Chapter 1 : Solutions to the exercises

Exercise 1.1 Solution

a) The probability is calculated by integrating the probability density function of X in the corresponding interval, obtaining for the uniform distribution

$$P(X>1)=\frac{1}{4},\ P(X>-1)=\frac{3}{4},\ P(X<-1)=\frac{1}{4},\ P(-1\leq X\leq 1)=\frac{1}{2}.$$

b) And for the Gaussian distribution

$$P(X>1)=Q\left(0\right)=\frac{1}{2},\ P(X>-1)=1-Q(1)=0.8414,$$
$$P(X<-1)=Q(1)=0.1586,\ P(-1\leq X\leq 1)=1-Q(1)-Q(0)=0.3414.$$

Exercise 1.2 Solution

a) The analytic expressions are:

$$F_X(x)=\begin{cases}0, & x<-1\\ \frac{x+1}{2}, & -1\leq x\leq 1\\ 1, & x>1\end{cases}.$$

$$f_X(x)=\frac{1}{2},\qquad -1\leq x\leq 1.$$

$$f_Y(y)=\begin{cases}0, & y<-1\\ 1+y, & -1\leq y\leq 0\\ 1-y, & 0\leq y\leq 1\\ 0, & y>1\end{cases}.$$

$$F_Y(y)=\begin{cases}0, & y<-1\\ \frac{1}{2}+y+\frac{y^2}{2}, & -1\leq y\leq 0\\ \frac{1}{2}+y-\frac{y^2}{2}, & 0\leq y\leq 1\\ 1, & y>1\end{cases}.$$

b) The variances of X and Y are:

$$\sigma_X^2=\frac{1}{3},\ \sigma_Y^2=\frac{1}{6}.$$

c) The required probabilities are:

$$P(Y>0)=\frac{1}{2},\ P\left(Y>-\frac{1}{2}\right)=\frac{7}{8},\ P\left(Y<-\frac{1}{2}\right)=\frac{1}{8},\ P\left(Y>-\frac{1}{2}\right)=\frac{9}{32}$$

Exercise 1.3 Solution

a) $m_X(t) = \frac{2}{3}$

b) $R_X(t+\tau,t) = \frac{1}{2} + \frac{(t+\tau)\,t}{3}$

c) This process is not a stationary process, in the wide sense. Despite the fact that the mean does not depend on time t, the autocorrelation function $R_X(t+\tau,t)$ depends explicitly on the value of each of the instants $t+\tau$ and t and not only on their difference τ.

Exercise 1.4 Solution

The answer is NO. In the first place, because the definition is limited to temporal instants greater than zero, with which the probability density function for negative instants is not symmetric and the stationarity condition is no longer fulfilled.

In any case, even if this restriction did not exist, it would not be WSS either.

$$\mathrm{Cov}(X(t_1)X(t_2)) = E[X(t_1)X(t_2)] - m_X(t_1)m_X(t_2).$$

Taking into account that the mean is zero,

$$R_X(t_1,t_2) = E[X(t_1)X(t_2)] = \sigma^2 \min(t_i,t_j),$$

it depends on the precise instants t_i and t_j and not on their difference $\tau = t_i - t_j$.

Exercise 1.5 Solution

a) $m_Y[n] = (1+a)\,m_X$

b) $R_Y(n+k,n) = (1+a^2)R_X[k] + aR_X[k+1] + aR_X[k-1]$

c) $S_Y(e^{j\omega}) = S_X(e^{j\omega})\left((1+a^2) + ae^{j\omega} + ae^{-j\omega}\right)$

Exercise 1.6 Solution

a) $m_Z(t) = 0$

b) La función de autocorrelación $R_X(t+\tau,t)$ vale

$$R_Z(t+\tau,t) = \frac{\sigma_X^2+\sigma_Y^2}{2}\cos(\omega_0\tau) + \frac{\sigma_X^2-\sigma_Y^2}{2}\cos(\omega_0(2t+\tau)).$$

c) The process is not stationary, since the autocorrelation depends on the concrete value of t and not just τ. But the process is cyclostationary since $R_Z(t+\tau+T_0,t+T_0) = R_Z(t+\tau,t)$ para $T_0 = \frac{\pi}{\omega_0}$.

d) The power spectral density is

$$S_Z(j\omega) = TF\{\widetilde{R}_X(\tau)\} = \pi\frac{\sigma_X^2+\sigma_Y^2}{2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)].$$

e) In the case $\sigma_X = \sigma_Y = \sigma$ the mean does not change, it is $m_Z(t) = 0$. Instead, the autocorrelation function is
$$R_Z(t+\tau,t) = \sigma^2\cos(\omega_0\tau) = R_Z(\tau).$$
Which means that the process is Wide Sense Stationary (WSS). This implies that the power spectral density is
$$S_Z(j\omega) = TF\{R_Z(\tau)\} = \sigma^2\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$$

Exercise 1.7 Solution

a) $m_Z(t) = \frac{1}{2}\sin(\omega_0 t)$

b) $R_X(t+\tau,t) = \frac{1}{3}\cos(\omega_0\tau)$

c) The random process is not stationary, but cyclostationary, since although the autocorrelation function only depends on τ, and not on t, the mean does depend on t, and also $m_Z(t+T_0) = m_Z(t)$ y $R_Z(t+\tau+T_0, t+T_0) = R_Z(t+\tau,t)$ for $T_0 = \frac{2\pi}{\omega_0}$.

d) $S_z(j\omega) = \frac{\pi}{3}\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right]$

Exercise 1.8 Solution

a)
$$m_X(t) = 0$$
$$R_X(t+\tau,t) = \frac{\sigma^2}{2}\left(\cos(\omega(2t+\tau)+2\theta)+\cos(\omega\tau)\right)$$
The process is not stationary since the autocorrelation depends on the origin of times. However, it is cyclostationary with period $T = \frac{\pi}{\omega}$ seconds.

b)
$$m_X(t) = \cos(\omega t+\theta)$$
$$R_X(t+\tau,t) = (\sigma^2+1)\cos(\omega(t+\tau)+\theta)\cos(\omega t+\theta)$$
The process is cyclostationary with period $T = \frac{2\pi}{\omega}$. Both the mean as the autocorrelation are periodic with said period.

c)
$$m_X(t) = A\frac{\sin(2\,\pi\,t+\theta)-\sin(\theta)}{2\pi t}$$
$$R_X(t+\tau,t) = \frac{A^2}{2}\left(\frac{\sin(2\,\pi\,\tau+4\,\pi\,t+2\,\theta)-\sin(2\,\theta)}{2\pi(\tau+2\,t)}+\frac{\sin(2\,\pi\,\tau)}{2\pi\tau}\right)$$
It is neither stationary nor cyclostationary.

d)
$$m_X(t) = 0$$
$$R_X(t+\tau,t) = \frac{A^2}{2}\cos(\omega\tau)$$
The process is stationary.

e)
$$m_X(t) = -\frac{2A}{\pi}\sin(\omega t)$$
$$R_X(t+\tau,t) = \frac{A^2}{2}\cos(\omega\tau)$$
The process is cyclostationary with period $T = \frac{2\pi}{\omega}$.

f)
$$m_X(t) = 0$$
$$R_X(t+\tau,t) = \frac{1}{4\pi}\frac{\sin(2\,\pi\,\tau)}{\tau} = \frac{1}{2}\text{sinc}(2\tau)$$
The process is stationary.

Exercise 1.9 Solution

a) $m_Y(t) = 0$.

b)
$$R_Y(t+\tau,t) = \frac{A_c^2}{2}R_X(\tau)\left[\cos(\omega_c\tau) + \cos(\omega_c(2t+\tau))\right].$$

c) $S_Y(j\omega) = \frac{A_c^2}{4}\left[S_X(j\omega - j\omega_c) + S_X(j\omega + j\omega)\right]$.

d) $P_Y = \frac{A_c^2}{2}\,P_X$.

Exercise 1.10 Solution

a) $m_S(t) = 0$

b) $R_S(t+\tau,t) = R_M(\tau)\ \cos(\omega_c\tau)$. Since the mean is a constant, and the autocorrelation function is a function that does not depend on time, but on the difference between time instants τ, the random process $S(t)$ is a stationary process (in the broad sense).

c)
$$S_S(j\omega) = \frac{1}{2}\,S_M(j\omega - j\omega_c) + \frac{1}{2}\,S_M(j\omega + j\omega_c).$$
$$P_S = P_M.$$

Exercise 1.11 Solution

a) $R_Y(\tau) = N_0B\ \text{sinc}(2B\tau)$.

b) The joint probability density function is a joint Gaussian distribution, defined by the vector of means $\boldsymbol{\mu} = [0,0]$ and by the covariance matrix
$$\boldsymbol{C} = \begin{bmatrix} N_0B & 0 \\ 0 & N_0B \end{bmatrix},$$
The variables are independent, since their covariance is zero and they are jointly Gaussian.

For $\tau = \frac{1}{4B}$, the joint pdf is now defined by the vector of means $\boldsymbol{\mu} = [0,0]$ and by the covariance matrix
$$\boldsymbol{C} = \begin{bmatrix} N_0B & N_0B\ \text{sinc}\left(\frac{1}{2}\right) \\ N_0B\ \text{sinc}\left(\frac{1}{2}\right) & N_0B \end{bmatrix}.$$
Now, they are not independent, since the covariance between both random variables is not null, the covariance matrix is no longer a diagonal matrix.

Exercise 1.12 Solution

a) The answer is yes. If a stationary process $X(t)$ passes through a linear and invariant system, not only is the output process $Y(t)$ stationary, but also $X(t)$ and $Y(t)$ are jointly stationary. And in this case, the complete system is a linear and invariant system (SLI). A retarder is linear, an adder is linear, and the derivative operation, which has the impulse response $h(t) = \delta'(t)$, is an SLI.

b) $S_Y(j\omega) = S_X(j\omega)2\omega^2\,[1+\cos(\omega T)]$.

Exercise 1.13 Solution

$$\frac{S}{N} = \underbrace{2\frac{P_0}{N_0}}_{cte}\,\frac{\arctan(B/f_0)}{(B/f_0)}.$$

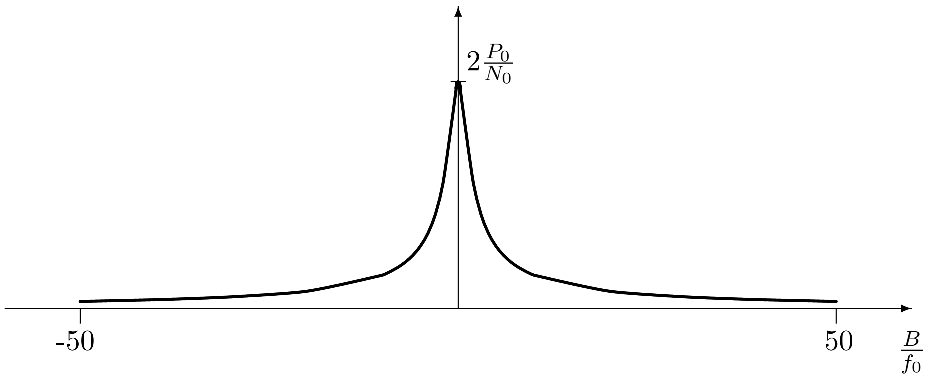


Figure 1.1: Representation of the signal to noise ratio for the Exercise 1.13.

The maximum of $\frac{S}{N}$ is obtained for $\frac{B}{f_0} \to 0$ (for zero there would be neither signal nor noise), where it tends to the value

$$\frac{S}{N} \to 2\frac{P_0}{N_0}.$$

One aspect to take into account is that although a value of B as low as possible improves the S/N ratio, when B is below f_0 the signal degrades by losing important frequency components. Therefore, an appropriate trade-off value in this case is $B = f_0$, where the signal-to-noise ratio is equal to

$$\frac{S}{N} = 2\frac{P_0}{N_0}0.785 = 1.57\frac{P_0}{N_0}.$$

Exercise 1.14 Solution

$$B_{eq} = \frac{\frac{1}{2\tau}}{2\times 1} = \frac{1}{4\tau} = \frac{1}{4RC}.$$

Exercise 1.15 Solution

a)

$$\frac{S}{N}\text{ (dB)} = 10\log_{10}\frac{P_Y}{P_Z} = 10\log_{10}\frac{10^{-11}}{2\times 10^{-14}} = 27 \text{ dB}.$$

b)

$$\frac{S}{N}\ (\text{dB}) = 10\log_{10}\frac{P_Y}{P_Z} = 10\log_{10}\frac{10^{-11}}{4\times 10^{-14}} = 24\ \text{dB},$$

c)

$$\frac{S}{N}\ (\text{dB}) = 10\log_{10}\frac{P_Y}{P_Z} = 10\log_{10}\frac{5\times 10^{-12}}{10^{-14}} = 27\ \text{dB}.$$

d)

$$\frac{S}{N}\ (\text{dB}) = 10\log_{10}\frac{P_Y}{P_Z} = 10\log_{10}\frac{8\times 10^{-12}}{3.33\times 10^{-14}} = 23.8\ \text{dB}.$$

Exercise 1.16 Solution

a)

$$\frac{S}{N}(\text{dB}) = 10\log_{10}\frac{P_Y}{P_Z} = 10\log_{10}\frac{10^{-11}}{2.001\times 10^{-14}} = 26.97 \approx 27\ \text{dB}.$$

b)

$$\frac{S}{N}(\text{dB}) = 10\log_{10}\frac{P_Y}{P_Z} = 10\log_{10}\frac{10^{-11}}{4.002\times 10^{-14}} = 23.97 \approx 24\ \text{dB}.$$

c)

$$\frac{S}{N}(\text{dB}) = 10\log_{10}\frac{P_Y}{P_Z} = 10\log_{10}\frac{0.818\times 10^{-11}}{1.0005\times 10^{-14}} = 29.12\ \text{dB}.$$

Exercise 1.17 Solution

a)

$$m_Y(t) = 1 + a$$

$$\begin{aligned} R_Y(\tau) = &(1+a^2)\ [\cos(\omega_A\tau) + \cos(\omega_B\tau)] \\ &+ a\ [\cos(\omega_A(\tau-b)) + \cos(\omega_B(\tau-b)) + \cos(\omega_A(\tau+b)) + \cos(\omega_B(\tau+b))]\,. \end{aligned}$$

Since the mean is a constant, and the autocorrelation function depends only on the difference between instants of time, τ, the random process is stationary.

$$S_Y(j\omega) = \pi\ (1 + 2\ a\ \cos(\omega b) + a^2)\ [\delta(\omega-\omega_A) + \delta(\omega+\omega_A) + \delta(\omega-\omega_B) + \delta(\omega+\omega_B)]\,.$$

<u>REMARK</u>: Since both ω_A and ω_B are even integer multiples of π, b being an integer, ωb is an even multiple of π for both $\omega = \omega_A$ and $\omega = \omega_B$, so $\cos(\omega(\tau \pm b)) = \cos(\omega\tau)$ and $\cos(\omega b) = 1$ for those two frequencies, so the previous expressions can be simplified, and also taking into account that $1 + 2\ a + a^2 = (1+a)^2$, would end up as

$$R_Y(\tau) = (1+a)^2\ [\cos(\omega_A\tau) + \cos(\omega_B\tau)]\,.$$

$$S_Y(j\omega) = \pi\ (1+a)^2\ [\delta(\omega-\omega_A) + \delta(\omega+\omega_A) + \delta(\omega-\omega_B) + \delta(\omega+\omega_B)]\,.$$

The power of the process is

$$P_Y = 2\ (1+a)^2\ \text{Watts}$$

b)

$$m_Z = 2\ m_Y = 2\ (1+a)$$

$$R_Z(\tau) = (1+a^2)\ \left[\cos(\omega_A \tau)\right].$$

The process is stationary, since neither the mean nor the autocorrelation function depend on time.

$$S_Z(j\omega) = \pi\ (1+a)^2\ \left[\delta(\omega-\omega_A)+\delta(\omega+\omega_A)\right].$$

$$P_Z = (1+a)^2 \text{ Watts.}$$

REMARK: In case you did not realize the simplifications in considering that ω_A and ω_B are even multiples of π, the expressions would be

$$R_Z(\tau) = TF^{-1}\left[S_Z(j\omega)\right] = (1+a^2)\ \left[\cos(\omega_A\tau)\right] + a\ \left[\cos(\omega_A(\tau-b))+\cos(\omega_A(\tau+b))\right].$$

$$S_Z(j\omega) = \pi\ (1+2\ a\ \cos(\omega b)+a^2)\ \left[\delta(\omega-\omega_A)+\delta(\omega+\omega_A)\right].$$

$$P_Z = 1+a^2+2\ a\ \cos(\omega_A b) \text{ Watts.}$$

Exercise 1.18 Solution

a) Power and power spectral density are, respectively

$$P_X = 1 \text{ Watt}, \qquad S_X(j\omega) = \Pi\left(\frac{\omega}{2\pi}\right)$$

b) Mean, power, autocorrelation function, and power spectral density of $Y(t)$

$$m_Y = 0, \quad P_Y = \frac{1}{2} \text{ Watts}, \quad R_Y(\tau) = \frac{1}{2}\text{sinc}\left(\frac{\tau}{2}\right), \quad S_Y(j\omega) = \Pi\left(\frac{\omega}{\pi}\right).$$

c) For the stochastic process $Z(t)$

i) Mean, autocorrelation function, and explain whether or not it is stationary

$$m_Z = 0, \quad R_Z(t+\tau,t) = \frac{1}{2}R_X(\tau)\left[\cos(\pi\tau)-\cos(\pi(2t+\tau)\right]$$

The process is cyclostationary, as the autocorrelation function of the temporal variable depends periodically (the mean is constant, which can also be considered as a periodic function of any period, which makes the total statistics periodic).

ii) Power and power spectral density

$$S_Z(j\omega) = \frac{1}{4}\Pi\left(\frac{\omega}{4\pi}\right), \quad P_Z = \frac{1}{2} \text{ Watts.}$$

Exercise 1.19 Solution

a) Potencia y densidad espectral de potencia de $X(t)$

$$P_X = \infty \text{ Watts}, \quad S_X(j\omega) = 1.$$

Resultados evidentes al tratarse de un proceso blanco.

b) Media, potencia, función de autocorrelación y densidad espectral de potencia de $Y(t)$

$$m_Y = 0, \quad P_X = 1 \text{ Watts}, \quad R_Y(\tau) = \text{sinc}\left(\tau\right), \quad S_Y(j\omega) = \Pi\left(\frac{\omega}{2\pi}\right).$$

c) Para el proceso $Z(t)$

i) Media, función de autocorrelación, y explicar si es o no estacionario

$$m_Z = 0, \quad R_Z(t+\tau,t) = R_Y(\tau)\left[\cos(\pi\tau)+\sin(\pi(2t+\tau)\right]$$

El proceso es cicloestacionario, al depender la función de autocorrelación de la variable temporal de forma períodica (la media es constante, lo que se puede también considerar como una función periódica de cualquier período, lo que hace que la estadística total sea periódica).

ii) Densidad espectral de potencia y potencia

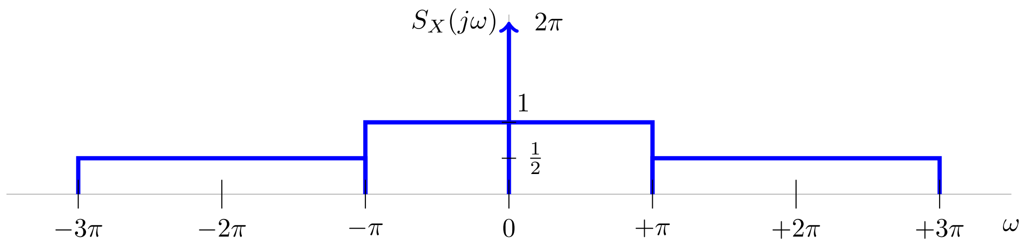
$$S_Z(j\omega) = \frac{1}{2}\Pi\left(\frac{\omega}{4\pi}\right), \quad P_Z = 1 \text{ Watts.}$$

Exercise 1.20 Solution

a) Densidad espectral de potencia

$$S_X(j\omega) = 2\pi\,\delta(\omega) + \Pi\left(\frac{\omega}{2\pi}\right) + \frac{1}{2}\left[\Pi\left(\frac{\omega-2\pi}{2\pi}\right)+\Pi\left(\frac{\omega+2\pi}{2\pi}\right)\right].$$

The response is plotted below (the amplitude of the delta is scaled in the figure)



The power of the process is

$$P_X = 3 \text{ Watts}$$

b) We now consider the filtered process

i) The mean of the output process is

$$m_Y = 0$$

The autocorrelation function

$$R_Y(\tau) = \text{sinc}\,(\tau)\cos(2\pi\tau),$$

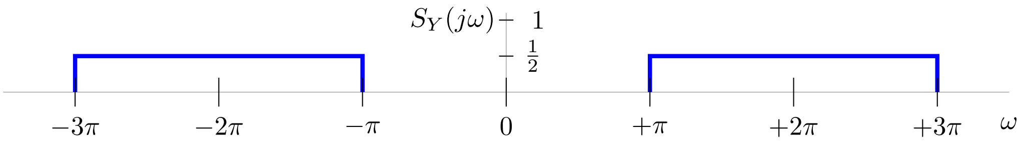
and the power

$$P_Y = 1 \text{ Watts.}$$

The power spectral density is

$$S_Y(j\omega) = \frac{1}{2}\left[\Pi\left(\frac{\omega-2\pi}{2\pi}\right)+\Pi\left(\frac{\omega+2\pi}{2\pi}\right)\right]$$

and is represented in the figure



ii) The signal-to-noise ratio at the input is zero (or minus infinity dB)

$$\left(\frac{S}{N}\right)_{entrada} = 0.$$

At the output of the filter

$$\left(\frac{S}{N}\right)_{salida} = 2.5 \times 10^{20} \text{ (in dB, 204 dB)} .$$

c) In this case we have the random process $Z(t) = \frac{X(t)}{2} - \frac{1}{2}$

i) The mean of $Z(t)$ is

$$m_Z(t) = 0$$

The autocorrelation function

$$R_Z(t+\tau,t) = \frac{1}{4}\text{sinc}(\tau)\ (1+\cos(2\pi\tau)) \equiv R_Z(\tau).$$

The process is stationary, since the mean is a constant and the autocorrelation function does not depend on time, but only on the difference between the two instants of time, τ.

ii) The cross-correlation function is defined as

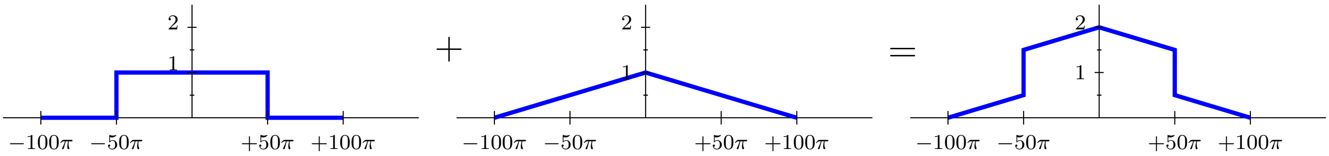
$$R_{X,Z}(t+\tau,t) = \frac{1}{2}\text{sinc}(\tau)\ (1+\cos(2\pi\tau)) \equiv R_{X,Z}(\tau)$$

The random processes $X(t)$ and $Z(t)$ are jointly stationary, since both are stationary, and their cross-correlation function does not depend on time, but only on the difference between the two time arguments, τ.

Exercise 1.21 Solution

a) As the power spectral density can be written as

$$S_X(j\omega) = \Pi\left(\frac{\omega}{100\pi}\right) + \Lambda\left(\frac{\omega}{100\pi}\right)$$



the autocorrelation function is

$$R_X(\tau) = \mathcal{FT}^{-1}\{S_X(j\omega)\} = 50\ \text{sinc}(50\tau) + 50\ \text{sinc}^2(50\tau).$$

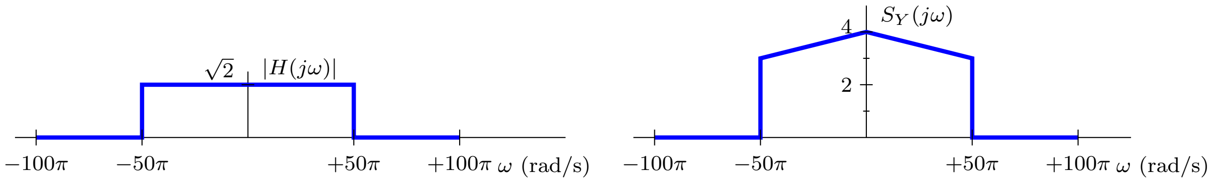
The power is

$$P_X = R_X(0) = 50\ \text{sinc}(0) + 50\ \text{sinc}^2(0) = 100 \text{ Watts}$$

It can also be calculated as follows

$$P_X = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(j\omega)\ d\omega = \frac{1}{2\pi}(100\ \pi + 100\ \pi) = 100 \text{ Watts}$$

b) The frequency response of the filter is shown in the figure (left)



i) The mean is

$$m_Y = m_X \times \int_{-\infty}^{\infty} h(t)\, dt = m_X \times H(0) = \sqrt{2}A,$$

because $H(0) = \sqrt{2}$. The power spectral density is

$$S_Y(j\omega) = S_X(j\omega)|H(j\omega)|^2 = 3\,\Pi\left(\frac{\omega}{100\pi}\right) + \Lambda\left(\frac{\omega}{50\pi}\right),$$

which is plotted in the figure (right).
The autocorrelation function

$$R_Y(\tau) = \mathcal{FT}^{-1}\left\{S_Y(j\omega)\right\} = 150\,\text{sinc}\,(50\tau) + 25\,\text{sinc}^2\,(25\tau)\,,$$

The power is

$$P_Y = R_Y(0) = 150 + 25 = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_Y(j\omega)\, d\omega = \frac{1}{2\pi}(300\pi + 50\pi) = 175 \text{ Watts.}$$

ii) The signal-to-noise ratio at the input is

$$\left(\frac{S}{N}\right)_{entrada} = \frac{P_X}{P_n} = \frac{150}{\infty} = 0 \Rightarrow \left(\frac{S}{N}\right)_{entrada} (dB) = 10\log_{10}(0) = -\infty \text{ dB.}$$

At the output

$$P_Z = N_0BG = kTBG = 1.38 \times 10^{-23} \times 290 \times 25 \times 2 = 2 \times 10^{-19} \text{ Watts.}$$

$$\left(\frac{S}{N}\right)_{salida} = \frac{P_Y}{P_Z} = 8.745 \times 10^{20} \Rightarrow \left(\frac{S}{N}\right)_{salida} (\text{dB}) = 10\log_{10}\frac{P_Y}{P_Z} = 209.41 \text{ dB}$$

c) For $Z(t) = X(t)\ \cos(200\pi t)$

i) The mean is

$$m_Z(t) = E\left[Z(t)\right] = E[X(t)]\ \cos(200\pi t) = A\cos(200\pi t)$$

because $E[X(t)]) = m_X(t) = m_X = A$.
The autocorrelation function

$$\begin{aligned} R_Z(t+\tau, t) =& E\left[Z(t+\tau)Z(t)\right] = E[X(t+\tau)\ X(t)]\ \cos(200\pi(t+\tau))\ \cos(200\pi t) \\ =& \frac{1}{2}R_X(\tau)\ \left[\cos(200\pi\tau) + \cos(200\pi(2t+\tau))\right] \end{aligned}$$

The process is cyclostationary, since both the mean and the autocorrelation function depend on time, but periodically, with period $T_0 = \frac{2\pi}{\omega_0} = \frac{1}{f_0} = \frac{1}{100}$ seconds.

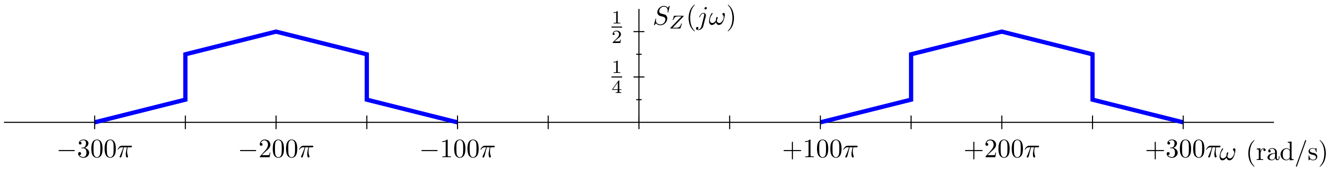
ii) Therefore,

$$\tilde{R}_Z(\tau) = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} R_S(t+\tau,t)\,dt = \frac{1}{2}R_X(\tau)\,\cos(200\pi\tau)$$

and

$$S_Z(j\omega) = \mathcal{TF}\{\tilde{R}_Z(\tau)\} = \frac{1}{4}\left[S_X(j\omega - j200\pi) + S_X(j\omega + j200\pi)\right]$$

Actually $Z(t)$ is a double sideband AM modulation with $X(t)$ as modulator, so this result is theoretically known. The density is plotted in the figure



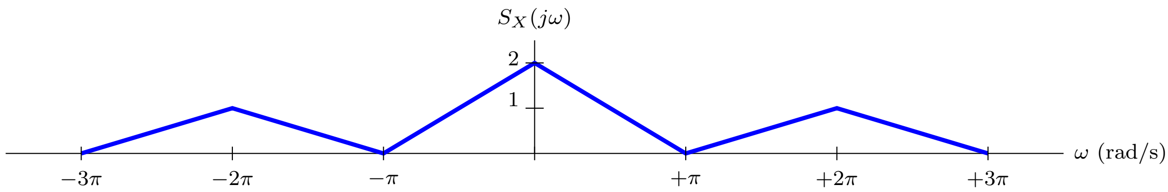
The power is

$$\begin{aligned}P_Z =& \frac{1}{2\pi}\int_{-\infty}^{\infty} S_Z(j\omega)\,d\omega\\ =& \frac{1}{4}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(j\omega - j200\pi)\,d\omega + \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(j\omega + j200\pi)\,d\omega\right] = \frac{1}{2}\left[P_X + P_X\right]\\ =& \tilde{R}_Z(0) = \frac{1}{2}R_X(0) = \frac{1}{2}P_X = 50 \text{ Watts.}\end{aligned}$$

Exercise 1.22 Solution

a) The power spectral density is

$$\begin{aligned}S_X(j\omega) =& \mathcal{TF}\{R_X(\tau)\} = 2\Lambda\left(\frac{\omega}{\pi}\right) + \frac{1}{2\pi}2\Lambda\left(\frac{\omega}{\pi}\right) * \left[\pi\delta(\omega - 2\pi) + \pi\delta(\omega + 2\pi)\right]\\ =& 2\Lambda\left(\frac{\omega}{\pi}\right) + \Lambda\left(\frac{\omega - 2\pi}{\pi}\right) + \Lambda\left(\frac{\omega + 2\pi}{\pi}\right)\end{aligned}$$



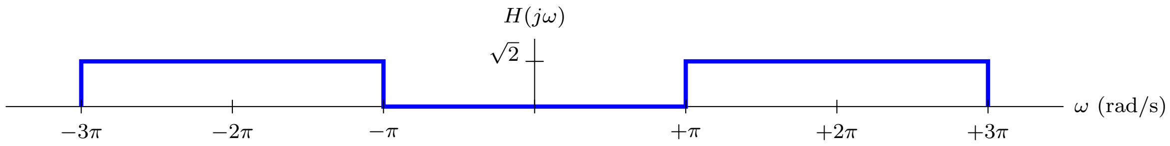
The power is

$$P_X = R_X(0) = \text{sinc}^2(0)[1 + \cos(0)] = 2 \text{ Watts.}$$

It can also be calculated as

$$P_X = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(j\omega)\,d\omega = \frac{1}{2\pi}\left(\pi + 2\,\pi + \pi\right) = 2 \text{ Watts.}$$

b) The figure plots the frequency response of the filter

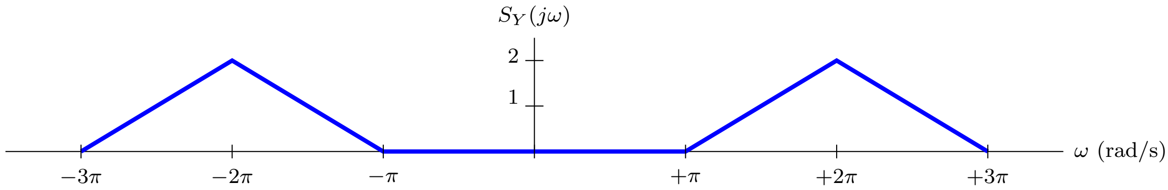


i) The mean is

$$m_Y = m_X \times \int_{-\infty}^{\infty} h(t)\, dt = m_X \times H(0) = m_X \times 0 = 0,$$

because $H(0) = 0$. The power spectral density is

$$S_Y(j\omega) = S_X(j\omega)|H(j\omega)|^2 = 2\Lambda\left(\frac{\omega - 2\pi}{\pi}\right) + 2\Lambda\left(\frac{\omega + 2\pi}{\pi}\right)$$



The autocorrelation function

$$R_Y(\tau) = \mathcal{TF}^{-1}\{S_Y(j\omega)\} = 2\text{sinc}^2\left(\frac{\tau}{2}\right)\ \cos(2\pi\tau),$$

The power is

$$P_Y = R_Y(0) = 2 \times 1 \times 1 = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_Y(j\omega)\, d\omega = \frac{1}{2\pi}(2\pi + 2\pi) = 2 \text{ Watts}.$$

ii) The signal-to-noise ratio at the input of the filter is

$$\left(\frac{S}{N}\right)_{entrada} = \frac{P_X}{P_n} = \frac{150}{\infty} = 0 \Rightarrow \left(\frac{S}{N}\right)_{entrada} (dB) = 10\log_{10}(0) = -\infty \text{ dB}.$$

At the output

$$P_Z = N_0 BG = kTBG = 1.38 \times 10^{-23} \times 290 \times 1 \times 2 = 8 \times 10^{-21} \text{ Watts}.$$

$$\left(\frac{S}{N}\right)_{salida} = \frac{P_Y}{P_Z} = 2.5 \times 10^{20} \Rightarrow \left(\frac{S}{N}\right)_{salida} (\text{dB}) = 10\log_{10}\frac{P_Y}{P_Z} = 203.9794 \approx 204 \text{ dB}.$$

c) Now $Z(t) = X(t) + Y(t)$

i) The cross-correlation funtion is

$$\begin{aligned} R_{X,Z}(t+\tau, t) =& E\left[X(t+\tau)Z(t)\right] = E[X(t+\tau)\ X(t)] + E[X(t+\tau)\ Y(t)] \\ =& R_Z(t+\tau, t) + R_{X,Y}(t+\tau, t) = R_Z(\tau) + R_{X,Y}(\tau) \end{aligned}$$

The cross-correlation function among X and Y, and the cross PSD are

$$R_{X,Y}(\tau) = R_X(\tau) * h(-\tau) \overset{\mathcal{TF}}{\leftrightarrow} S_{X,Y}(j\omega) = S_X(j\omega)\ H^*(j\omega).$$

For two random processes to be jointly stationary, they must both be stationary, and the cross-correlation must depend solely on the difference between the two time arguments.

$$R_{X,Z}(t+\tau, t) \equiv R_{X,Z}(\tau)$$

as it happens in this case.

ii) The mean of $Z(t)$ is

$$m_Z = m_X + m_Y = 2 + 0 = 2,$$

and its autocorrelation function

$$\begin{aligned} R_Z(\tau) =& R_X(\tau) + R_Y(\tau) + R_{X,Y}(\tau) + R_{Y,X}(\tau) \\ =& \text{sinc}^2\left(\frac{\tau}{2}\right)\ [1+\cos(2\pi\tau)] + 2\ \text{sinc}^2\left(\frac{\tau}{2}\right)\ \cos(2\pi\tau) + 2\times\sqrt{2}\,\text{sinc}^2\left(\frac{\tau}{2}\right)\ \cos(2\pi\tau) \\ =& \text{sinc}^2\left(\frac{\tau}{2}\right)\ \left[1+(3+2\sqrt{2})\ \cos(2\pi\tau)\right] = \text{sinc}^2\left(\frac{\tau}{2}\right)\ [1+8.485\ \cos(2\pi\tau)] \end{aligned}$$

because $H^*(j\omega) = H(j\omega)$, and $S_{Y,X}(j\omega) = S_X(j\omega)H(j\omega) = S_{X,Y}(j\omega)$, $R_{Y,X}(\tau) = R_{Y,X}(\tau)$. Finally,

$$\begin{aligned} S_Z(j\omega) =& S_X(j\omega) + S_Y(j\omega) + S_{X,Y}(j\omega) + S_{Y,X}(j\omega) \\ =& \mathcal{TF}\left\{R_Z(\tau)\right\} = 2\Lambda\left(\frac{\omega}{\pi}\right) + 4.242\ \Lambda\left(\frac{\omega-2\pi}{\pi}\right) + 4.242\ \Lambda\left(\frac{\omega+2\pi}{\pi}\right). \end{aligned}$$

The process is stationary because it is the sum of two jointly stationary random processes, and this implies that its mean is constant, and that its autocorrelation function depends only on the difference between the two temporal arguments.