Chapter 1 : Exercises

Exercise 1.1 A random variable X is under analysis.

a) If X is a random variable with a uniform distribution in the interval $[-2, 2]$, find the probabilities

  i) $P(X > 1)$
  ii) $P(X > -1)$
  iii) $P(X < -1)$
  iv) $P(-1 \le X \le 1)$

b) Repeat the previous section if X has a Gaussian distribution with mean 1 and variance 4.

Exercise 1.2 Two random variables, X and Y, have the distribution function $F_X(x)$ and the probability density function $f_Y(y)$, respectively, which are represented in the Figure 1.1.

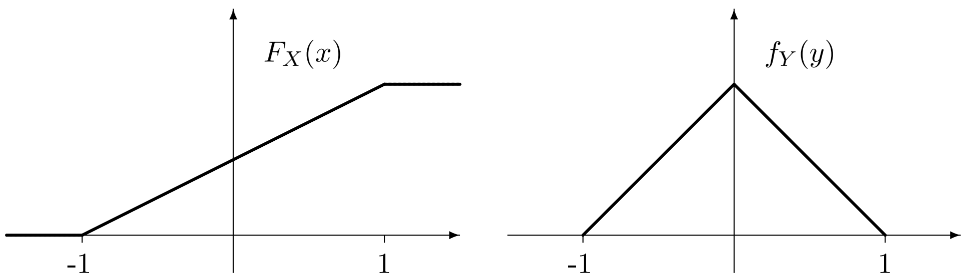


Figure 1.1: Distribution function, $F_X(x)$, and probability density function, $f_Y(y)$, for Exercise 1.2.

a) Write the analytical expressions of the distribution and probability density functions for each of the random variables: $F_X(x)$, $f_X(x)$, $F_Y(y)$, $f_Y(y)$).

b) Compute the variance of the two random variables.

c) Compute the following probabilities on the random variable Y: $P(Y > 0)$, $P\left(Y > -\frac{1}{2}\right)$, $P\left(Y < -\frac{1}{2}\right)$, $P\left(Y > \frac{1}{4}\right)$.

Exercise 1.3 A random process $X(t)$ is defined as

$$X(t) = A + B\,t,$$

where A and B are two independent random variables, the first with a pdf $f_A(a)$ as shown in Figure 1.2, and the second with a uniform pdf on the interval $[-1, +1]$, i.e. $f_B(b) = \mathcal{U}(-1, +1)$.

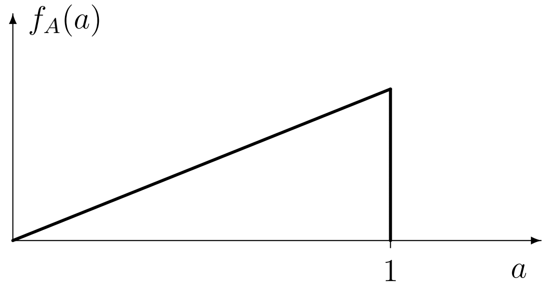


Figure 1.2: Probability density function of the random variable A.

a) Compute the mean, $m_X(t)$, of the process $X(t)$.

b) Calculate the autocorrelation function, $R_X(t+\tau, t)$, of the process $X(t)$.

c) Is this process (wide sense) stationary?

Exercise 1.4 A stochastic process $X(t)$ is described as follows: For $t \geq 0$ has the property that for every n and every set of n instants of time, $(t_1, t_2, \cdots, t_n) \in \mathbf{R}^n$, the joint probability density function of $\{X(t_i)\}_{i=1}^n$ is a joint Gaussian probability of zero mean and covariance matrix given by

$$C_{i,j} = \text{Cov}(X(t_i), X(t_j)) = \sigma^2 \min(t_i, t_j)$$

Is this process (wide sense) stationary?

Exercise 1.5 Let $X[n]$ be a discrete time random process, stationary, with mean m_X, and autocorrelation function $R_X[k]$. Consider the process

$$Y[n] = X[n] + a\, X[n-1],$$

where a is a constant.

a) Compute the mean, $m_Y[n]$, of process $Y[n]$.

b) Compute the autocorrelation function, $R_Y[n+k, n]$, of the process $Y[n]$.

c) Obtain the power spectral density, $S_Y(e^{j\omega})$, of the process $Y[n]$.

Exercise 1.6 The random process $Z(t)$ is defined as

$$Z(t) = X\, \cos(\omega_0 t) + Y\, \sin(\omega_0 t),$$

where X and Y are independent Gaussian random variables, with zero mean and variances σ_X^2 y σ_Y^2.

a) Compute the mean $m_Z(t)$.

b) Compute the autocorrelation function $R_Z(t+\tau, t)$.

c) Is $Z(t)$ stationary or cyclostationary?

d) Calculate the power spectral density $S_Z(j\omega)$.

e) Repeat the questions above for $\sigma_X = \sigma_Y$.

Exercise 1.7 The stochastic process $Z(t)$ has the following analytic description

$$Z(t) = X\ \cos(\omega_0 t) + Y\ \sin(\omega_0 t),$$

where X is a uniform random variable in the interval (-1,1), and Y is a uniform random variable in the interval (0,1). Furthermore, X and Y are independent.

a) Compute the mean $m_Z(t)$.

b) Compute the autocorrelation function $R_Z(t+\tau, t)$.

c) Is $Z(t)$ stationary or cyclostationary?

d) Calculate the power spectral density $S_Z(j\omega)$.

Exercise 1.8 Calculate the mean and autocorrelation function of the random process

$$X(t) = A\ \cos(\omega t + \theta) \tag{1.1}$$

and determine whether it is stationary or cyclostationary (in this case, obtain the period) in the following cases:

a) A is a real Gaussian random variable with zero mean and unit variance and ω and θ are real non null constants.

b) A is a real Gaussian random variable with mean equal to 1 and unit variance and ω and θ are real non null constants.

c) ω is a uniformly distributed random variable in the interval $(0, 2\pi]$ and A and θ are real non null constants.

d) θ is a random variable with uniform distribution in the interval $(0, 2\pi]$ and A and ω are real not null constants.

e) θ is a random variable with uniform distribution in the interval $(0, \pi]$ and A and ω are real not null constants.

f) A is a real Gaussian random variable with zero mean and unit variance, ω and θ are random variables with uniform distribution in the interval $(0, 2\pi]$, and they are statistically independent.

Exercise 1.9 The random process $Y(t)$ is

$$Y(t) = A_c\ \cos(\omega_c t)\ X(t),$$

where A_c and ω_c are two constants that determine the amplitude and frequency of the sinusoid, respectively, and $X(t)$ is a zero-mean stationary random process, with autocorrelation function $R_X(\tau)$, power spectral density $S_X(j\omega)$, and power P_X.

a) Compute the process mean $Y(t)$.

b) Calculate the autocorrelation function of the process $Y(t)$.

c) Compute the power spectral density of $Y(t)$.

d) Calculate the power of the process $Y(t)$.

Exercise 1.10 A certain analog modulation that simultaneously modulates two modulating signals can be defined by the following process random

$$S(t) = M_A(t)\ \cos(\omega_c t) + M_B(t)\ \sin(\omega_c t),$$

where $M_A(t)$ and $M_B(t)$ are two random processes that model the two modulating signals. Both are assumed to be independent random processes, stationary, mean zero, and identical autocorrelation function $R_{M_A}(\tau) = R_{M_B}(\tau) = R_M(\tau)$, power spectral density $S_{M_A}(j\omega) = S_{M_B}(j\omega) = S_M(j\omega)$, and power $P_{M_A} = P_{M_B} = P_M$.

a) Compute the mean of the random process $S(t)$, $m_S(t)$.

b) Calculate the autocorrelation function of the process $S(t)$, $R_S(t+\tau, t)$, and say if the process is stationary or cyclostationary

c) Compute the power spectral density of the process, $S_S(j\omega)$, and find its power, P_S.

Exercise 1.11 A noise signal that is modeled as a stationary, white, Gaussian random process with zero mean and power spectral density $N_0/2$ passes through an ideal low-pass filter with bandwidth B Hz.

a) Compute the autocorrelation function of the output process, $Y(t)$.

b) Assuming that $\tau = \frac{1}{2B}$, calculate the joint pdf of the random variables $Y(t)$ and $Y(t+\tau)$. Are these random variables independent? And for $\tau = \frac{1}{4B}$?

Exercise 1.12 $X(t)$ is a stationary process with power spectral density $S_X(j\omega)$. This process goes through the system shown in Fig. 1.3

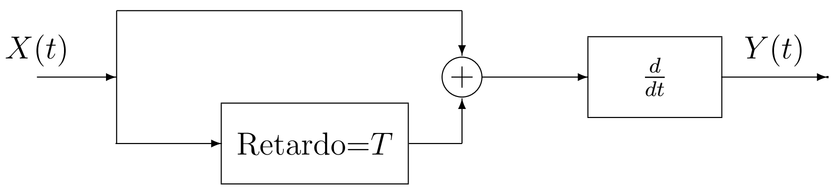


Figure 1.3: Syistem for Exercise 1.12.

a) Is the process $Y(t)$ stationary? Explain clearly the reason.

b) Calculate the power spectral density of the process $Y(t)$.

Exercise 1.13 The signal received in a communications system is obtained as the sum of two components, one of signal and the other of noise.

$$r(t) = s(t) + n(t).$$

The signal $s(t)$ is modeled with a random process with a power spectral density

$$S_s(j\omega) = \frac{P_0}{1 + (\omega/\omega_0)^2},$$

where $\omega_0 = 2\pi f_0$ and f_0 is the 3 dB bandwidth (in Hz) of the signal. The noise term, $n(t)$ is modeled with a white random process that has a power spectral density $N_0/2$ for all frequencies. The received signal is filtered with an ideal low-pass filter with bandwidth B Hz and unity gain. Calculate and draw the signal to noise ratio at the output of the filter as a function of the ratio B/f_0. For what filter bandwidth, B, is the maximum signal-to-noise ratio obtained?

Exercise 1.14 Calculate the equivalent noise bandwidth of an RC low-pass filter, taking into account that the frequency response of this type of filter is

$$H(j\omega) = \frac{1}{1 + j\omega\tau}$$

where the time constant τ takes the value $\tau = R\,C$.

Exercise 1.15 The information signal that is received at the input of a receiver of communications is modeled by a random process $X(t)$ whose power spectral density is

$$S_X(j\omega) = 10^{-18}\,\Pi\left(\frac{\omega}{2W_X}\right),$$

where W_X is the bandwidth of the signal in rad/s, which can be written as $W = 2\pi B_X$, and B_X is the bandwidth in Hz, which in this case it is $B_X = 5$ MHz. During transmission, white and Gaussian noise is added to the signal with power spectral density $N_0/2$, with $N_0 = 4\ 10^{-21}$Watt/Hz. In the receiver, the received signal is filtered with filter with impulse response $h(t)$. Calculate the signal to noise ratio in dB at the output of the receiver, if thes filter is:

a) An ideal low pass filter with bandwidth $B = B_X = 5$ MHz.

b) An ideal low-pass filter with bandwidth $B = 2B_X = 10$ MHz.

c) An ideal low-pass filter with bandwidth $B = B_X/2 = 2.5$ MHz.

d) A low-pass RC filter with time constant $\tau = R\,C = 3\ 10^{-8}$.

Exercise 1.16 The information signal that is received at the input of a receiver of communications is modeled by a random process $X(t)$ whose power spectral density is

$$S_X(j\omega) = \begin{cases} 10^{-18}\ \left[1 + \cos\left(\frac{\omega}{2B_X}\right)\right], & \text{if } |\omega| \le W_X = 2\pi B_X \\ 0, & \text{if } |\omega| > W_X = 2\pi B_X. \end{cases}$$

where W_X is the bandwidth of the signal in rad/s, which can be written as $W_X = 2\pi B_X$, and B_X is the bandwidth in Hz, which in this case it is $B_X = 5$ MHz. During transmission, thermal noise is added to the signal, being the temperature $T = 290$ ºK. In the receiver, the received signal is filtered. Calculate the signal to noise ratio in dB at the output of the receiver filter, when the filter is:

a) An ideal low pass filter with bandwidth $B = B_X = 5$ MHz.

b) An ideal low-pass filter with bandwidth $B = 2B_X = 10$ MHz.

c) An ideal low-pass filter with bandwidth $B = B_X/2 = 2.5$ MHz.

Exercise 1.17 Let $X(t)$ be a stationary random process, with mean $m_X = 1$, and autocorrelation function $R_X(\tau) = \cos(\omega_A\tau) + \cos(\omega_B\tau)$. The process $Y(t)$ is obtained as

$$Y(t) = X(t) + a\,X(t-b),$$

where a and b are two constant integer values. This process goes through a linear and invariant system with the following frequency response

$$H(j\omega) = 2\,\Lambda\left(\frac{\omega}{W}\right) = \begin{cases} 2\ \left(1 - \frac{|\omega|}{W}\right), & \text{if } |\omega| \le W \\ 0 & \text{if } |\omega| > W \end{cases},$$

giving rise to the process $Z(t)$. Consider that $\omega_A = 4\pi$, $\omega_B = 10\pi$ and $W = 8\pi$.

a) Compute the mean, $m_Y(t)$, the autocorrelation function, $R_Y(t+\tau, t)$, the power spectral density, $S_Y(j\omega)$, and the power, P_Y, of the process $Y(t)$, and discuss whether or not this process is stationary and why.

b) Compute the mean, $m_Z(t)$, the autocorrelation function, $R_Z(t+\tau, t)$, the power spectral density, $S_Z(j\omega)$, and the power, P_Z, of the process $Z(t)$, and discuss whether or not this process is stationary and why, and whether the random processes $Y(t)$ and $Z(t)$ are jointly stationary and why.

Exercise 1.18 A random process $X(t)$, stationary, has zero mean and autocorrelation function

$$R_X(\tau) = \mathrm{sinc}(\tau)$$

a) Calculate the power and power spectral density of the process $X(t)$.

b) The random process $X(t)$ passes through an ideal low-pass filter of bandwidth $\pi/2$ rad/s. Compute the mean, power, autocorrelation function, and power spectral density of the output process, $Y(t)$.

c) The random process $Z(t)$ is defined as follows:

$$Z(t) = X(t) \times \sin(\pi t)$$

 i) Compute the mean and the autocorrelation function of the process $Z(t)$, and explain whether the process is stationary or cyclostationary.

 ii) Calculate the power spectral density and the power of the process $Z(t)$

Exercise 1.19 A random process $X(t)$, stationary, has zero mean and autocorrelation function

$$R_X(\tau) = \delta(\tau)$$

a) Calculate the power and power spectral density of the process $X(t)$. Explain the results obtained taking into account the type of random process that is analyzed.

b) The random process $X(t)$ passes through an ideal low-pass filter of bandwidth π rad/s. Compute the mean, power, autocorrelation function, and power spectral density of the output process, $Y(t)$.

c) The random process $Z(t)$ is defined from the process $Y(t)$ obtained in the previous section as follows:

$$Z(t) = Y(t)\ \cos(\pi t) + Y(t)\ \sin(\pi t)$$

 i) Compute the mean and the autocorrelation function of $Z(t)$, and explain whether the process is stationary or cyclostationary.

 ii) Calculate the power spectral density and the power of $Z(t)$

Exercise 1.20 A random process $X(t)$, stationary, has mean $m_X = 1$ and autocorrelation function

$$R_X(\tau) = 1 + \mathrm{sinc}(\tau)\ (1 + \cos(2\pi\tau)).$$

a) Calculate and plot the power spectral density of the process $X(t)$, and obtain its power.

b) The random process $X(t)$ is filtered with an ideal bandpass filter, with center frequency 2π rad/s and bandwidth also 2π rad/s.

i) Compute the mean, power, and autocorrelation function of the output process, $Y(t)$, and compute and plot its power spectral density, $S_Y(j\omega)$.

ii) If thermal noise is added to the process $X(t)$, and the noise temperature is 290 Kelvin degrees, calculate the signal-to-noise ratio before and after filtering with the specified filter the sum of $X(t)$ and the thermal noise.

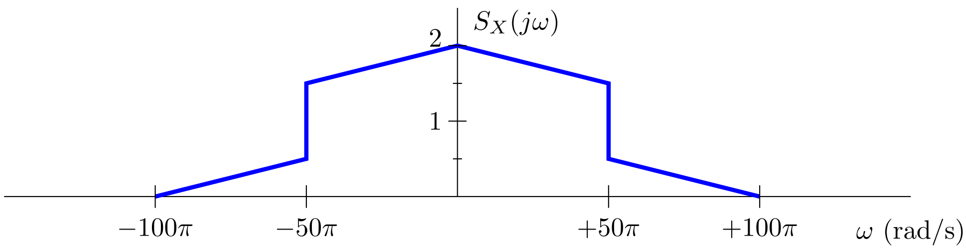
c) The random process $Z(t)$ is defined as follows:

$$Z(t) = \frac{X(t)}{2} - \frac{1}{2}.$$

i) Compute the mean and the autocorrelation function of $Z(t)$, and explain whether the process is stationary or cycle-stationary and why.

ii) Compute the cross-correlation function between $X(t)$ and $Z(t)$, $R_{X,Z}(t+\tau, t)$, and explain whether or not the two processes are jointly stationary and why.

Exercise 1.21 A random process $X(t)$, stationary, has mean $m_X = A$ and the power spectral density of the figure



a) Calculate the power of the process $X(t)$, and its autocorrelation function.

b) The random process $X(t)$ is filtered with a low-pass filter of 25 Hz bandwidth and power gain $G = 2$.

i) Compute the mean, power, and autocorrelation function of the output process, $Y(t)$, and compute and plot its power spectral density, $S_Y(j\omega)$.

ii) If thermal noise is added to the process $X(t)$, and the noise temperature is 290º Kelvin, calculate the signal-to-noise ratio before and after filtering with the specified filter the sum of $X(t)$ and the thermal noise.

c) The random process is defined $Z(t) = X(t)\ \cos(200\pi t)$

i) Compute the process mean $m_Z(t)$, the process autocorrelation function, $R_Z(t+\tau, t)$, and explain whether the process is stationary or cyclostationary and why.

ii) Compute the cross-correlation function between $X(t)$ and $Z(t)$, $R_{X,Z}(t+\tau, t)$, and explain whether or not the two processes are jointly stationary and why.

iii) Compute the power, and compute and plot the power spectral density of $Z(t)$.

Exercise 1.22 A random process $X(t)$, stationary, has mean $m_X = 2$ and autocorrelation function

$$R_X(\tau) = \text{sinc}^2\left(\frac{\tau}{2}\right)\ [1 + \cos(2\pi\tau)]\,.$$

a) Calculate the power of the process $X(t)$, and its power spectral density.

b) The random process $X(t)$ is filtered with an ideal band-pass filter with central frequency $\omega_c = 2\pi$ rad/s, bandwidth $W = 2\pi$ rad/s and power gain $G = 2$.

   i) Compute the mean, power, and autocorrelation function of the output process, $Y(t)$, and compute and plot their power spectral density.

   ii) If thermal noise is added to the process $X(t)$, and the noise temperature is 290º Kelvin, calculate the signal-to-noise ratio, expressed in decibels, before and after filtering with the specified filter the sum of $X(t)$ and the thermal noise.

c) The random process $Z(t) = X(t) + Y(t)$ is defined.

   i) Calculate the cross-correlation function between $X(t)$ and $Z(t)$, $R_{X,Z}(t+\tau, t)$ and answer the following question: What conditions must be fulfilled for the processes $X(t)$ and $Z(t)$ to be jointly stationary?

   ii) Compute the mean $m_Z(t)$, the autocorrelation function, $R_Z(t+\tau, t)$, and the power spectral density, $S_Z(j\omega)$, of the process, and explain whether it is a stationary or cyclostationary process and why.