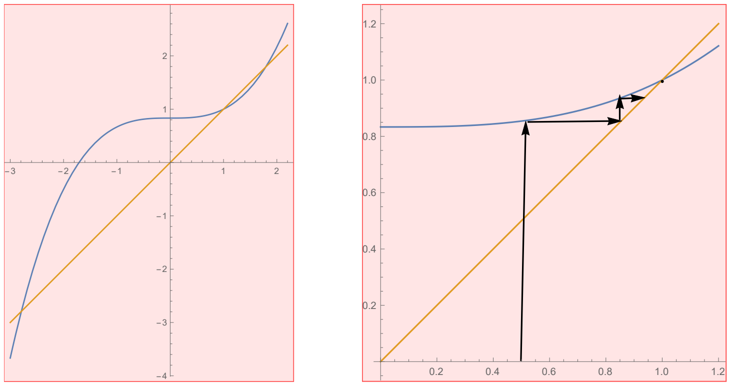
so, the sequence is increasing. If there is a limit it must satisfy:

$$x = \frac{x^3+5}{6} \implies x^3 - 6x + 5 = 0 \implies x = 1, \quad x = \frac{-1}{2} \pm \frac{\sqrt{21}}{2}.$$

Observe that $\dfrac{\sqrt{21}}{2} \sim 2.29$, so the three possible limits are $x = 1$, $x \sim 1{,}79$ and $x \sim -2{,}79$. Since $a_0 < 1$, $a_1 < 1$ we use induction to prove that all the sequence is bounded above by 1. If $a_n < 1$, then:

$$a_{n+1} = \frac{a_n^3+5}{6} < \frac{1+5}{6} = 1.$$

Then, the sequence converges to 1, because it is increasing and it is bounded above by 1.



Problem 5 (**0,5 + 0,5 + 1 = 2 points**)

a) Study the convergence of the series $\displaystyle\sum_{n=1}^{\infty}(-1)^n \tan\left(\frac{1}{\sqrt{n}}\right)$.

b) Sum and obtain the interval of convergence of the series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{2^n n!}$.

b) Obtain the Taylor series and the interval of convergence of $f(x) = \ln\left(\dfrac{1}{1-2x}\right) - 2x$.

SOLUTION:

a) This is an alternating series. We study the absolute convergence comparing with the $\frac{1}{2}$-harmonic series. We compute the limit:

$$\lim_{n\to\infty}\frac{\tan\left(\frac{1}{\sqrt{n}}\right)}{\frac{1}{\sqrt{n}}} = \lim_{t\to 0^+}\frac{\tan t}{t} = \lim_{t\to 0^+}\frac{1}{\cos^2 t} = 1,$$

with the change $\frac{1}{\sqrt{n}} = t$ and using L'Hôpital. Then, both series have the same character, that is, both diverge. So our series does not converge absolutely. Now we study the conditional convergence. Since:

$$\lim_{n\to\infty}\tan\left(\frac{1}{\sqrt{n}}\right) = 0,$$

and it is decreasing because the derivative of $f(x) = \tan\left(\frac{1}{\sqrt{x}}\right)$ is negative:

$$f'(x) = \left(1 + \tan^2\left(\frac{1}{\sqrt{n}}\right)\right)\frac{(-1/2)}{n^{3/2}} < 0.$$

So, by Leibniz criterion, the series converges conditionally.

b) This series is related to the exponential:

$$\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{2^n n!} = \sum_{n=1}^{\infty}\left(\frac{-x^2}{2}\right)^n\frac{1}{n!} = \mathrm{e}^{-x^2/2} - 1.$$

The series converges on $\mathbf{R}$, because the exponential is convergent on $\mathbf{R}$.

b) Observe that $f(x) = -\ln(1-2x) - 2x$, so we use the series of the logarithm:

$$-\ln(1-2x) - 2x = -\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(-2x)^n}{n} - 2x = -2x + \sum_{n=1}^{\infty}\frac{(2x)^n}{n} = \sum_{n=2}^{\infty}\frac{(2x)^n}{n}.$$

This series is convergent for $2x \in [-1, 1)$, that is, it converges for $x \in [-\frac{1}{2}, \frac{1}{2})$.

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