uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

DIFFERENTIAL CALCULUS
FINAL EXAM
Degree in Applied Mathematics and Computation

Time: 3 hours

Problem 1 (**1,5 points**)

Minimize the function $f(x) = \dfrac{x^p}{p} - bx$ for $x \in (0, \infty)$ and prove the following inequality:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \qquad \text{where} \quad a, b > 0, \quad p, q > 1, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Problem 2 (**2 + 1 = 3 points**)

a) Plot the graph of this function, studying the derivative at the left of 0, but without f''.

$$f(x) = \frac{\mathrm{e}^{1/x}}{1+x}, \qquad x \neq 0; \qquad f(0) = 0,$$

b) Study in a reasoned way how many solutions the equation $\dfrac{\mathrm{e}^{1/x}}{1+x} = x^3$ has in $\mathbb{R}$.

Problem 3 (**1 point**)

Calculate the Taylor polynomial of degree 3 at the origin of $f(x) = \sin(2x) - \mathrm{e}^{2x}$ and a bound of the error when we approximate at $x = 1/2$ the function by the polynomial.

Problem 4 (**1 + 1,5 = 2.5 points**)

a) Compute the limit: $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^2}{n^2} \sin\frac{1}{k}$.

b) Study the convergence of the sequence defined by: $a_{n+1} = \dfrac{a_n^3 + 5}{6}, \quad a_0 = 1/2.$

Problem 5 (**0,5 + 0,5 + 1 = 2 points**)

a) Study the convergence of the series $\displaystyle\sum_{n=1}^{\infty} (-1)^n \tan\left(\frac{1}{\sqrt{n}}\right)$.

b) Sum and obtain the interval of convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^n n!}$.

b) Obtain the Taylor series and the interval of convergence of $f(x) = \ln\left(\dfrac{1}{1-2x}\right) - 2x$.
