Chapter 4 : Exercises

Exercise 4.1 A random variable X has an alphabet $\mathcal{A}_X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ with probabilities

$$p_X(x_1) = 0.1,\ p_X(x_2) = 0.2,\ p_X(x_3) = 0.3,\ p_X(x_4) = 0.05,\ p_X(x_5) = 0.15,\ p_X(x_6) = 0.2.$$

Calculate the entropy of the random variable and compare this entropy with that of a random variable with the same alphabet but uniformly distributed.

Exercise 4.2 X is a binary random variable with alphabet $\mathcal{A}_X = \{0, 1\}$ and with probabilities $p_X(0) = p$ and $p_X(1) = 1 - p$. Y is a binary random variable with the same alphabet that is related with X with the following conditional distribution

$$p_{Y|X}(1|0) = p_{Y|X}(0|1) = \varepsilon.$$

a) Calculate $H(X)$, $H(Y)$, $H(Y|X)$, $H(X,Y)$, $H(X|Y)$ and $I(X,Y)$).

b) For a fixed value ε, what value of p maximizes $I(X,Y)$?

c) For a fixed value p, what value of ε minimizes $I(X,Y)$?

Exercise 4.3 The input of a memoryless discrete channel has a 7-symbol alphabet

$$\mathcal{A}_X = \{x_0, x_1, x_2, x_3, x_4, x_5, x_6\}$$

with probabilities

$$p_X(x_0) = 0.05,\ p_X(x_1) = 0.1,\ p_X(x_2) = 0.1,\ p_X(x_3) = 0.15,\ p_X(x_4) = 0.05,\ p_X(x_5) = 0.25,\ p_X(x_6) = 0.3.$$

respectively. The memoryless discrete channel has the following channel matrix

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

a) Calculate the entropy of the channel input, X, and compare it to that of a random variable with the same alphabet but equiprobable symbols.

b) Calculate the entropy of the random variable at the output of the channel, Y, and compare it with that of a random variable with the same alphabet but equiprobable symbols.

c) Calculate the joint entropy $H(X,Y)$, the conditional entropies, $H(X|Y)$ and $H(Y|X)$, and the mutual information $I(X,Y)$.

Exercise 4.4 The channel shown in Figure 4.1 models the so-called Binary Erasure Channel (BEC). It represents a model in which a symbol is decided when there is some certainty about it, but it is marked as doubtful (deleted) when there is not enough certainty. Calculate the channel capacity and represent it as a function of the parameter ε that determines the probability of erasure.

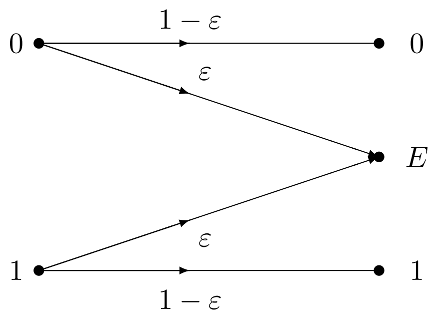

Figure 4.1: Arrow diagram for the BEC channel of Exercise 4.4.

Exercise 4.5 A discrete memoryless channel is plotted in Figure 4.2.

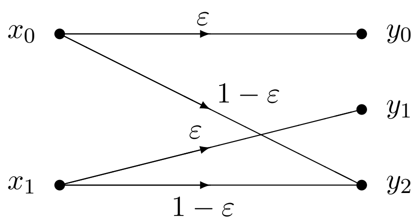

Figure 4.2: Channel for Exercise 4.5.

a) Calculate the channel capacity (value and input probabilities for which it is reached).

b) Assuming that the symbol x_0 is transmitted with probability p, calculate the joint input-output entropy, $H(X,Y)$, and the conditional entropy of the input given the output, $H(X|Y)$ as functions of ε and p. Draw these information measures, as a function of p, for a value $\varepsilon = 0.25$.

Exercise 4.6 Two DMC channels are shown in Figure 4.3.

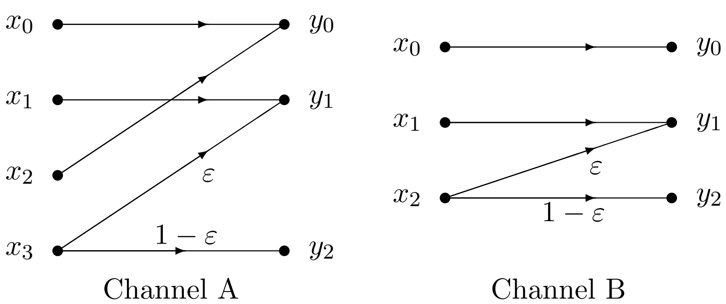

Figure 4.3: Channels for Exercise 4.6.

a) Calculate the capacity of Channel A for $\varepsilon = 0$ and for $\varepsilon = 1$.

b) For Channel A, if the input symbols are equiprobable, $p_X(x_i) = 1/4$, $i = 0, \cdots, 3$, it is known that the Mutual information between the input and output of the channel is $I(X,Y) = 1.069$ and that the entropy of the channel output is $H(Y) = 1.272$. Calculate the value of the parameter ε and the conditional entropy $H(X|Y)$.

c) For Channel B, it is known that for the *a priori* probabilities, $p_X(x_i)$, which maximize the mutual information between the input and output of the channel, $I(X,Y)$, it has a conditional entropy $H(Y|X) = 0.207$ bits for $\varepsilon = 1/4$. Calculate the channel capacity and the probabilities of the input symbols for which it is obtained.

Exercise 4.7 Figure 4.4 shows two DMCs.

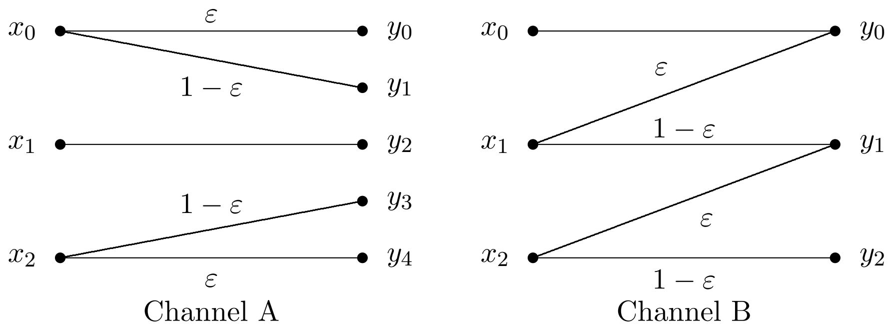

Figure 4.4: Channels for Exercise 4.7.

a) Calculate the capacity of Channel A (value and probabilities of the input symbols for which it is reached).

b) For Channel B, if the input symbols are equiprobable, it is known that the joint entropy is $H(X,Y) = 2.1258$ bits/symbol. Calculate the value of ε, $H(Y)$, $I(X,Y)$, and $H(X|Y)$.

Exercise 4.8 A DMC is shown in Figure 4.5.

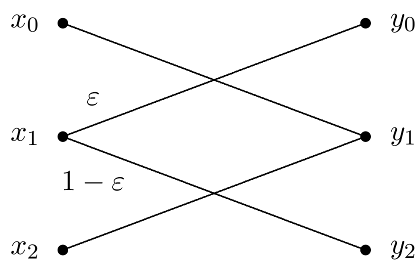

Figure 4.5: Channels for Exercise 4.8.

a) Calculate the capacity of the channel and plot it as a function of ε for $\varepsilon \in [0,1]$.

b) If $p = p_X(x_1)$ is defined and $p_X(x_2) = 0$ is set, calculate the expressions for the conditional entropy $H(X|Y)$ and the joint entropy, $H(X,Y)$, as a function of p and ε, and represent them as a function of p for $\varepsilon = 1/2$.

Exercise 4.9 A channel is shown in Figure 4.6.

a) Calculate the capacity of the channel plot it as a function of ε for $\varepsilon \in [0,1]$.

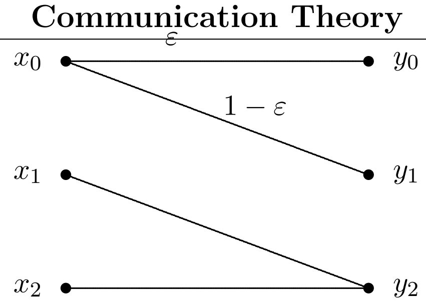

Figure 4.6: Channels for Exercise 4.9.

b) If $p_X(x_1) = p_X(x_2) = p$ is set, calculate the expressions for the conditional entropy $H(X|Y)$ and of the joint entropy, $H(X,Y)$, as a function of p and ε, and plot them as a function of p for $\varepsilon = 1/2$.

Exercise 4.10 The Discrete Memoryless Channel (DMC) shown in Figure 4.7 corresponds to the statistical channel model of a communications system that uses the constellation of four symbols of the same figure.

a) Write the channel matrix associated with the schematic DMC representation, compare it with the channel matrix for that constellation, explain what approximation has been considered for a communications system with that constellation, and obtain the value of ε for this approximation (assume transmission over a Gaussian channel of noise with power spectral density $N_0/2$).

b) Calculate $H(Y|X)$, $H(X|Y)$, $H(X,Y)$ and $I(X,Y)$ for the channel of the figure if the input symbols are equiprobable, and from the analytical expressions obtain the value of ε that makes $H(Y|X)$ and $H(X|Y)$ minimum, and $I(X,Y)$ maximum.

c) Calculate the channel capacity for the minimum and maximum values that ε can take taking into account the structure of the DMC.

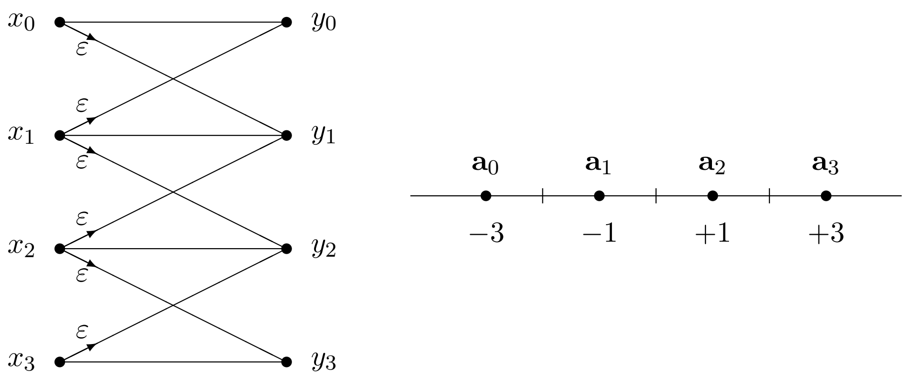

Figure 4.7: Channels for the Exercise 4.10.

Exercise 4.11 The Discrete Memoryless Channel (DMC) shown in Figure 4.8 corresponds to the statistical channel model of a communications system that uses the constellation of four symbols that is plotted in the same figure.

a) Obtain the channel matrix associated with the schematic DMC representation, compare it with the channel matrix for that constellation, explain what approximation has been considered for a communications system with that constellation, and obtain the value of ε for this approximation (assume transmission over a Gaussian channel and noise with power spectral density $N_0/2$).

b) Calculate $H(Y|X)$, $H(X|Y)$, $H(X,Y)$ and $I(X,Y)$ for the DMC if the input symbols are equiprobable, represent them as a function of ε, and obtain the value of ε that makes maximum $H(X,Y)$, $H(Y|X)$ and $H(X|Y)$, and minimum $I(X,Y)$.

c) Calculate the channel capacity as a function of ε.

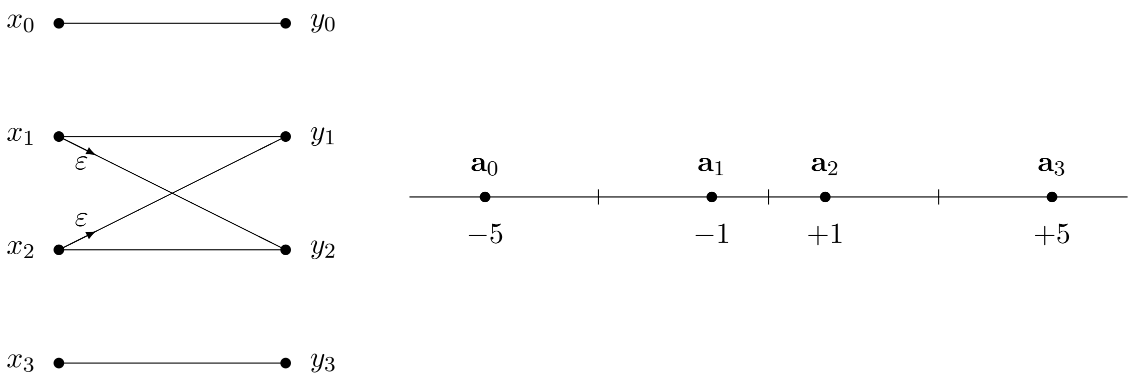

Figure 4.8: Channels for Exercise 4.11.

Exercise 4.12 A DMC with input alphabet $\mathcal{A}_X = \{x_0, x_1, x_2, x_3\}$ and output alphabet $\mathcal{A}_Y = \{y_0, y_1, y_2, y_3\}$ has the following channel matrix

$$\mathbf{P} = \begin{bmatrix} 1-\varepsilon_0 & \varepsilon_0 & 0 & a \\ 0 & 1-\varepsilon_1 & b & 0 \\ 0 & c & 1-\varepsilon_1 & 0 \\ d & 0 & \varepsilon_0 & 1-\varepsilon_0 \end{bmatrix}.$$

a) Determine the value of the constants a, b, c and d, and calculate $H(Y)$, $H(X,Y)$, $H(Y|X)$, $H(X|Y)$, and $I(X,Y)$ for the channel if the input symbols are equiprobable.

b) Calculate the channel capacity as a function of ε_0 and ε_1.

Exercise 4.13 A Discrete Memoryless Channel (DMC) with input alphabet $\mathcal{A}_X = \{x_0, x_1, x_2\}$ and output alphabet $\mathcal{A}_Y = \{y_0, y_1, y_2\}$ is given by the following channel matrix

$$\mathbf{P}_{Y|X} = \begin{bmatrix} 1-\varepsilon_0 & \varepsilon_0 & 0 \\ \varepsilon_1 & 1-2\varepsilon_1 & \varepsilon_1 \\ 0 & \varepsilon_0 & 1-\varepsilon_0 \end{bmatrix}$$

a) Calculate $H(Y)$, $H(Y|X)$, $H(X|Y)$, and $I(X,Y)$ for the channel if input symbols are equiprobable.

b) Calculate the channel capacity as a function of ε_0 if $p_X(x_1)$=0.

c) Calculate the channel capacity as a function of ε_0 and ε_1 and the input distribution for which it is obtained.

d) If $\varepsilon_0 = \varepsilon_1 = \varepsilon$, calculate the value of ε for the maximum capacity . Explain why this capacity is the maximum achievable on a channel with 3 input symbols.

Exercise 4.14 A discrete memoryless channel (DMC) with input alphabet $\mathcal{A}_X = \{x_0, x_1, x_2, x_3\}$ and output alphabet $\mathcal{A}_Y = \{y_0, y_1, y_2, y_3\}$ is given by the following channel matrix

$$\mathbf{P} = \begin{bmatrix} 1-\varepsilon_1 & \varepsilon_1 & a & b \\ \varepsilon_1 & 1-\varepsilon_1 & c & d \\ a & c & 1-\varepsilon_0 & \varepsilon_0 \\ b & d & \varepsilon_0 & 1-\varepsilon_0 \end{bmatrix}.$$

a) Determine the value of the constants a, b, c and d, and give an example of a digital communications system (taking into account the transmitted constellation counts) for which this channel matrix be a good approximation.

b) For the case in which the symbols x_0 and x_1 are transmitted with equal probability, $p_X(x_0) = p_X(x_1) = p/2$, and the symbols x_2 and x_3 also have the same probability, different in general of those of the other two, calculate $H(X)$, $H(Y)$, $H(X,Y)$, $H(Y|X)$, and $H(X|Y)$, as a function of ε_0, ε_1 and p.

c) Calculate the channel capacity as a function of ε_0 and ε_1 and indicate the input distribution for which it is obtained.

Exercise 4.15 Two communication systems based on two discrete memoryless channels (DMC) are defined. System A has as input alphabet $\mathcal{A}_X = \{x_0, x_1\}$ and as output alphabet $\mathcal{A}_Y = \{y_0, y_1\}$. The behavior of the entire system is given by the joint probability of the input and output symbols $P_{X,Y}(x_i, y_j)$ where $i, j \in \{0, 1\}$. Similarly, System B has as input alphabet $\mathcal{A}_Y = \{y_0, y_1\}$ and as output alphabet $\mathcal{A}_Z = \{z_0, z_1\}$. The behavior of the system is given by the joint probability of the input and output symbols $P_{Y,Z}(y_i, z_j)$ where $i, j \in \{0, 1\}$

$P_{X,Y}(x_i, y_j)$	x_0	x_1
y_0	α	$(1-\alpha)\,\epsilon$
y_1	0	$(1-\alpha)(1-\epsilon)$

$P_{Y,Z}(y_i, z_j)$	y_0	y_1
z_0	$\beta(1-\epsilon')$	0
z_1	$\beta\epsilon'$	$1-\beta$

If we want to study the performance of each of the systems independently in order to compare them:

a) Get $H(X)$, $H(Y)$, $H(X,Y)$, $H(Z)$ and $H(Y|Z)$.

b) For each of the channels, obtain respectively $P_{Y|X}(y_j|x_i)$ and $P_{Z|Y}(z_j|y_i)$ and draw the arrpw diagram of the DMC associated in each case. Identify input symbols, output symbols, and transition probabilities.

c) Compute the mutual information of input and output for System A.

If the two channels are concatenated obtaining a new system (A-B):

d) Obtain the equivalent DMC by identifying the transition probabilities in each case.

e) Determine the values of ϵ and ϵ' that lead the equivalent channel to behave as a binary symmetric channel (BSC).

Exercise 4.16 For the DMC that is shown in Figure 4.9, the probability distribution of the input symbols is denoted as $p_X(x_i) \equiv p_i$ with $i \in \{0, 1, 2\}$.

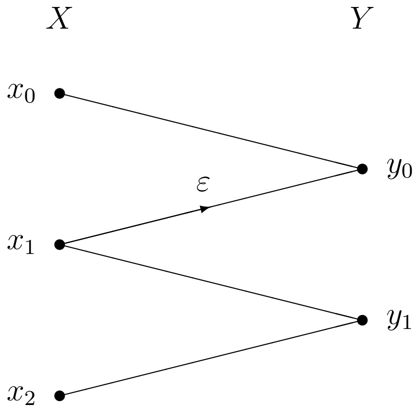

Figure 4.9: Arrow diagram representation for the DMC of the Exercise 4.16.

a) Write the channel matrix corresponding to this equivalent discrete channel.

b) Calculate the entropies $H(X)$, $H(Y)$, $H(X,Y)$, $H(Y|X)$ and $H(X|Y)$ and the mutual information $I(X,Y)$ for the following input distribution:

   i) Equiprobable input symbols: $p_0 = p_1 = p_2 = \frac{1}{3}$.

   ii) Only the symbol x_1 is transmitted: $p_0 = p_2 = 0,\ p_1 = 1$.

c) Obtain the channel capacity, indicating the input distribution for which it is obtained, in the following cases

   i) For the case $\varepsilon = \frac{1}{2}$.

   ii) If the symbol x_2 is removed from the alphabet of X (i.e., for $p_X(x_2) = 0$).
   NOTE: In this section, ε does not take the value from the previous section, it is an arbitrary constant value.

Exercise 4.17 Two mutually dependent binary random variables, X and Y, are related by the following joint probabilities:

$$p_{X,Y}(x_0, y_0) = 0.1 \quad p_{X,Y}(x_0, y_1) = 0.2 \quad p_{X,Y}(x_1, y_0) = 0.3 \quad p_{X,Y}(x_1, y_1) = 0.4$$

a) Get the joint entropy between the two variables

b) Get $H(X)$, $H(Y)$ and $H(X/Y)$

c) Calculate the mutual information between the two variables.

Exercise 4.18 Two communication systems based on two discrete memoryless channels (DMC) are defined. DMC A has as input alphabet $\mathcal{X} = \{x_1, x_2\}$ and as output alphabet $\mathcal{Y} = \{y_1, y_2\}$. The behavior of DMC A is given by the probability of the output symbols conditioned on the input symbols $p_{Y|X}(y_j|x_i)$ where $i, j \in \{1, 2\}$. Similarly, DMC B has as input alphabet $\mathcal{Y} = \{y_1, y_2\}$ and as output alphabet $\mathcal{Z} = \{z_1, z_2, z_3, z_4\}$. The behavior of DMC B is given by the probability of the output symbols conditioned on the input symbols $p_{Z|Y}(z_\ell|y_k)$ where $k \in \{1, 2\}$ and $\ell \in \{1, 2, 3, 4\}$

$p_{Y\|X}(y_j\|x_i)$	x_1	x_2
y_1	$(1-\epsilon_1)$	ϵ_2
y_2	ϵ_1	$(1-\epsilon_2)$

$p_{Z\|Y}(z_\ell\|y_k)$	y_1	y_2
z_1	$(1-\epsilon_3)$	0
z_2	ϵ_3	0
z_3	0	ϵ_4
z_4	0	$(1-\epsilon_4)$

We want to study the performance of each of the systems independently in order to compare them. Suppose in this case that the probabilities of the input symbols to each DMC are $p_X(x_1) = \alpha$ and $p_Y(y_1) = \beta$:

a) Draw the arrow diagram of the two DMCs. Identify input symbols, output symbols, and transition probabilities.

b) Obtain for DMC A: $H(X)$, $H(Y)$ and $H(X,Y)$.

c) For DMC B obtain $H(Z)$ and $H(Y|Z)$.

d) Compute the mutual information of input and output for DMC A and DMC B.

If the two channels are concatenated, obtaining a new DMC (A-B) and with $\epsilon_1 = \epsilon_2 = 0$:

e) Obtain the equivalent DMC by identifying the transition probabilities in each case.

f) Get the channel capacity of the concatenated DMC system.