Communication Theory

English Grades

Chapter 1

Noise in communications systems

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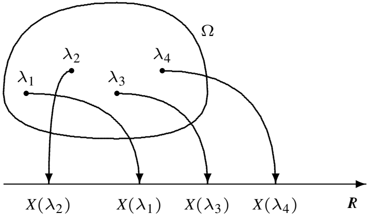
Random Variable (Real)

Function that assigns a numeric (real) value to the output of a random experiment

$$\Omega \to \boldsymbol{R}$$
$$\lambda \in \Omega \to X(\lambda) \in \boldsymbol{R}$$



- Range (domain) of X: $\mathcal{D}_X = \{x \in \boldsymbol{R} : \exists\, \lambda \in \Omega,\, X(\lambda) = x\}$
  - Discrete r.v.: domain formed by a discrete set of values
  - Continuous r.v.: continuous range of values
- Description (probabilistic):
  - Distribution function: $F_X(x)$
  - Probability density function: $f_X(x)$

Distribution Function or Cumulative Distribution Function (CDF)

- Definition

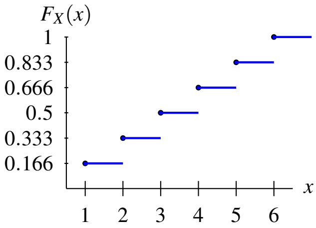
$$F_X(x) = P(X \leq x)$$

- Frequency interpretation (probabilistic)

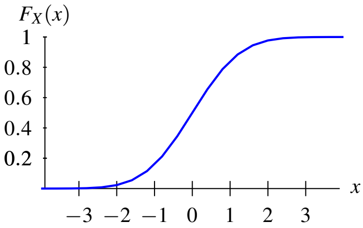
$$F_X(x) = P(X \leq x) = \lim_{n\to\infty} \frac{n_x}{n}$$

n : number of realizations of the random variable X

n_x : number of results in the n realizations with $X \leq x$

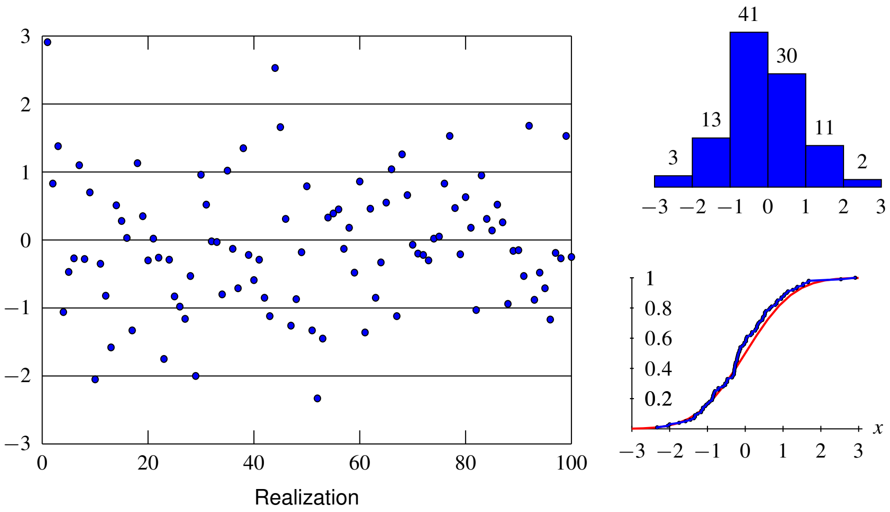


(a) Discrete (Die)

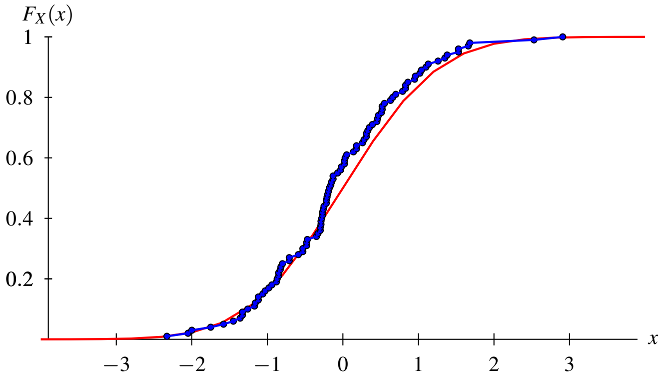


(b) Continuous (Gaussian: $\mathcal{N}(0,1)$)

Realizations of a Gaussian random variable

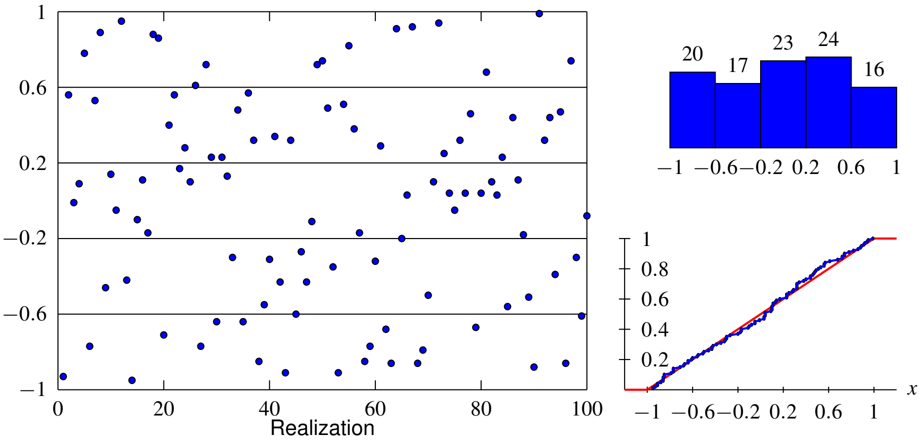


Estimate of the distribution function (Gaussian)

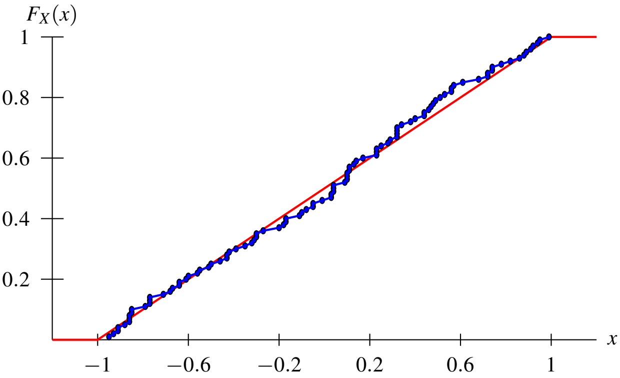


Gaussian: $\mathcal{N}(0,1)$

Realizations of a Uniform random variable



Estimate of the distribution function (Uniform)



Uniform: $\mathcal{U}(-1, 1)$

Properties of the distribution function

1. $0 \leq F_X(x) \leq 1$
2. $x_1 < x_2 \rightarrow F_X(x_1) \leq F_X(x_2)$ ($F_X(x)$ is non-decreasing)
3. $F_X(-\infty) = 0$ and $F_X(\infty) = 1$ $(\lim_{x\to-\infty} F_X(x) = 0, \lim_{x\to\infty} F_X(x) = 1)$
4. $F_X(x^+) = F_X(x)$ ($F_X(x)$ is continuous from the right)
5. $F_X(b) - F_X(a) = P(a < X \leq b)$

$$P(a \leq X \leq b) = F_X(b) - F_X(a^-)$$
$$P(a < X < b) = F_X(b^-) - F_X(a)$$
$$P(a \leq X < b) = F_X(b^-) - F_X(a^-)$$

6. $P(X = a) = F_X(a) - F_X(a^-)$
7. $P(X > x) = 1 - F_X(x)$

Probability density function (PDF)

- Definition

$$f_X(x) = \frac{d}{dx} F_X(x)$$

  - Discrete r.v.: ground points $p_i = P(X = x_i)$
  - Notation for discrete r.v.: $p_X(x_i) = p_i$

- Frequency interpretation (probabilistic)

$$f_X(x) = \lim_{\Delta_x \to 0} \frac{P(x \leq X \leq x + \Delta_x)}{\Delta_x}$$

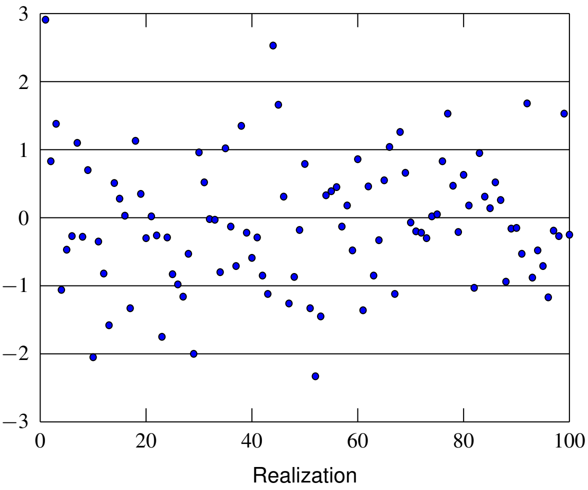
$$f_X(x) = \lim_{\Delta_x \to 0} \left\{ \frac{1}{\Delta_x} \lim_{n \to \infty} \frac{n_x}{n} \right\}$$

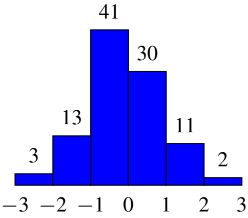
n : number of realizations of the random variable X
n_x : number of results in the n realizations such that $x \leq X \leq x + \Delta_x$

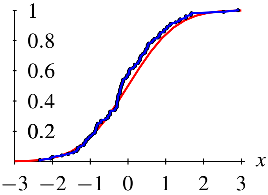
Interpretation of the probability density function

- The PDF indicates how the realizations of a random variable are distributed
  - Large values of $f_X(x)$: high probability that the random variable takes values in that range
  - Allows the calculation of probabilities on the possible values of a random variable
- A PDF can be interpreted as a histogram pushed to the limit
- Several examples are shown below
  - Uniform random variable
  - Gaussian random variable

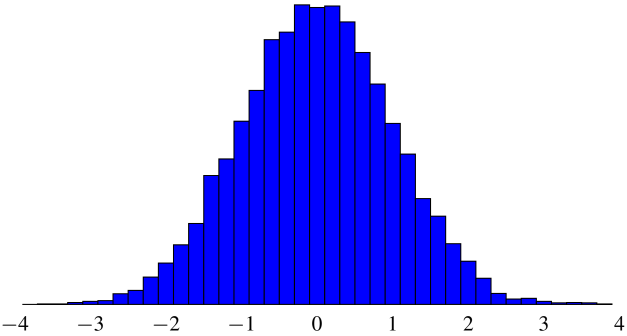
Realizations of a Gaussian random variable



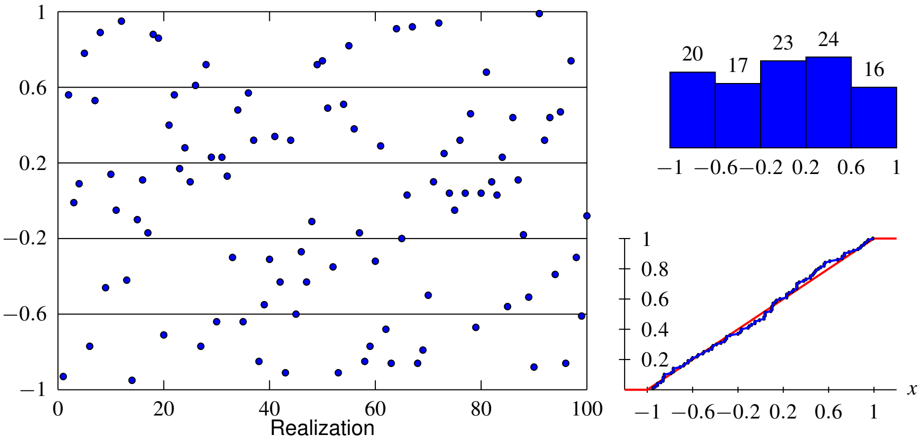




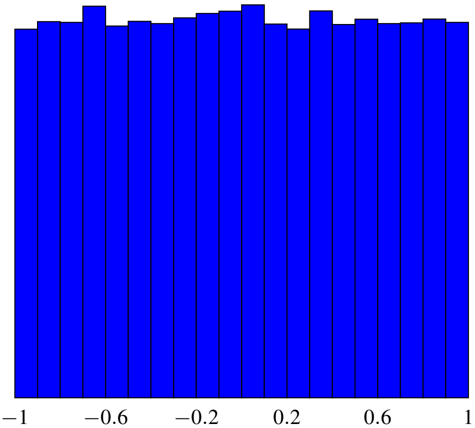
Histogram of 10000 realizations of a Gaussian random variable



Realizations of a Uniform random variable



Histogram of 10000 realizations of a Uniform random variable.



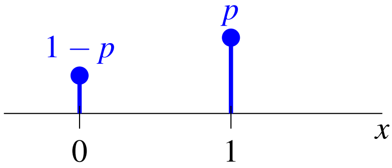
Properties of $f_X(x)$

1. $f_X(x) \geq 0$
2. $$\int_{-\infty}^{\infty} f_X(x)\ dx = 1$$
3. $$\int_{a^+}^{b^+} f_X(x)\ dx = P(a < X \leq b)$$
4. In general, $P(X \in A) = \int_A f_X(x)\ dx$
5. $$F_X(x) = \int_{-\infty}^{x^+} f_X(u)\ du$$

Bernoulli random variable

- Discrete random variable with $\mathcal{D}_X = \{0, 1\}$
- Parameter: $p = P(X = 1)$

$$f_X(x) = \begin{cases} 1-p, & x = 0 \\ p, & x = 1 \\ 0, & \text{else} \end{cases}$$



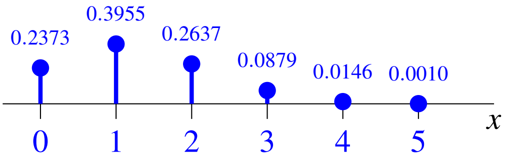
- Application examples in communications
  - Binary data generator
  - Error model

Binomial Random Variable

- Number of 1's in n Bernoulli experiments (indep.)
  - Parameters: n, p
    - ★ n: number of Bernoulli experiments
    - ★ p: probability of "1" in each experiment
- $\mathcal{D}_X = \{0, 1, \ldots, n\}$

$$f_X(x) = \begin{cases} \binom{n}{x}\, p^x\, (1-p)^{n-x}, & 0 \leq x \leq n \text{ y } x\, in\mathbb{Z} \\ 0, & \text{else} \end{cases}$$

Example: $n = 5$, $p = 0.25$



- Example of application in communications
  - Number of bits received in error in a block of n bits

Binomial Coefficients - Pascal's Triangle

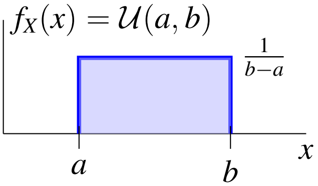
$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$n = 0$	1
$n = 1$	1 1
$n = 2$	1 2 1
$n = 3$	1 3 3 1
$n = 4$	1 4 6 4 1
$n = 5$	1 5 10 10 5 1
$n = 6$	1 6 15 20 15 6 1
$n = 7$	1 7 21 35 35 21 7 1
$n = 8$	1 8 28 56 70 56 28 8 1
$n = 9$	1 9 36 84 126 126 84 36 9 1
$n = 10$	1 10 45 120 210 252 210 120 45 10 1
$n = 11$	1 11 55 165 330 462 462 330 165 55 11 1
$n = 12$	1 12 66 220 495 792 924 792 495 220 66 12 1

Uniform Random Variable

- Continuous random variable with parameters a and b
  - Notation: $\mathcal{U}(a,b)$
  - Domain: $\mathcal{D}_X = (a,b)$

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & \text{else} \end{cases}$$

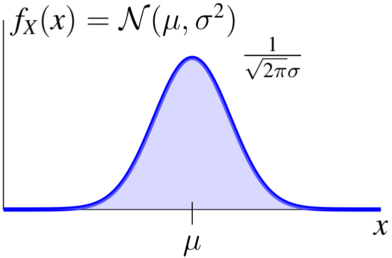


- Example of application in communications
  - Random phase in a sinusoid: r.v. uniform between 0 and 2π

Gaussian (Normal) random variable

- Parameters: mean (μ), and variance (σ^2)
  - Notation: $\mathcal{N}(\mu, \sigma^2)$
  - Domain: $\mathcal{D}_X = \boldsymbol{R}$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$



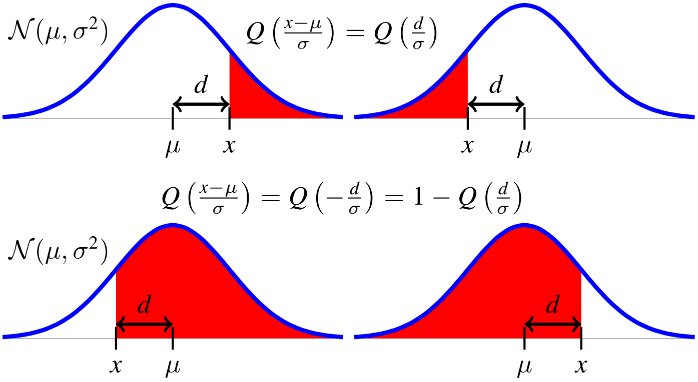
- Example of application in communications
  - Thermal noise modeling

Integrals over Gaussian distributions $\mathcal{N}(\mu, \sigma^2)$

- If the Gaussian distribution has mean μ and variance σ^2

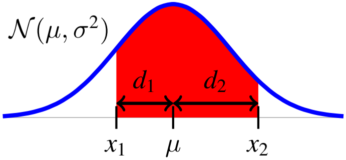
$$P(X > x) = Q\left(\frac{x-\mu}{\sigma}\right)$$

- Graphical interpretation (considering definition and symmetry)

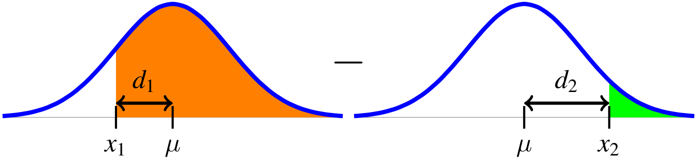


Integrals over $\mathcal{N}(\mu, \sigma^2)$ in intervals

- In general, they can be written as sums or differences of different terms involving integrals from a point to $\pm\infty$, which can be obtained using the function $Q(x)$
- An illustrative example



$$\int_{x_1}^{x_2} \mathcal{N}(\mu, \sigma^2) = \int_{x_1}^{\infty} \mathcal{N}(\mu, \sigma^2) - \int_{x_2}^{\infty} \mathcal{N}(\mu, \sigma^2) = \left[1 - Q\left(\frac{d_1}{\sigma}\right)\right] - \left[Q\left(\frac{d_2}{\sigma}\right)\right]$$



Functions of a random variable

- A function $Y = g(X)$ of a random variable is another random variable
- Cumulative Distribution Function (CDF)

$$F_Y(y) = P(Y \leq y) = P(g(X) \leq y)$$

$$F_Y(y) = P(x \in B_X^g(y)), \; B_X^g(y) = \{x \in \boldsymbol{R} : g(x) \leq y\}$$

- Probability density function (PDF)

$$f_Y(y) = \sum_{i=1}^{N_r} \frac{f_X(x_i)}{|g'(x_i)|}$$

  - $\{x_i\}$: roots of the equation $y = g(x)$
  - $g'(x)$: derivative of the function $g(x)$
  - Conditions: finite number of roots, N_r and $g(x_i) \neq 0 \; \forall x_i$

Statistical moments

- Expected value (mean, or mathematical expectation)

$$m_X = E[X] = \int_{-\infty}^{\infty} x\, f_X(x)\, dx$$

- Expected value of a function of X ($g(X)$)

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx$$

- Moment of order n

$$m_X^n = \int_{-\infty}^{\infty} x^n\, f_X(x)\, dx$$

- Variance

$$\sigma_X^2 = E\left[(X - m_X)^2\right] = \int_{-\infty}^{\infty} (x - m_X)^2\, f_X(x)\, dx$$

NOTE: $\sigma_X^2 = E\left[(x - m_X)^2\right] = E[X^2] - (E[X])^2 = E[X^2] - (m_X)^2$

Moment properties

Considering random variables X, Y, and constant c:

- $E[X+Y] = E[X] + E[Y] = m_X + m_Y$ (Linear operator)
- $E[c] = c$
- $E[c\,X] = c\,E[X]$
- $E[X+c] = E[X] + c$
- $\text{Var}(c) = 0$
- $\text{Var}(c\,X) = c^2\,\text{Var}(X)$
- $\text{Var}(X+c) = \text{Var}(X)$

Multidimensional Random Variables

- It is possible to work jointly with two random variables defined on the same sample space Ω
- Joint probabilistic modeling
  - Joint distribution function

$$F_{X,Y}(x,y) = P(X \leq x, Y \leq y)$$

  - Joint probability density function

$$f_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y)$$

Properties of $F_{X,Y}(x,y)$ and $f_{X,Y}(x,y)$

- $$F_X(x) = F_{X,Y}(x,\infty)$$
- $$F_Y(y) = F_{X,Y}(\infty,y)$$
- $$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dy$$
- $$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dx$$
- $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dx\ dy = 1$$
- $$P((X,Y)\in A) = \int\int_{(x,y)\in A} f_{X,Y}(x,y)\ dx\ dy$$
- $$F_{X,Y}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(u,v)\ du\ dv$$

Functions of random variables

- Functions of random variables: $Z=g(X,Y),\ W=h(X,Y)$

$$\left\{\begin{array}{l} Z=g(X,Y)\\ W=h(X,Y)\end{array}\right.$$

- Joint distribution function, $F_{Z,W}(z,w)$

$$F_{Z,W}(z,w)=P(Z\leq z,W\leq w)=P\left((x,y)\in B_{X,Y}^{g,h}(z,w)\right)$$

$$B_{X,Y}^{g,h}(z,w)=\{(x,y)\in \boldsymbol{R}^2: g(x,y)\leq z, h(x,y)\leq w\}$$

- Joint probability density function

$$f_{Z,W}(z,w)=\sum_i \frac{f_{X,Y}(x_i,y_i)}{|\det\ \mathbf{J}(x_i,y_i)|},\quad \mathbf{J}(x,y)=\left[\begin{array}{cc} \frac{\partial z(x,y)}{\partial x} & \frac{\partial z(x,y)}{\partial y}\\ \frac{\partial w(x,y)}{\partial x} & \frac{\partial w(x,y)}{\partial y}\end{array}\right]$$

  - $\{x_i,y_i\}$: roots of the system of equations $z=g(x,y), w=h(x,y)$
  - Finite number of roots and non-zero determinant for all of them

Conditional probability density function

- Meaning of a conditional probability:
  - Knowing the value of one variable may vary the (prior) probabilities on the other

$$f_{Y|X}(y|x) = \begin{cases} \frac{f_{X,Y}(x,y)}{f_X(x)}, & f_X(x) \neq 0 \\ 0, & \text{else} \end{cases}$$

    - ★ In general, it may happen that $f_{Y|X}(y|x) \neq f_Y(y)$
- Definition of statistical independence:

$$f_{Y|X}(y|x) = f_Y(y)$$

$$f_{X|Y}(x|y) = f_X(x)$$

  - Implication: for independent random variables

$$f_{X,Y}(x,y) = f_X(x)\, f_Y(y)$$

Statistical moments

- Expected value of a function $g(X, Y)$

$$E[g(X,Y)] = \int_{\infty}^{\infty} \int_{\infty}^{\infty} g(x,y)\, f_{X,Y}(x,y)\; dx\; dy$$

- Particular cases
  - Correlation: $g(X,Y) = X\ Y$
  - Covariance: $g(X,Y) = (X - m_X)\ (Y - m_Y)$
- Implication of independence: if $g(X,Y) = g_1(X)\ g_2(Y)$

$$E\left[g_1(X)\ g_2(Y)\right] = E\left[g_1(X)\right]\ E\left[g_2(Y)\right]$$

REMARK: Only under independence between X and Y !!!!

Uncorrelation

- Correlation coefficient

$$\rho_{X,Y} = \frac{\text{Cov}(X,Y)}{\sigma_X \; \sigma_Y}, \quad 0 \leq |\rho_{X,Y}| \leq 1$$

  - If $\rho_{X,Y} = 0$: r.v.'s **uncorrelated**
    - ⋆ Independence implies uncorrelation
    - ⋆ Uncorrelation does not imply independence
  - If $\rho_{X,Y} = \pm 1$: linear relationship $Y = aX + b$

$$\rho_{X,Y} = +1 \rightarrow a > 0; \quad \rho_{X,Y} = -1 \rightarrow to < 0$$

- Uncorrelation only implies independence for Jointly-Gaussian r.v.'s
  - Except for this case, in general, uncorrelation does not imply independence !!!

Jointly-Gaussian random variables

Variables characterized by a Joint-Gaussian (multivariate normal) PDF

- For two variables, X, Y:
  - 2-D Gaussian

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} e^{-\frac{1}{(1-\rho^2)}\left(\frac{(x-\mu_X)^2}{2\sigma_X^2}+\frac{(y-\mu_Y)^2}{2\sigma_Y^2}-\frac{\rho(x-\mu_x)(y-\mu_Y)}{\sigma_X\sigma_Y}\right)}$$

- For n random variables $\mathbf{X} = X_1, X_2, \ldots, X_n$
  - Variables with a Joint-Gaussian PDF n-D

$$f_{\mathbf{X}}(x_1, x_2, \ldots, x_n) = \frac{1}{\sqrt{(2\pi)^n \det(\mathbf{C})}} e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})\mathbf{C}^{-1}(\mathbf{x}-\boldsymbol{\mu})^T}$$

    - ⋆ Vector of means: $\boldsymbol{\mu} = [\mu_1, \mu_2, \ldots, \mu_n]^T, \qquad \mu_i = E[X_i]$
    - ⋆ Covariance matrix: $\mathbf{C}$, given by

$$C_{i,j} = \text{Cov}(X_i, X_j) = \rho_{i,j}\,\sigma_i\,\sigma_j$$

Jointly Gaussian properties of r.v.'s

- Fully characterized by $\boldsymbol{\mu}$ and $\mathbf{C}$ (2nd order statistics)
- If n random variables are jointly Gaussian, then any subset is also jointly Gaussian distributed
  - In particular, all individual variables are Gaussian
- Any subset of jointly-Gaussian r.v.'s, conditioned on another subset of the same original joint-Gaussian r.v.'s, has a jointly-Gaussian distribution
  - The parameters of the multivariate normal distribution are modified
- Any set of linear combinations of $(X_1, X_2, \ldots, X_n)$ is joint Gaussian
  - In particular, individually any linear combination Y_i is Gaussian
- Two uncorrelated variables are independent
  - For jointly-Gaussian random variables independence is equivalent to uncorrelation
- If the variables are uncorrelated, $\rho_{i,j} = 0 \; \forall i \neq j$
  - $\mathbf{C}$ is a diagonal matrix

Sum of random variables

- **Law of large numbers (weak)**: If $(X_1, X_2, \ldots, X_n)$ are *uncorrelated* and they all have the same mean m_X and variance $\sigma_X^2 < \infty$, regardless of its distribution, for any $\varepsilon > 0$,

$$\text{if } Y = \frac{1}{n}\sum_{i=1}^{n} X_i, \quad \lim_{n\to\infty} P(|Y - m_X| > \varepsilon) = 0$$

- **Central Limit Theorem**: If $(X_1, X_2, \ldots, X_n)$ are *independent* with means $m_1, m_2, \ldots, m_n$, and variances $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$, then the distribution of

$$Y = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} \frac{X_i - m_i}{\sigma_i}$$

converges to a Gaussian distribution with mean 0 and variance 1

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \equiv \mathcal{N}(0,1)$$

Sum of random variables (II)

- Particular case: variables *independent and identically distributed (i.i.d.)*, that is, they all have the same distribution with the same mean m and the same variance σ^2; the average

$$Y = \frac{1}{n}\sum_{i=1}^{n} X_i,$$

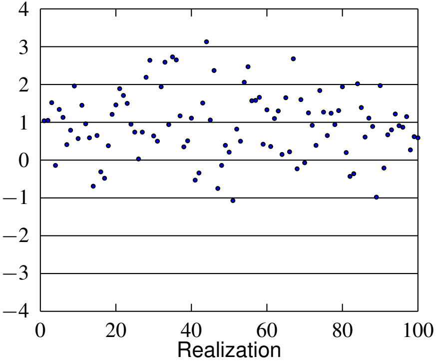
converges to a distribution

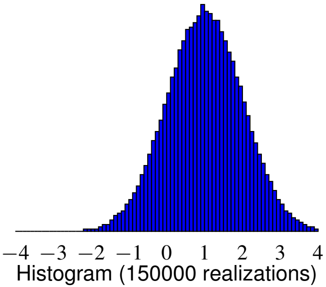
$$Y \sim \mathcal{N}(m, \frac{\sigma^2}{n})$$

  - This is so even if the distribution of each X_i is not Gaussian
- Reminder: conditions to satisfy
  - Law of large numbers (weak): non-correlation
  - Central Limit Theorem: independence

Realizations of 1 Gaussian random variable

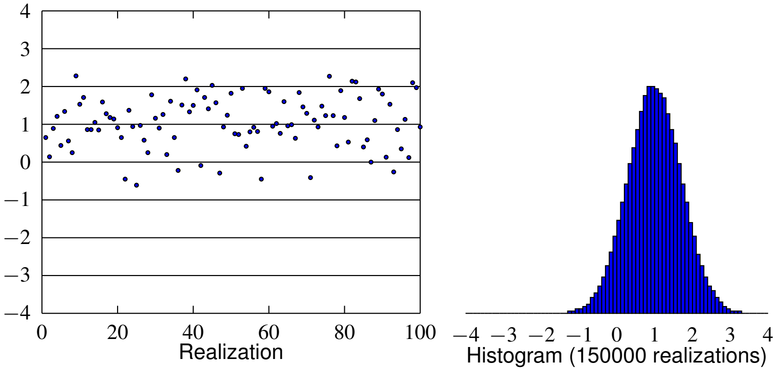
- Gaussian random variable: mean $m_X = 1$, variance $\sigma_X^2 = 1$





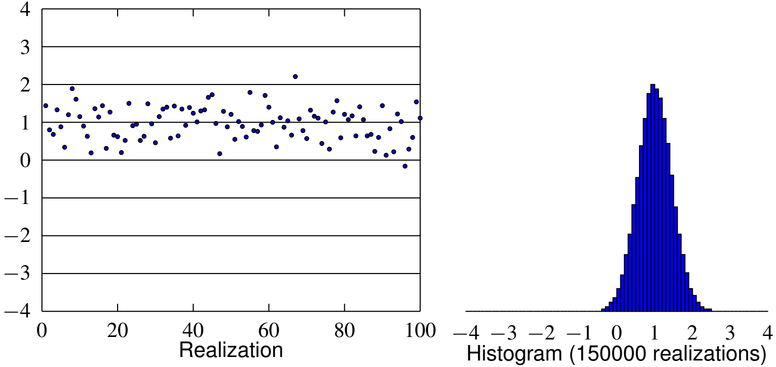
Average of 2 Gaussian random variables

- Gaussian random variables: mean $m_X = 1$, variance $\sigma_X^2 = 1$



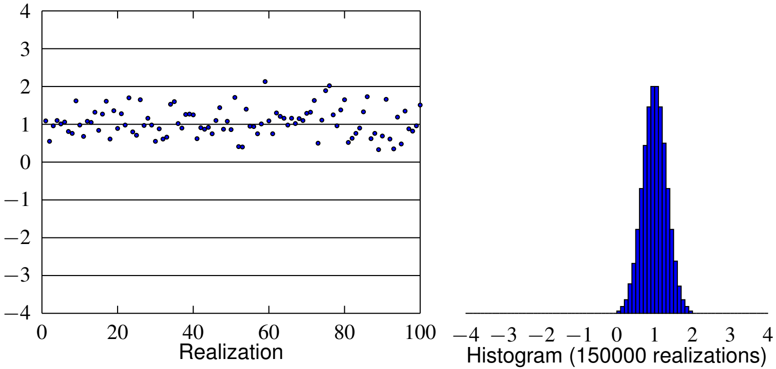
Average of 5 Gaussian random variables

- Gaussian random variables: mean $m_X = 1$, variance $\sigma_X^2 = 1$



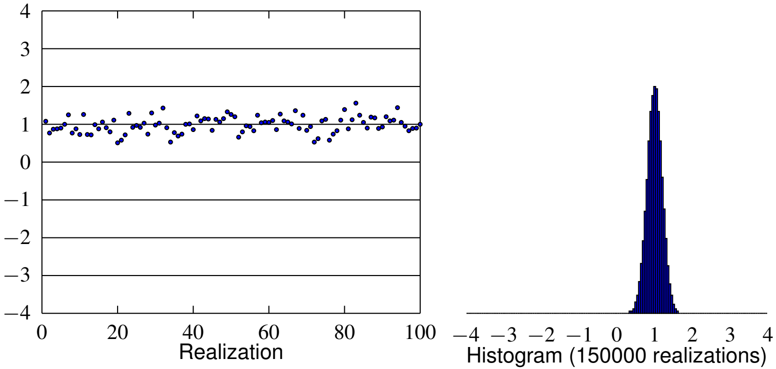
Average of 10 Gaussian random variables

- Gaussian random variables: mean $m_X = 1$, variance $\sigma_X^2 = 1$



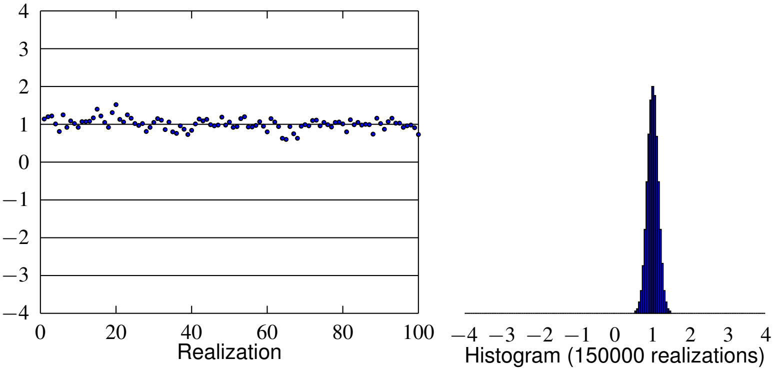
Average of 25 Gaussian random variables

- Gaussian random variables: mean $m_X = 1$, variance $\sigma_X^2 = 1$



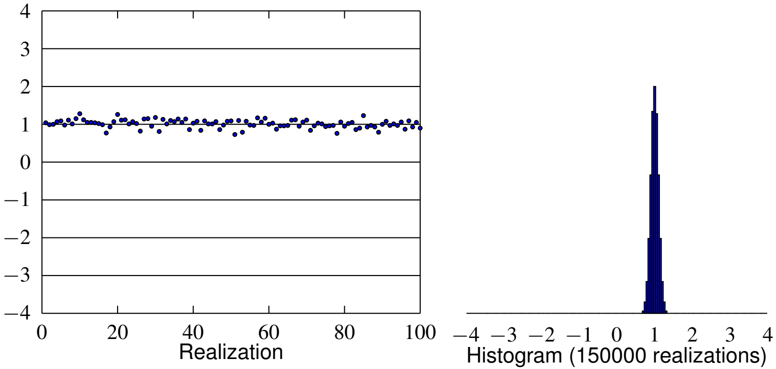
Average of 50 Gaussian random variables

- Gaussian random variables: mean $m_X = 1$, variance $\sigma_X^2 = 1$



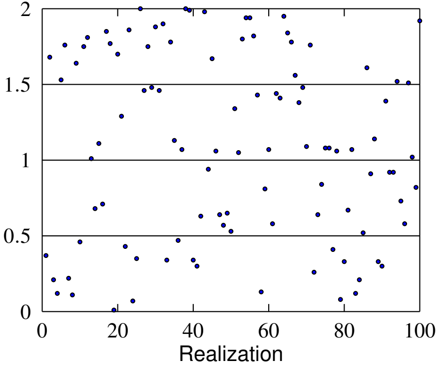
Average of 100 Gaussian random variables

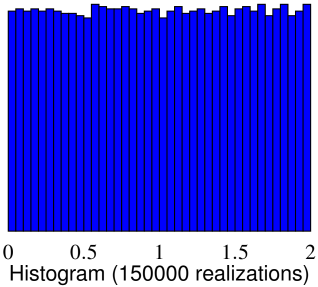
- Gaussian random variables: mean $m_X = 1$, variance $\sigma_X^2 = 1$



Realizations of 1 Uniform random variable

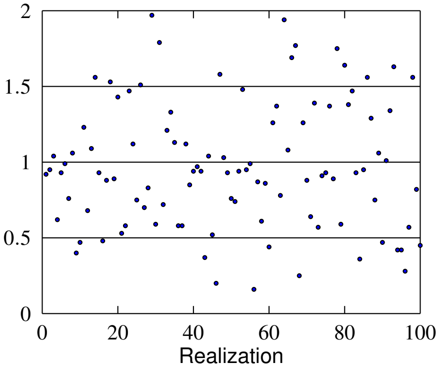
- Uniform random variable: $\mathcal{U}(-1, 1)$

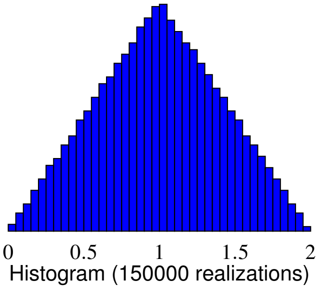




Average of 2 Uniform random variables

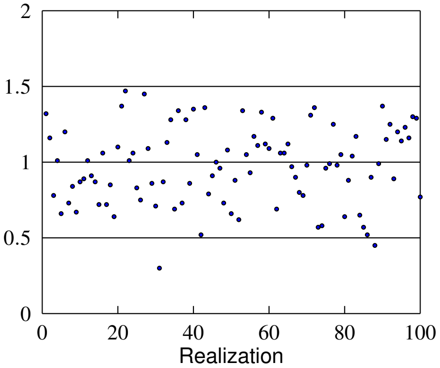
- Uniform random variables: $\mathcal{U}(-1,1)$

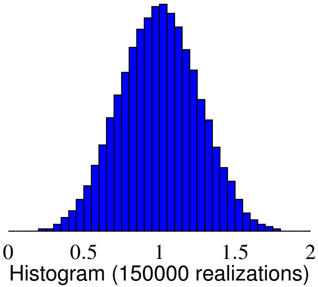




Average of 5 Uniform random variables

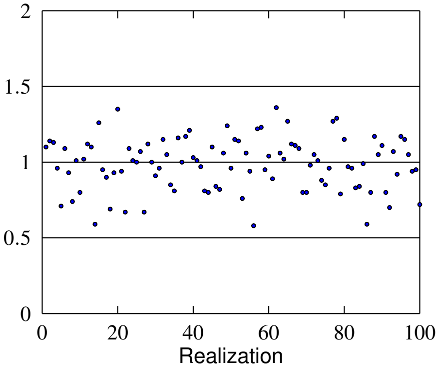
- Uniform random variables: $\mathcal{U}(-1, 1)$

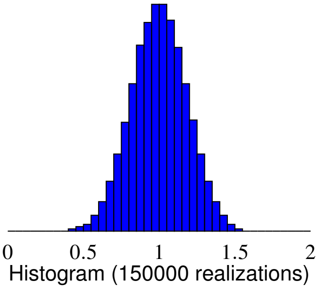




Average of 10 Uniform random variables

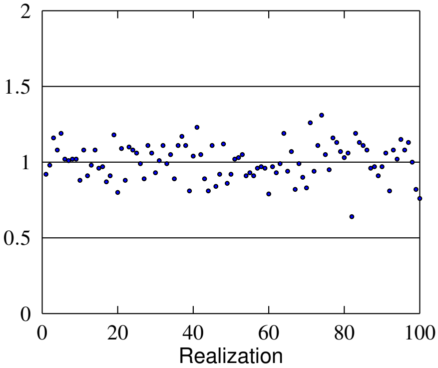
- Uniform random variables: $\mathcal{U}(-1, 1)$

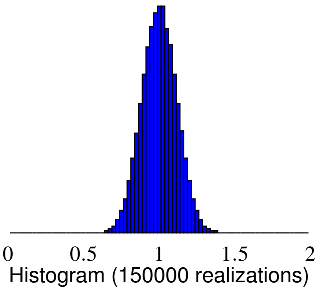




Average of 25 Uniform random variables

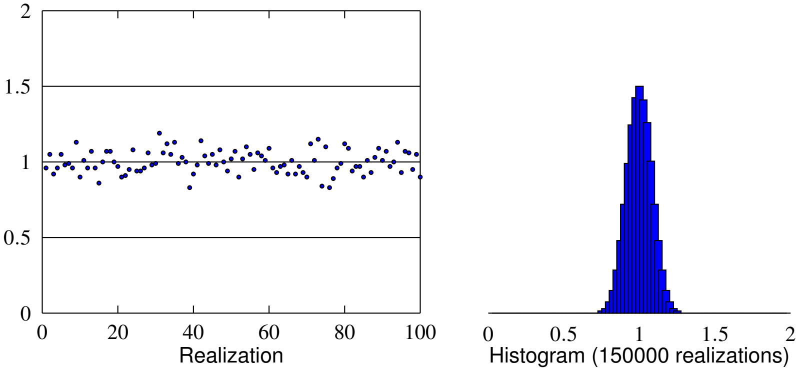
- Uniform random variables: $\mathcal{U}(-1,1)$





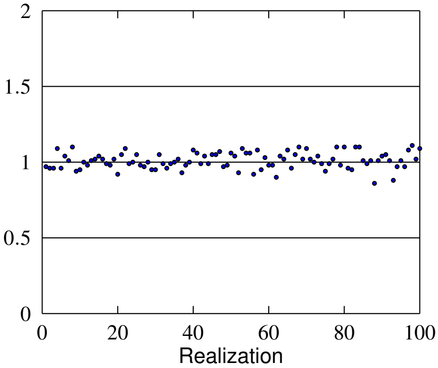
Average of 50 Uniform random variables

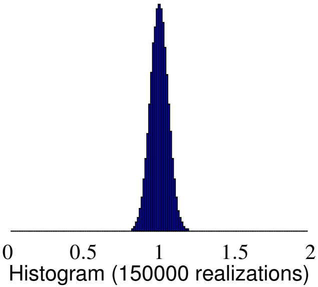
- Uniform random variables: $\mathcal{U}(-1, 1)$



Average of 100 Uniform random variables

- Uniform random variables: $\mathcal{U}(-1,1)$





Stochastic or random processes (real)

- Extension of the concept of random variable including time dependency
  - Random variable: $\lambda \in \Omega \rightarrow X(\lambda)$
  - Stochastic process (continuous or discrete time): $\lambda \in \Omega \rightarrow X(t, \lambda)$ or $X[n, \lambda]$
- Particularizations
  - $X(t, \lambda_i)$ or $X[n, \lambda_i]$: time signal associated with λ_i
  - $X(t_i, \lambda)$ or $X[n_i, \lambda]$: random variable $X(\lambda)$
  - $X(t_i, \lambda_j)$ or $X[n_i, \lambda_i]$: individual realization of a r.v.
- Notation: $X(t)$ or $X[n]$
- Interpretation: indexed set of random variables
  - Continuous index ($t \in \boldsymbol{R}$): continuous time stochastic process
  - Discrete index ($n \in \mathbb{Z}$): discrete-time random process

Random process $X(t,\lambda) \equiv X(t)$ - Example I

- Random experiment: throwing a dice
  - 6 possible outcomes

$$\lambda \in \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6\}$$

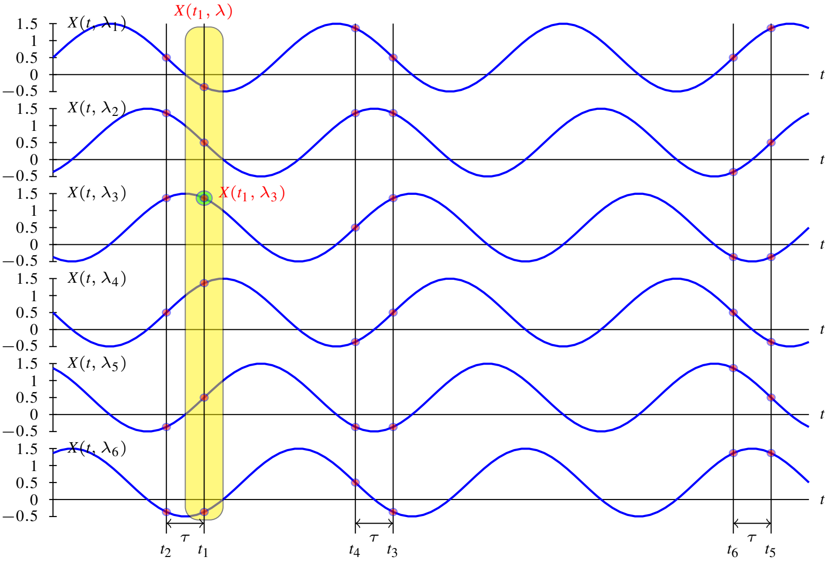
- Random variable
  - Usual assignment: $\lambda_i = i$
  - Domain $\mathcal{D}_X = \{1, 2, 3, 4, 5, 6\}$
- Stochastic process
  - A signal for each possible outcome of the experiment

$$X(t,\lambda_i) = \frac{1}{2} + \sin(\omega_0 t - \theta_i)$$

with $\theta_i = (i-1)\dfrac{2\pi}{6}$

for $i \in \{1, 2, 3, 4, 5, 6\}$

Random process $X(t, \lambda) \equiv X(t)$ - Example I



Random process $X(t, \lambda) \equiv X(t)$ - Example II

- Random experiment: throwing a dice
  - 6 possible outcomes

$$\lambda \in \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6\}$$

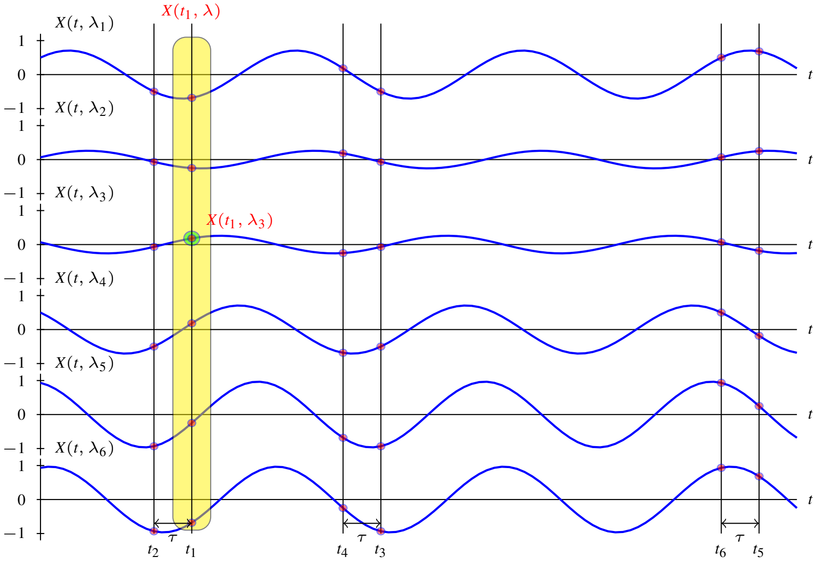
- Random process
  - A signal for each possible outcome of the experiment

$$X(t, \lambda_i) = \frac{1}{2}\cos(\omega_0 t) + \frac{1}{2}\sin(\omega_0 t - \theta_i)$$

with $\theta_i = (i-1)\frac{2\pi}{6}$

for $i \in \{1, 2, 3, 4, 5, 6\}$

Random process $X(t, \lambda) \equiv X(t)$ - Example II



Description of a random process

- Analytical description

$$X(t) = f(t, \boldsymbol{\theta})$$

$\boldsymbol{\theta} = \{\theta_1, \theta_2, \ldots, \theta_n\}$: vector of n random variables

  - Equation $f(t, \boldsymbol{\theta})$
  - Statistical description of $\boldsymbol{\theta}$ (joint PDF)

$$f_{\boldsymbol{\theta}}(\boldsymbol{x}) = f_{\theta_1,\theta_2,\ldots,\theta_n}(x_1, x_2, \cdots, x_n)$$

- Statistical description
  - Complete: $\forall\ (t_1, t_2, \ldots, t_n) \in \boldsymbol{R}^n$, $\forall\ n$

$$f_{X(t_1),X(t_2),\ldots,X(t_n)}(x_1, x_2, \ldots, x_n)$$

  - Of order M: $\forall\ n \leq M$, $\forall\ (t_1, t_2, \ldots, t_n) \in \boldsymbol{R}^n$

$$f_{X(t_1),X(t_2),\ldots,X(t_n)}(x_1, x_2, \ldots, x_n)$$

Expectations of processes (statistical averages)

- Mean of a random process

$$m_X(t) = E[X(t)] = \int_{-\infty}^{\infty} x\, f_{X(t)}(x) dx$$

- Autocorrelation function of a random process

$$R_X(t_1, t_2) = E[X(t_1)\; X(t_2)]$$

$$R_X(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1\; x_2\; f_{X(t_1),X(t_2)}(x_1, x_2)\; dx_1\; dx_2$$

NOTE: Definition for complex random processes

$$R_X(t_1, t_2) = E[X(t_1)\; X^*(t_2)]$$

$$R_X(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1\; x_2^*\; f_{X(t_1),X(t_2)}(x_1, x_2)\; dx_1\; dx_2$$

Some examples of description

- Analytical description

$$X(t) = A\ \cos(\omega_0 t) + B\ \sin(\omega_0 t)$$

  - $A \sim \mathcal{N}(0, \sigma_A^2)$
  - $B \sim \mathcal{N}(0, \sigma_B^2)$
  - A and B are independent: $f_{A,B}(a,b) = f_A(a) \times f_B(b)$

- Statistical Description (of order 2)

$$f_{X(t_1),X(t_2)}(x_1,x_2) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{(1-\rho^2)}\left(\frac{(x_1-1)^2}{2}+\frac{(x_2+1)^2}{2}-\rho(x_1-1)(x_2+1)\right)}$$

Examples I and II: mean and autocorrelation function

- Example I
  - Mean $m_X(t) = \frac{1}{2}$
  - Autocorrelation function

$$R_X(t_1, t_2) = \frac{1}{4} + \frac{1}{2}\cos(\omega_0(t_1 - t_2))$$

$$R_X(t+\tau, t) = \frac{1}{4} + \frac{1}{2}\cos(\omega_0 \tau)$$

- Example II
  - Mean $m_X(t) = \frac{1}{2}\cos(\omega_0 t)$
  - Autocorrelation function

$$R_X(t_1, t_2) = \frac{1}{4}\cos(\omega_0(t_1 - t_2)) + \frac{1}{4}\cos(\omega_0(t_1 + t_2))$$

$$R_X(t+\tau, t) = \frac{1}{4}\cos(\omega_0 \tau) + \frac{1}{4}\cos(\omega_0(2t + \tau))$$

Stationarity and cyclostationarity

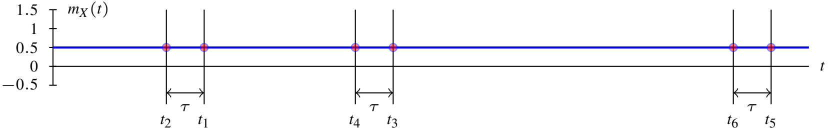
- (Strict Sense) Stationarity (SSS): $\forall\ (t_1, t_2, \ldots, t_n)$, $\forall n$, $\forall \Delta$

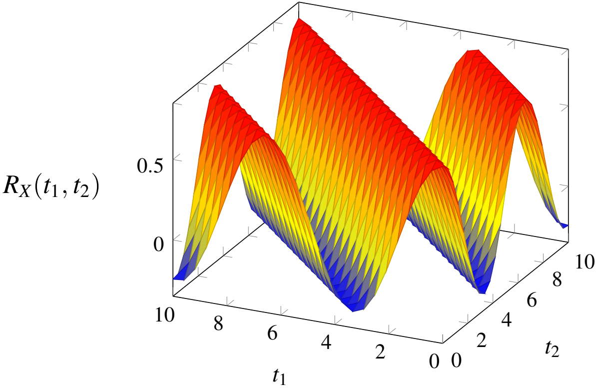
$$f_{X(t_1),X(t_2),\ldots,X(t_n)}(x_1, x_2, \ldots, x_n) = f_{X(t_1+\Delta),X(t_2+\Delta),\ldots,X(t_n+\Delta)}(x_1, x_2, \ldots, x_n)$$

  - Stationarity of order M: for $n \leq M$
- (Wide Sense) Stationarity (WSS)
  1. $m_X(t) = m_X$ (does not depend on t)
  2. $R_X(t_1, t_2) = R_X(t_1 - t_2) = R_X(\tau)$ (defining $\tau = t_1 - t_2$)

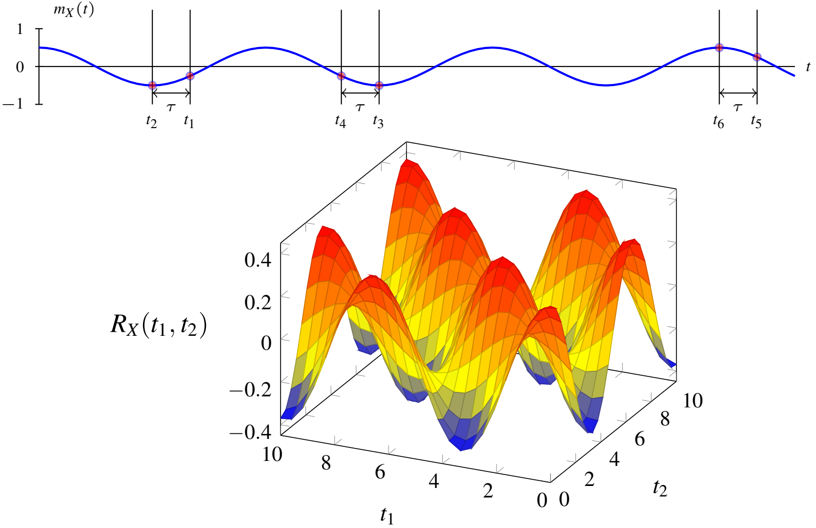
     It is also usually denoted $R_X(t+\tau, t) = R_X(\tau)$
- (Wide Sense) Cyclostationarity (WSCS)
  1. $m_X(t + T_o) = m_X(t)$
  2. $R_X(t + \tau + T_o, t + T_o) = R_X(t + \tau, t)$, for all t and τ

Mean and autocorrelation function: $m_X(t)$, $R_X(t_1, t_2)$ - Ex. I

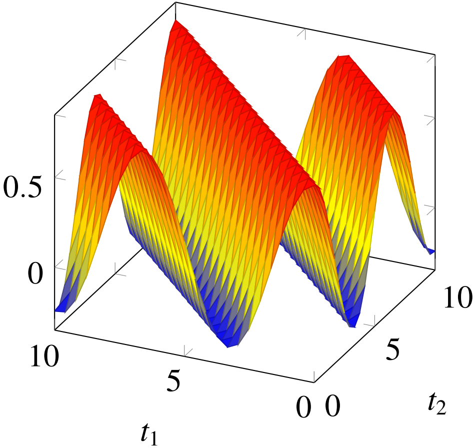


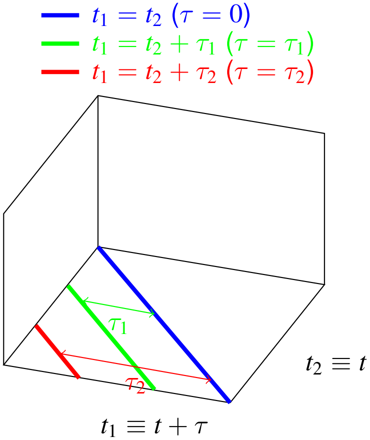


Mean and autocorrelation function: $m_X(t)$, $R_X(t_1,t_2)$ - Ex. II

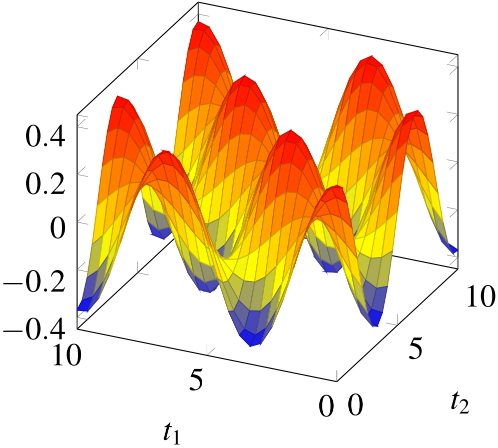


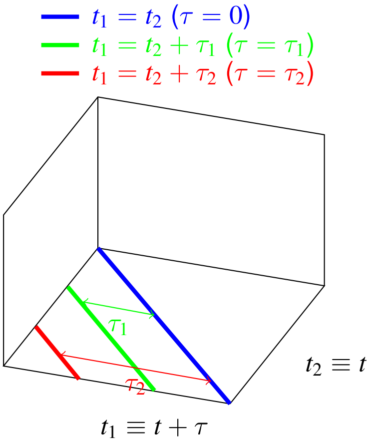
Mean and autocorrelation function: $m_X(t)$, $R_X(t_1, t_2)$ - Ex. I





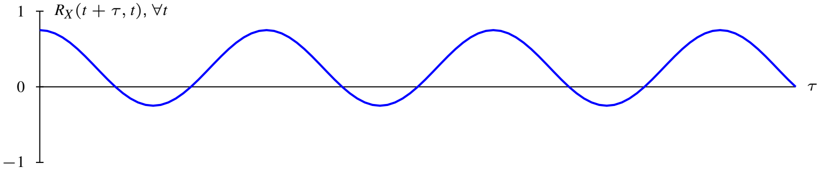
Mean and autocorrelation function: $m_X(t)$, $R_X(t_1, t_2)$ - Ex. II



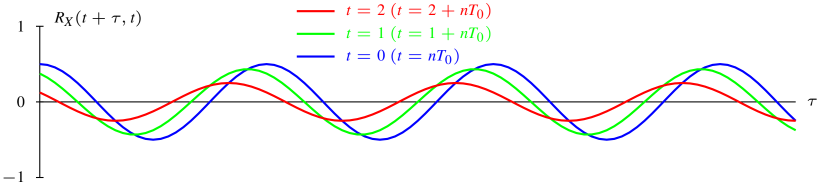


Autocorrelation function: $R_X(t_1, t_2) \equiv R_X(t+\tau, t)$

- Example I



- Example II



Random process $X(t,\lambda) \equiv X(t)$ - Example III

- Random experiment: throwing a dice
  - 6 possible outcomes

$$\lambda \in \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6\}$$

- Random process
  - A signal for each possible outcome of the experiment

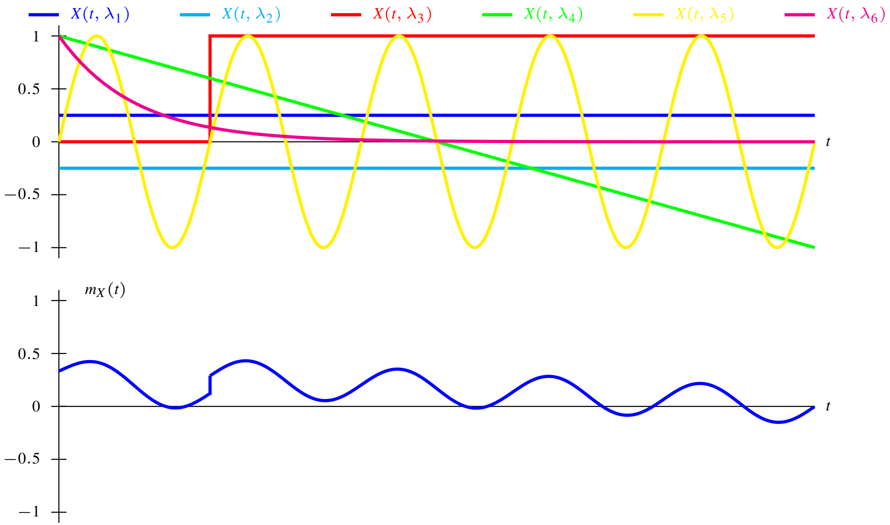
$$X(t,\lambda_1) = \frac{1}{4}, \qquad X(t,\lambda_2) = -\frac{1}{4}$$

$$X(t,\lambda_3) = u(t-2) = \begin{cases} 1, & \text{if } t \geq 2 \\ 0, & \text{if } t < 2 \end{cases}$$

$$X(t,\lambda_4) = 1 - \frac{t}{5}$$

$$X(t,\lambda_5) = e^{-t}, \qquad X(t,\lambda_6) = \sin(\pi t)$$

Random process $X(t, \lambda) \equiv X(t)$ - Example III



Autocorrelation of stationary processes

Properties of $R_X(\tau)$ for a stationary stochastic process $X(t)$:

- Is an even function

$$R_X(-\tau) = R_X(\tau)$$

- The maximum value (in module) is obtained at $\tau = 0$

$$|R_X(\tau)| \leq R_X(0)$$

- If for some T_o, $R_X(T_o) = R_X(0)$ holds, then for every integer k

$$R_X(kT_o) = R_X(0)$$

- It is a positive semidefinite function

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(t)\, R_X(t-s)\, g(s)\, dt\, ds \geq 0, \forall g(t)$$

  - Implication: $S_X(j\omega) = \mathcal{FT}\{R_X(\tau)\} \geq 0, \quad \forall \omega$

Ergodicity

- Averages for a process $X(t)$ and a function $g(x)$:
  1. Statistical average
  $$E[g(X(t))] = \int_{-\infty}^{\infty} g(x)\, f_{X(t)}(x)\, dx$$
  This value is, in general, dependent on t.
  2. Time average of the function for $X(t, \lambda_i)$
  $$< g(x) >_i = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} g(X(t, \lambda_i))\, dt$$
  Independent of t, but in general dependent on λ_i
- The stationary process $X(t)$ is ergodic if $\forall\ g(x)$ and $\forall\ \lambda_i \in \Omega$
$$< g(x) >_i = E[g(X(t))]$$

Energy and power

- Energy of the random process $X(t)$, E_X

$$E_X = E[\mathcal{E}_X], \quad \mathcal{E}_X = \int_{-\infty}^{\infty} |X(t)|^2 \, dt$$

- Power of the random process $X(t)$, P_X

$$P_X = E[\mathcal{P}_X], \quad \mathcal{P}_X = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |X(t)|^2 \, dt$$

  - A random process has energy if $E_X < \infty$
  - A random process is of power if $0 < P_X < \infty$

Energy and power (II)

$$E_X = E\left[\int_{-\infty}^{\infty} |X(t)|^2 \, dt\right] = \int_{-\infty}^{\infty} E[|X(t)|^2] \, dt = \int_{-\infty}^{\infty} R_X(t,t) \, dt$$

$$P_X = E\left[\lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |X(t)|^2 \, dt\right]$$
$$= \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} E[|X(t)|^2] \, dt = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} R_X(t,t) \, dt$$

For *stationary random processes* $R_X(t,t) = R_X(0)$

$$P_X = R_X(0), \quad E_X = \int_{-\infty}^{\infty} R_X(0) \, dt = \infty$$

Stationary processes of interest: **POWER processes**

Multidimensional (multiple) random processes

- Independence:
  - $X(t)$ and $Y(t)$ are *independent* if $\forall\ t_1, t_2$, the random variables $X(t_1)$ and $Y(t_2)$ are independent
- Uncorrelation:
  - $X(t)$ and $Y(t)$ are *uncorrelated* if $\forall\ t_1, t_2$, the random variables $X(t_1)$ and $Y(t_2)$ are uncorrelated
- Cross-correlation function

$$R_{X,Y}(t_1, t_2) = E[X(t_1)\ Y^*(t_2)]$$

  - In general $R_{X,Y}(t_1, t_2) = R^*_{Y,X}(t_2, t_1)$
- Joint stationarity: $X(t)$ and $Y(t)$ are *jointly stationary* if
  - Both are individually stationary
    - $m_X(t) = m_X,\ R_X(t+\tau, t) = R_X(\tau)$
    - $m_Y(t) = m_Y,\ R_Y(t+\tau, t) = R_Y(\tau)$
  - $R_{X,Y}(t_1, t_2) = R_{X,Y}(\tau)$, with $\tau = t_1 - t_2$
    - Alternative notation: $R_{X,Y}(t+\tau, t) = R_{X,Y}(\tau)$

Time-autocorrelation function (time-ambiguity)

- Definition for a DETERMINISTIC function/signal $x(t)$

$$x(t) \in R \quad \equiv \quad r_x(t) = x(t) * x(-t)$$

$$x(t) \in \mathcal{C} \quad \equiv \quad r_x(t) = x(t) * x^*(-t)$$

$$R_x(j\omega) = |X(j\omega)|^2$$

  - "*Matched*" signal for $x(t)$: time-reversed and complex conjugated signal, $x^*(-t)$
    - In frequency: if $x(t) \stackrel{\mathcal{FT}}{\leftrightarrow} X(j\omega)$, for the matched signal $x^*(-t) \stackrel{\mathcal{FT}}{\leftrightarrow} X^*(j\omega)$

- Properties
  - Symmetric function with maximum at zero
  - The value at $t = 0$ provides the energy

$$\mathcal{E}\{x(t)\} = r_x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_x(j\omega)\, d\omega$$

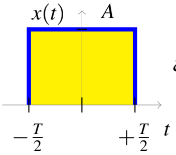
This property is evident considering the definition of energy (Parseval's relation)

$$\mathcal{E}\{x(t)\} = \int_{-\infty}^{\infty} |x(t)|^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2\, d\omega$$

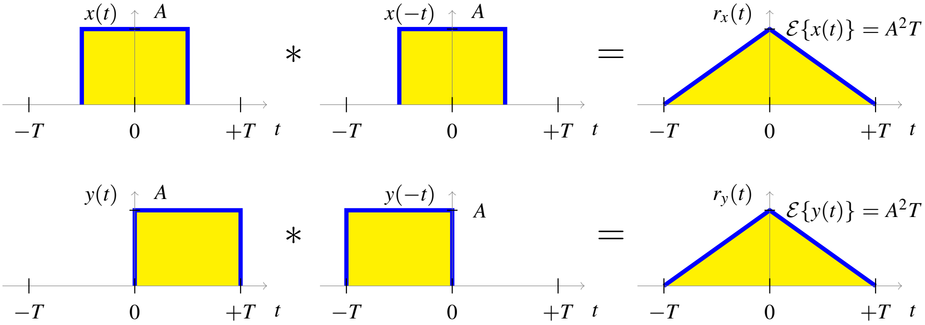
  - Translation-invariant function of $x(t)$

$$y(t) = x(t - t_0) \rightarrow r_y(t) = r_x(t)$$

Example



$$\mathcal{E}\{x(t)\} = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = A^2 T$$



Random processes in the frequency domain

- Spectrum of one of the signals of the random process

$$x_i(t) = X(t,\lambda_i) \rightarrow X_i(j\omega) = \mathcal{FT}\{x_i(t)\} = \int_{-\infty}^{\infty} x_i(t)\ e^{-j\omega t}\ dt$$

  - Not all signals have a Fourier transform defined
- Definition of truncated signals of duration T

$$x_i^{[T]}(t) = \begin{cases} x_i(t), & |t| < T/2 \\ 0, & \text{else} \end{cases}$$

- The truncated signals do have the Fourier transform well defined

$$X_i^{[T]}(j\omega) = \mathcal{FT}\left\{x_i^{[T]}(t)\right\} = \int_{-\infty}^{\infty} x_i^{[T]}(t)\ e^{-j\omega t}\ dt$$

$$= \int_{-\frac{T}{2}}^{\frac{T}{2}} x_i(t)\ e^{-j\omega t}\ dt$$

Power Spectral Density

- Truncated random process of duration T

$$X^{[T]}(t)\text{: Process whose signals are } X^{[T]}(t,\lambda_i) = x_i^{[T]}(t) \text{ (truncated)}$$

- Truncated random process in the frequency domain

$$X^{[T]}(j\omega)\text{: Process whose signals are the FTs of } x_i^{[T]}(t)\text{, i.e., } X_i^{[T]}(j\omega)$$

- Power spectral density of $X(t)$

$$S_X(j\omega) \stackrel{def}{=} E\left[\lim_{T\to\infty} \frac{\left|X^{[T]}(j\omega)\right|^2}{T}\right] = \lim_{T\to\infty} \frac{E\left[\left|X^{[T]}(j\omega)\right|^2\right]}{T}$$

Representation of the mean behavior of the squared modulus of the Fourier transform of all the signals that constitute the random process (with the "*trick*" of truncating to ensure the existence of said Fourier transform for all signals, and taking the truncation length to the limit)

  - Implication: $S_X(j\omega) \geq 0, \quad \forall\omega$

Wiener-Khinchin theorem

If for any finite value τ and any interval $\mathcal{A}$, of length $|\tau|$, and the autocorrelation function satisfies

$$\left|\int_{\mathcal{A}} R_X(t+\tau,t)dt\right| < \infty$$

the power spectral density of $X(t)$ is the Fourier transform of the time-average of the autocorrelation function

$$S_X(j\omega) = \mathcal{FT}\{< R_X(t+\tau,t) >\}$$

The time-average of the autocorelation function is

$$< R_X(t+\tau,t) > \stackrel{def}{=} \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} R_X(t+\tau,t)\ dt$$

Wiener-Khinchin theorem - Corollaries

- Corollary 1: If $X(t)$ is a stationary random process and $\tau\, R_X(\tau) < \infty$ for all $\tau < \infty$

$$S_X(j\omega) = \mathcal{FT}\left\{R_X(\tau)\right\}$$

- Corollary 2: If $X(t)$ is cyclostationary and $\left|\int_0^{T_o} R_X(t+\tau,t)\, dt\right| < \infty$ then

$$S_X(j\omega) = \mathcal{FT}\left\{\widetilde{R}_X(\tau)\right\}$$

where

$$\widetilde{R}_X(\tau) = \frac{1}{T_o}\int_{T_o} R_X(t+\tau,t)\, dt$$

and T_o is the cyclostationarity period

Power of a random process

- In the frequency domain

$$P_X = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(j\omega)\, d\omega \text{ Watts}$$

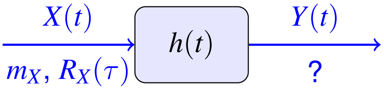
- In the time domain
  - Stationary process

$$P_X = R_X(0) \text{ Watts}$$

  - Cyclostationary process

$$P_X = \tilde{R}_X(0) \text{ Watts}$$

Stationary Random Processes and Linear Systems



Theorem: $X(t)$ is stationary, with mean m_X and autocorrelation function $R_X(\tau)$. The process passes through a linear time-invariant system with impulse response $h(t)$. In this case, *the input and output processes, $X(t)$ and $Y(t)$, are jointly stationary*, where

$$
\begin{aligned}
m_Y &= m_X \int_{-\infty}^{\infty} h(t)\, dt \\
R_Y(\tau) &= R_X(\tau) * h(\tau) * h(-\tau) = R_X(\tau) * r_h(\tau) \\
R_{X,Y}(\tau) &= R_X(\tau) * h(-\tau) \\
R_{Y,X}(\tau) &= R_{X,Y}(-\tau) = R_X(\tau) * h(\tau)
\end{aligned}
$$

Additionally $R_Y(\tau) = R_{X,Y}(\tau) * h(\tau) = R_{Y,X}(\tau) * h(-\tau)$

Output Process Mean

Definition of the output process by the convolution

$$Y(t) = \int_{-\infty}^{\infty} X(s)\ h(t-s)\ ds$$

The mean of this process is

$$\begin{aligned} m_Y(t) &= E\left[\int_{-\infty}^{\infty} X(s)\ h(t-s)\ ds\right] \\ &= \int_{-\infty}^{\infty} E[X(s)]\ h(t-s)\ ds \\ &= \int_{-\infty}^{\infty} m_X\ h(t-s)\ ds \\ &\stackrel{u=t-s}{=} m_X \int_{-\infty}^{\infty} h(u)\ du \end{aligned}$$

Cross correlation $R_{X,Y}(\tau)$

$$
\begin{aligned}
R_{X,Y}(t_1,t_2) &= E[X(t_1)\ Y(t_2)]\\
&= E\left[X(t_1)\left(\int_{-\infty}^{\infty} X(s)\ h(t_2-s)\ ds\right)\right]\\
&= \int_{-\infty}^{\infty} E[X(t_1)\ X(s)]\ h(t_2-s)\ ds\\
&= \int_{-\infty}^{\infty} R_X(t_1-s)\ h(t_2-s)\ ds\\
&\overset{u=s-t_2}{=} \int_{-\infty}^{\infty} R_X(t_1-t_2-u)\ h(-u)\ du\\
&= \int_{-\infty}^{\infty} R_X(\tau-u)\ h(-u)\ du\\
&= R_X(\tau) * h(-\tau)
\end{aligned}
$$

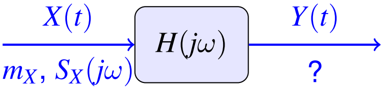
Cross correlation $R_{Y,X}(\tau)$

$$
\begin{aligned}
R_{Y,X}(t_1,t_2) &= E[Y(t_1)\; X(t_2)] \\
&= E\left[\left(\int_{-\infty}^{\infty} X(s)\; h(t_1-s)\; ds\right)\; X(t_2)\right] \\
&= \int_{-\infty}^{\infty} E[X(s)\; X(t_2)]\; h(t_2-s)\; ds \\
&= \int_{-\infty}^{\infty} R_X(s-t_2)\; h(t_1-s)\; ds \\
&\overset{u=s-t_2}{=} \int_{-\infty}^{\infty} R_X(u)\; h(t_1-t_2-u)\; du \\
&= \int_{-\infty}^{\infty} R_X(u)\; h(\tau-u)\; du \\
&= R_X(\tau) * h(\tau)
\end{aligned}
$$

Autocorrelation function $R_Y(\tau)$

$$\begin{aligned}
R_Y(t_1,t_2) &= E[Y(t_1)\ Y(t_2)]\\
&= E\left[\left(\int_{-\infty}^{\infty} X(s)\ h(t_1-s)\ ds\right)\ Y(t_2)\right]\\
&= \int_{-\infty}^{\infty} E[X(s)\ Y(t_2)]\ h(t_1-s)\ ds\\
&= \int_{-\infty}^{\infty} R_{X,Y}(s-t_2)\ h(t_1-s)\ ds\\
&\overset{u=s-t_2}{=} \int_{-\infty}^{\infty} R_{X,Y}(u)\ h(t_1-t_2-u)\ du\\
&= \int_{-\infty}^{\infty} R_{X,Y}(u)\ h(\tau-u)\ du\\
&= R_{X,Y}(\tau) * h(\tau)\\
&= R_X(\tau) * h(-\tau) * h(\tau)\\
&= R_X(\tau) * r_h(\tau)
\end{aligned}$$

Relationships in the frequency domain



- Mean of the output process

$$\boxed{m_Y = m_X \; H(0)}$$

- Power spectral density of the output process

$$\boxed{S_Y(j\omega) = S_X(j\omega) \; |H(j\omega)|^2}$$

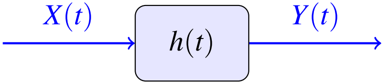
- Cross spectral densities

$$S_{X,Y}(j\omega) \stackrel{def}{=} \mathcal{FT}\left\{R_{X,Y}(\tau)\right\}$$

$$S_{X,Y}(j\omega) = S_X(j\omega) \; H^*(j\omega)$$

$$S_{Y,X}(j\omega) = S_{X,Y}^*(j\omega) = S_X(j\omega) \; H(j\omega)$$

Relationships between power spectral densities



$$S_X(j\omega) \times \begin{cases} H(j\omega) = S_{Y,X}(j\omega) \\ H^*(j\omega) = S_{X,Y}(j\omega) \\ |H(j\omega)|^2 = S_Y(j\omega) \end{cases}$$

$$S_{X,Y}(j\omega) = S_{Y,X}^*(j\omega)$$
$$S_Y(j\omega) = S_{X,Y}(j\omega)\ H(j\omega)$$
$$S_Y(j\omega) = S_{Y,X}(j\omega)\ H^*(j\omega)$$

Discrete-time Random Processes

- Notation: $X[n]$
- Statistical averages
  - Mean: $m_X[n] = E\left[X[n]\right]$
  - Autocorrelation: $R_X[n+k,n] = E\left[X[n+k]\ X[n]\right]$
    - ★ For complex random processes $R_X[n+k,n] = E\left[X[n+k]\ X^*[n]\right]$
- Stationarity:
  - Statistics are independent of time index n
  - Mean: $m_X[n] = m_X$
  - Autocorrelation: $R_X[n+k,n] = R_X[k]$
- Cyclostationarity:
  - Statistics are periodical in n, with period N
  - Mean: $m_X[n+N] = m_X[n]$
  - Autocorrelation: $R_X[n+k+N,n+N] = R_X[n+k,n]$

Discrete-time Random Processes - Spectrum and Power

- Power spectral density
  - Stationary processes

$$S_X(e^{j\omega}) = \mathcal{FT}\left\{R_X[k]\right\}$$

  - Cyclostationary processes

$$S_X(e^{j\omega}) = \mathcal{FT}\left\{\tilde{R}_X[k]\right\}, \qquad \tilde{R}_X[k] = \frac{1}{N}\sum_{n=0}^{N-1} R_X[n+k,n]$$

  - Power

$$P_X = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_X(e^{j\omega})\, d\omega = \begin{cases} R_X[0], & X[n] \text{ stationary} \\ \tilde{R}_X[0], & X[n] \text{ cyclostationary} \end{cases}$$

Discrete-time Random Processes : Linear Systems

- Average output process

$$m_Y = m_X \sum_n h[n] = m_X \; H(e^{j0})$$

- Autocorrelation of the output process

$$R_Y[k] = R_X[k] * h[k] * h[-k] = R_X[k] * r_h[k]$$

- Power spectral density of the output process

$$S_Y(e^{j\omega}) = S_X(e^{j\omega}) \; \left|H(e^{j\omega})\right|^2$$

- Cross statistics

$$R_{X,Y}[k] = R_X[k] * h[-k]$$

$$S_{X,Y}(e^{j\omega}) = S_X(e^{j\omega}) \; H^*(e^{j\omega})$$

Addition of jointly stationary random processes

- Sum of jointly stationary processes $X(t)$ and $Y(t)$: $Z(t) = X(t) + Y(t)$
  - Mean of $Z(t)$
    $$\begin{aligned} m_Z(t) &= E[Z(t)] = E[X(t) + Y(t)] \\ &= E[X(t)] + E[Y(t)] = m_X + m_Y = m_Z \end{aligned}$$
  - Autocorrelation function of $Z(t)$
    $$\begin{aligned} R_Z(t+\tau, t) =& E[Z(t+\tau)\ Z(t)] \\ =& E[(X(t+\tau) + Y(t+\tau))\ (X(t) + Y(t))] \\ =& E[X(t+\tau)\ X(t)] + E[X(t+\tau)\ Y(t)] \\ & + E[Y(t+\tau)\ X(t)] + E[Y(t+\tau)\ Y(t)] \\ =& R_X(\tau) + R_{X,Y}(\tau) + R_{Y,X}(\tau) + R_Y(\tau) \\ =& R_X(\tau) + R_Y(\tau) + R_{X,Y}(\tau) + R_{Y,X}(\tau) = R_Z(\tau) \end{aligned}$$
    - Random process $Z(t)$ is stationary
  - Power spectral density of $Z(t)$
    $$\begin{aligned} S_Z(j\omega) =& S_X(j\omega) + S_Y(j\omega) + S_{X,Y}(j\omega) + S_{Y,X}(j\omega) \\ =& S_X(j\omega) + S_Y(j\omega) + 2\ \text{Re}[S_{X,Y}(j\omega)] \end{aligned}$$

Sum of Random Processes - Uncorrelated

- Ratio (covariance / correlation) for jointly stationary processes

$$\begin{aligned}
\text{Cov}(X(t+\tau),Y(t)) =& E[(X(t+\tau)-m_X)\;(Y(t)-m_Y)] \\
=& \underbrace{E[X(t+\tau)\;Y(t)]}_{R_{X,Y}(\tau)} - m_Y\;\underbrace{E[X(t+\tau)]}_{m_X} \\
& - m_X\;\underbrace{E[Y(t)]}_{m_Y} + m_X\;m_Y \\
=& R_{X,Y}(\tau) - m_X\;m_Y
\end{aligned}$$

- Uncorrelated processes:

$$\begin{aligned}
\text{By definition : } & \text{Cov}(X(t+\tau),Y(t)) = 0,\ \forall\tau \\
\text{Consequence : } & R_{X,Y}(\tau) = m_X\;m_Y
\end{aligned}$$

  - If at least one of the (uncorrelated) processes has a zero mean

$$\begin{aligned}
R_Z(\tau) &= R_X(\tau) + R_Y(\tau) \\
S_Z(j\omega) &= S_X(j\omega) + S_Y(j\omega)
\end{aligned}$$

★ Usual case of sum of signal and noise: uncorrelation, noise with zero mean

Gaussian processes

- Definition: $X(t)$ is a *Gaussian process* if for all n and all $\{t_1, t_2, \ldots, t_n\}$, the joint distribution of RV's $\{X(t_i)\}_{i=1}^{n}$ is a Multivariate Normal (Gaussian) distribution
- Properties of Gaussian processes
  - $m_X(t)$ y $R_X(t_1, t_2)$: full statistical description of $X(t)$
    - Vector of means and matrix of covariances:
      - For $X(t_i)$, $\Rightarrow \mu_i = m_X(t_i)$
      - $X(t_i), X(t_j)$, $\Rightarrow C_{i,j} =$Cov($X(t_i)$,$X(t_j)$)=$R_X(t_i, t_j) - m_X(t_i)\, m_X(t_j)$
  - Strict sense and wide sense stationarity are equivalent
  - If $X(t)$ is a zero mean stationary process, a sufficient condition for ergodicity is

$$\int_{-\infty}^{\infty} |R_X(\tau)|\, d\tau < \infty$$

  - If $X(t)$ passes through a linear and time invariant system:
    - The output process $Y(t)$ is Gaussian
    - $X(t)$ and $Y(t)$ are jointly Gaussian

Jointly-Gaussian stochastic processes

- Definition: The processes $X(t)$ and $Y(t)$ are *jointly Gaussian*, if for all n, m, and all $\{t_1, t_2, \ldots, t_n\}$ and $\{\tau_1, \tau_2, \ldots, \tau_m\}$, the joint distribution of the following random variables

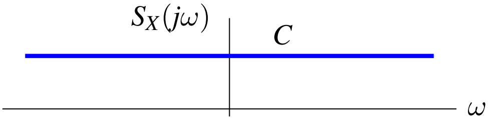
$$\{X(t_1), X(t_2), \ldots, X(t_n), Y(\tau_1), Y(\tau_2), \ldots, Y(\tau_m)\}$$

  is a Multivariate Normal (Gaussian) distribution (of dimension $n+m$)
- Property: For joint Gaussian processes, incorrelation and independence are equivalent

White process (stationary)

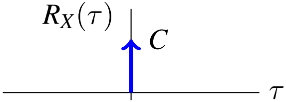
- A process is white if its power spectral density is constant for all frequencies

$$S_X(j\omega) = C$$



  - Consequences
    - Autocorrelation function of a stationary white process

$$R_X(\tau) = \mathcal{TF}^{-1}\left\{C\right\} = C\,\delta(\tau)$$



    - The power of a white process is infinite

$$P_X = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(j\omega)\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} C\,d\omega = \infty \text{ Watts.}$$

$$X(t) \text{ stationary : } P_X = R_X(0) = C\,\delta(0) = \infty \text{ Watts.}$$

Filtering a white process

$X(t)$ → $h(t) \overset{\mathcal{FT}}{\leftrightarrow} H(j\omega)$ → $Y(t)$

$S_X(j\omega) = C,\ R_X(\tau) = C\,\delta(\tau)$ $\qquad$ $S_Y(j\omega),\ R_Y(\tau)$?

- Power spectral density at the filter output ($Y(t)$)

$$S_Y(j\omega) = S_X(j\omega)\,|H(j\omega)|^2 = C\,|H(j\omega)|^2$$

  - In general the process $Y(t)$ is not white
    - $|H(j\omega)|$ is constant only for an all-pass system: atenuator or amplifier
- Autocorrelation function

$$R_Y(\tau) = R_X(\tau) * h(\tau) * h(-\tau) = R_X(\tau) * r_h(\tau) = C\,r_h(\tau)$$

- Power

$$P_Y = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_Y(j\omega)\,d\omega = C\,\frac{1}{2\pi}\int_{-\infty}^{\infty} |H(j\omega)|^2\,d\omega, \quad P_Y = R_Y(0) = C\,r_h(0)$$

  - As by definition $\frac{1}{2\pi}\int_{-\infty}^{\infty} |H(j\omega)|^2\,d\omega = \mathcal{E}\{h(t)\} = r_h(0)$

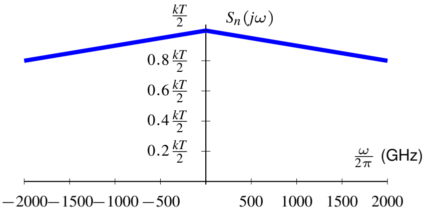
$$P_Y = C\,\mathcal{E}\{h(t)\} \text{ Watts}$$

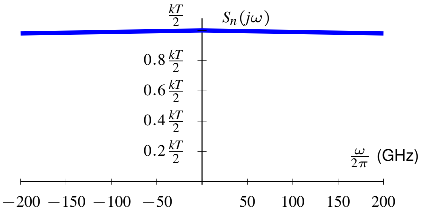
Thermal noise

- Gaussian statistics
- Power spectral density of thermal noise (quantum mechanics)

$$S_n(j\omega) = \frac{h\omega}{4\pi(e^{\frac{h\omega}{2\pi kT}} - 1)}$$

where
- h: Planck's constant (6.6$\times 10^{-34}$ Joules $\times$ second)
- k: Boltzmann's constant (1.38$\times 10^{-23}$ Joules/ºKelvin)
- T: Temperature in Kelvin degrees
- ω: Frequency in rad/s

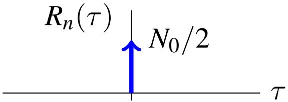




Thermal noise model

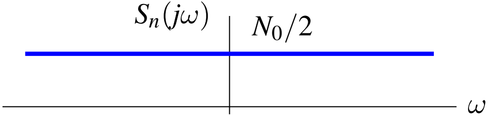
- Stochastic process $n(t)$
  - White, Gaussian, stationary and ergodic
  - Zero mean $(m_n = 0)$
  - Autocorrelation function

$$R_n(\tau) = \frac{N_0}{2}\,\delta(\tau)$$



  - Power spectral density

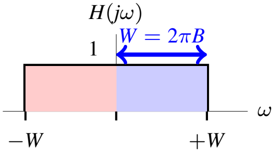
$$S_n(j\omega) = \frac{N_0}{2}$$



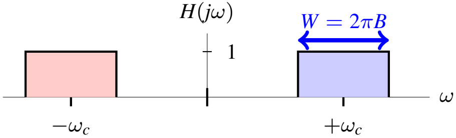
- Value of the constant N_0 $N_0 = k\,T$ Watts/Hz
- Power $$P_n = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_n(j\omega)\,d\omega = R_n(0) = \infty$$

Thermal noise power at the output of ideal filters

- Ideal filters with bandwidth B Hz (or $W = 2\pi B$ rad/s): low-pass or band-pass filter with center frequency f_c Hz (or $\omega_c = 2\pi f_c$ rad/s)



Low Pass Filter



Bandpass filter (center frequency ω_c rad/s)

- Filter output process

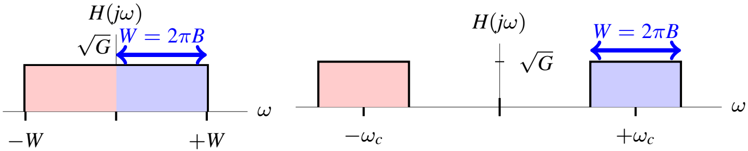
$$\text{Process } Z(t) \text{ with } S_Z(j\omega) = \frac{N_0}{2}\,|H(j\omega)|^2$$

- Filter output power

$$P_Z = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_z(j\omega)\,d\omega = \frac{N_0}{2}\,\underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty}|H(j\omega)|^2\,d\omega}_{\mathcal{E}\{h(t)\}=2B} = N_0\,B \text{ Watts}$$

Thermal noise power at the output of ideal filters with gain

- Ideal filters (low pass/band pass) of bandwidth B Hz (or $W = 2\pi B$ rad/s) and with power gain G (voltage gain $\sqrt{G}$)



- Filter output process

$$\text{Process } Z(t) \text{ with } S_Z(j\omega) = \frac{N_0}{2}\,|H(j\omega)|^2$$

- Filter output power

$$P_Z = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_z(j\omega)\,d\omega = \frac{N_0}{2}\,\underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty}|H(j\omega)|^2\,d\omega}_{\mathcal{E}\{h(t)\}=2BG} = N_0\,B\,G \text{ Watts}$$

Noise-Equivalent Bandwidth

- Noise at the output of a linear system
  - System response: $h(t) \stackrel{\mathcal{FT}}{\leftrightarrow} H(j\omega)$

$$\text{Process } Z(t) \text{ with } S_Z(j\omega) = \frac{N_0}{2}\,|H(j\omega)|^2$$

- Noise power at the output of the linear system

$$P_Z = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_Z(j\omega)\;d\omega = \frac{N_0}{2}\;\underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} |H(j\omega)|^2\;d\omega}_{\mathcal{E}\{h(t)\}}$$

- Noise power as a function of the noise-equivalent bandwidth

$$P_Z = N_0\;B_{eq}\;G_{eq}\;\text{Watts}$$

  - B_{eq}: Noise-equivalent bandwidth
  - G_{eq}: Equivalent power gain

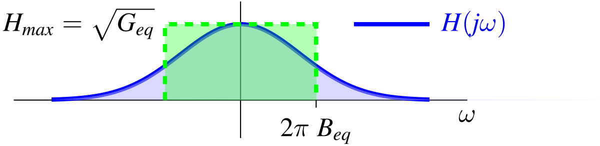
Noise-Equivalent Bandwidth - Identification

- Definition: $G_{eq} = H_{max}^2$, with $H_{max} = \max_{\omega} |H(j\omega)|$
- Identification of the value of B_{eq}

$$B_{eq} = \frac{\mathcal{E}\{h(t)\}}{2\,G_{eq}} \text{ Hz}$$

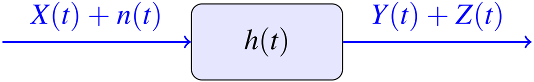
$$\mathcal{E}\{h(t)\} = \int_{-\infty}^{\infty} |h(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |H(j\omega)|^2\,d\omega \text{ J}$$

- Interpretation: An ideal linear system, of bandwidth B_{eq} and amplitude H_{max} ($\sqrt{G_{eq}}$) has the same noise power at the output than the filter $h(t)$



Signal to Noise Ratio (SNR): need for noise filtering

- Approach
  - Signal to be transmitted: process $X(t)$, with power P_X
  - Additive noise: thermal noise model $n(t)$
  - Filter (usually on the receiver): responses $h(t)$ and $H(j\omega)$
    - ★ Signal at the output of the receiver filter: process $Y(t)$
    - ★ Noise at the output of the receiver filter: process $Z(t)$



$$Y(t) = X(t) * h(t)$$
$$Z(t) = n(t) * h(t)$$

- Unfiltered signal-to-noise ratio

$$\left.\frac{S}{N}\right|_{in} = \frac{P_X}{P_n}, \quad \left.\frac{S}{N}\right|_{in} \text{(dB)} = 10\log_{10}\frac{P_X}{P_n} \text{ dB}$$

- Signal-to-noise ratio at the filter output

$$\left.\frac{S}{N}\right|_{out} = \frac{P_Y}{P_Z}, \quad \left.\frac{S}{N}\right|_{out} \text{(dB)} = 10\log_{10}\frac{P_Y}{P_Z} \text{ dB}$$

Signal to Noise Ratio (SNR): need for noise filtering (II)

- Signal to noise ratio before filtering

$$\left.\frac{S}{N}\right|_{in} = \frac{P_X}{P_n} = \frac{P_X}{\infty} = 0, \quad \left.\frac{S}{N}\right|_{in} (\text{dB}) = 10 \log_{10} \frac{P_X}{P_n} = -\infty \text{ dB}$$

Filtering is necessary to limit the power of thermal noise !!!

- Signal-to-noise ratio at the filter output

$$\left.\frac{S}{N}\right|_{out} = \frac{P_Y}{P_Z}, \quad \left.\frac{S}{N}\right|_{out} (\text{dB}) = 10 \log_{10} \frac{P_Y}{P_Z} \text{ dB}$$

  - Signal power: $P_Y = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_Y(j\omega)\, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(j\omega)\, |H(j\omega)|^2\, d\omega$
  - Noise power: $P_Z = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_Z(j\omega)\, d\omega = \frac{N_0}{2} \frac{1}{2\pi} \int_{-\infty}^{\infty} |H(j\omega)|^2\, d\omega = \frac{N_0}{2} \mathcal{E}\{h(t)\}$
    - Ideal filters (without gain | with gain)

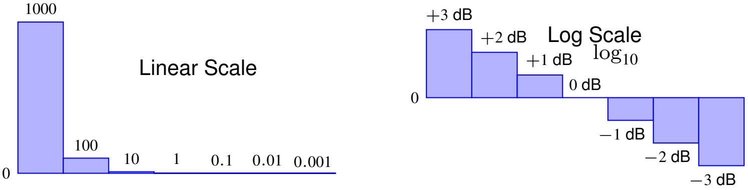
$$P_Z = N_0\, B \quad \Big| \quad P_Z = N_0\, B\, G$$

    - Filter with noise-equivalent bandwidth B_{eq} Hz and gain G_{eq}

$$P_Z = N_0\, B_{eq}\, G_{eq}$$

Log scale: decibels

- Relationship between quantities (relative measure, dimensionless)
  - Logarithmic scale to jointly visualize quantities with very different values from each other



  - Examples
    - ★ Power ratio

$$\text{dB} \equiv 10 \log_{10} \frac{P_1}{P_2}$$

    - ★ Relationship of voltages or currents

$$\text{dB} \equiv 20 \log_{10} \frac{V_1}{V_2} \qquad \text{dB} \equiv 20 \log_{10} \frac{I_1}{I_2}$$

    - ★ Representation of the frequency response of a system

$$\text{dB} \equiv 20 \log_{10} |H(j\omega)| = 10 \log_{10} |H(j\omega)|^2$$

Properties of the logarithmic representation

- Reverse operation

$$\text{dB} \equiv 10\ \log_{10}\frac{P_1}{P_2} \qquad \text{dB} \equiv 20\ \log_{10}\frac{V_1}{V_2}$$

$$\frac{P_1}{P_2} = 10^{\frac{dB}{10}} \qquad \frac{V_1}{V_2} = 10^{\frac{dB}{20}}$$

- Products in linear scale become sums in dB

$$A \times B \leftrightarrow 10\log_{10}(A \times B) = 10\log_{10} A + 10\log_{10} B = a\ \text{dB} + b\ \text{dB}$$

$$\frac{A}{B} \leftrightarrow 10\log_{10}\frac{A}{B} = 10\log_{10} A - 10\log_{10} B = a\ \text{dB} - b\ \text{dB}$$

- Doubling/dividing by two, in linear, is equivalent to
  - Add/subtract 3 dB $(10\log_{10} 2)$
  - Add/subtract 6 dB $(20\log_{10} 2)$
- Multiply/divide by 10, in linear, is equivalent to
  - Add/subtract 10 dB $(10\log_{10} 10)$
  - Add/subtract 20 dB $(20\log_{10} 20)$

Units based on decibel

- Measure of the relative difference with respect to a reference unit
  - Example: Sound intensity (ref.: hearing threshold, 1 pW/m^2)

$$\text{dB}_{\text{SI}} \equiv 10 \log_{10} \frac{P_1}{P_0} = 10 \log_{10} \frac{P_1}{10^{-12}}$$

    - ★ Reference: 0 dB$_{\text{SI}}$ are equivalent to an intensity equal to the hearing threshold
- Other logarithmic units
  - dBW: reference 1 W (power)
  - dBm: reference 1 mW (power)
  - dBu: reference $\sqrt{\frac{3}{5}} = 0.7746$ V (voltage)
    - ★ It is the voltage that applied to an impedance of 600 Ω develops a power of 1 mW
  - dBc: takes as reference the value of the carrier (to measure the value of its harmonics)
  - dB$_{\text{SPL}}$: reference sound pressure level
    - ★ In the air: 10 μPa
    - ★ In water: 1 μPa
  - dBI: takes an isotropic antenna as reference