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Communication Theory

English Grades

Chapter 2

Analog Modulations

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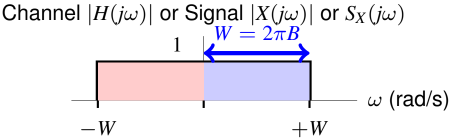
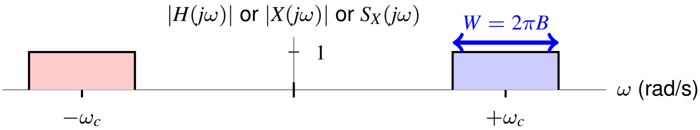
Bandwidth of a signal/system

- Baseband signal/system
  - Frequency response is centered at 0 Hz
- Bandpass signal/system
  - Frequency response is centered at center frequency f_c Hz
  - Or, equivalently, at $\omega_c = 2\pi f_c$ rad/s
- Bandwidth (B Hz, or $W = 2\pi B$ rad/s)
  - Range of positive frequencies that are used or available



Signal/system in baseband



Bandpass signal/system (center frequency ω_c rad/s)

Analog signal to be transmitted: modulating signal $m(t)$

- Deterministic signal: characteristics of signal $m(t)$
  - Low pass signal of bandwidth B Hz (or $W = 2\pi B$ rad/s)
    - ★ Fourier transform $M(j\omega)$ with $M(j\omega) = 0$ for $|\omega| > 2\pi B$
  - It is a power signal: Its power is P_m Watts

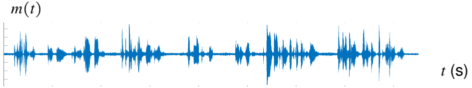
$$P_m = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{+T/2} |m(t)|^2 dt$$

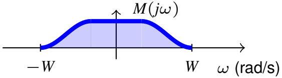
- Random (stochastic) signal: statistical analysis ("*average*")
  - Model for the signal: stochastic process $M(t)$
    - ★ Wide Sense Stationary Process (WSS)
    - ★ Zero mean $m_M = 0$
    - ★ Autocorrelation function $R_M(\tau)$
    - ★ Power spectral density $S_M(j\omega)$
    - ★ Bandwidth B Hz : $S_M(j\omega) = 0$ for $|\omega| > 2\pi B$
    - ★ Power: P_M Watts

$$P_M = R_M(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_M(j\omega)\, d\omega$$

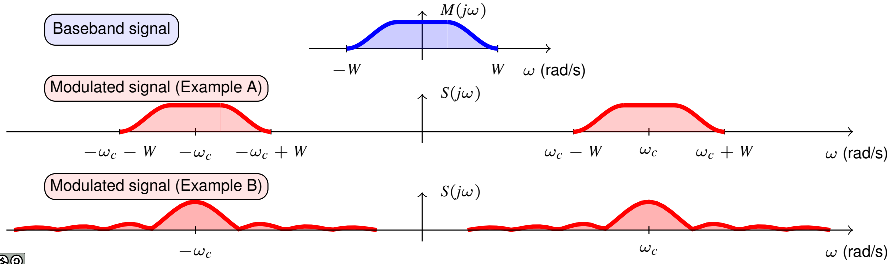
Analog Communications Systems

- Information signal (modulating signal) $m(t)$
  - Analog signal: the information is in the waveform $m(t)$
  - Baseband and bandlimited: Bandwidth W rad/s





- Types of transmission in analog systems
  - Baseband transmission (unmodulated)
  - Modulated transmission : $s(t)$
    - Spectrum of the signal is shifted (center or carrier frequency ω_c)
    - The shape can be maintained or changed (usually bandwidth can be spread)

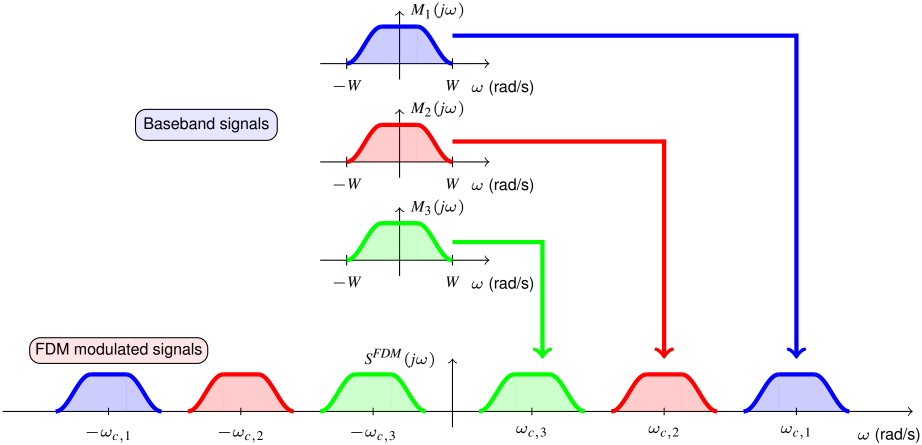


Purpose of modulating an analog signal

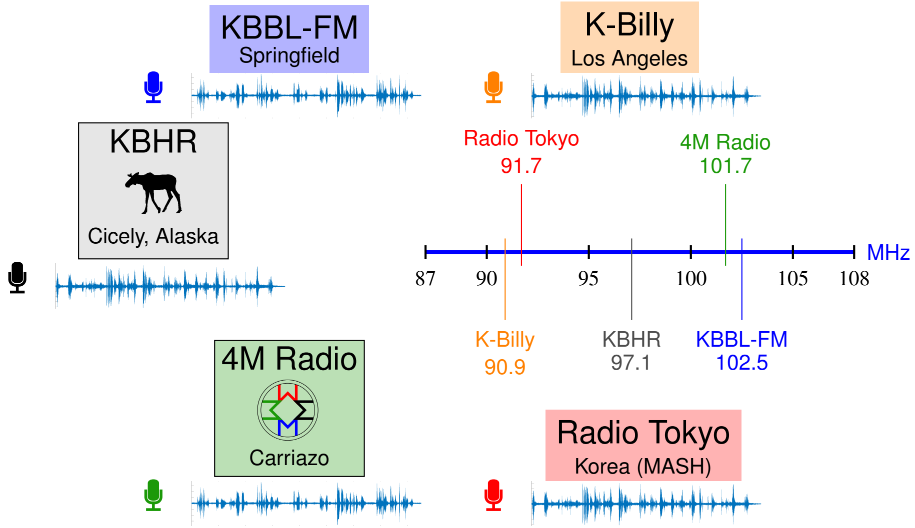
1. Adapting the signal to the characteristics of the channel
   - Choose the appropriate and/or available frequency range
     - The spectrum of the signal is shifted to the corresponding band
2. Expanding the bandwidth
   - Increase the noise immunity of the modulated signal during transmission
3. Multiplexing / medium access
   - Accommodating the simultaneous transmission of different signals on the same medium
     - Frequency Division Multiplexing (FDM)
     - Frequency Division Multiple Access (FDMA)

Frequency Division Multiplexing (FDM)

- The spectrum of different signals is shifted to non-overlapping frequency bands

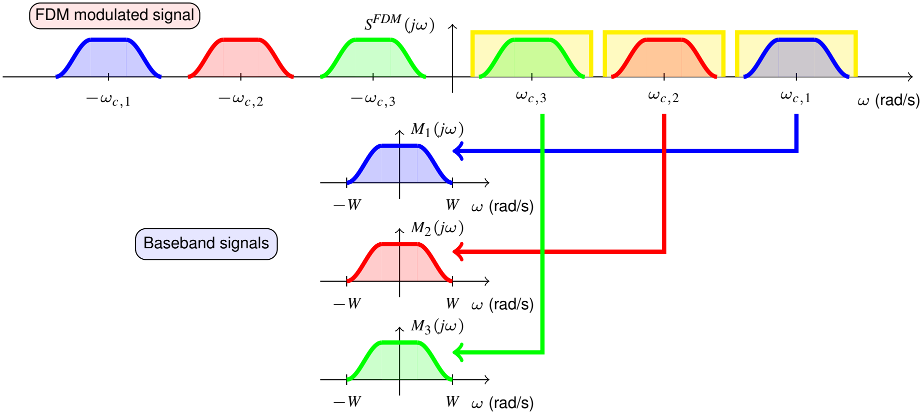


FDM - Example: Commercial Radio



FDM - Demultiplexing

- At the receiver, the spectrum of each signal is filtered and shifted back to baseband, allowing each signal to be recovered



Introduction to the concept of modulation

- Analog modulation:
  - The analog information (modulating) signal is printed in a sinusoidal carrier

$$c(t) = A_c \ \cos(\underbrace{2\pi f_c}_{\omega_c} t + \phi_c) \begin{cases} \text{Amplitude} : A_c \\ \text{Phase} : \phi_c \\ \text{Frequency}: f_c \text{ or } \omega_c \end{cases}$$

  - Parameters of a sinusoidal carrier:
    - Amplitude: A_c V
    - Phase: ϕ_c rad
    - Frequency: f_c Hz (or equivalently, $\omega_c = 2\pi\, f_c$ rad/s)
- Types of analog modulations
  - Amplitude Modulation (AM) $A_c \rightarrow A_c(t) = f(m(t))$
  - Angle modulations
    - Phase Modulation (PM) $\phi_c \rightarrow \phi_c(t) = f(m(t))$
    - Frequency Modulation (FM) $f_i(t) = f_c \rightarrow f_i(t) = f(m(t))$

      $f_i(t)$: instantaneous frequency of the carrier signal

Amplitude Modulations (AM)

- The modulating signal (or message) $m(t)$ is encoded in the amplitude of the carrier signal $c(t)$

$$c(t) = A_c \cos(\omega_c t + \phi_c)$$

$$A_c \rightarrow A_c(t) = f(m(t))$$

- There are different variants of AM modulation
  - AM: Conventional AM modulation (double sideband with carrier)
  - DSB: Double SideBand (without carrier)
  - SSB: Single SideBand
  - VSB: Vestigial SideBand

Conventional AM modulation

- Double SideBand (DSB) + carrier

$$s(t) = \underbrace{A_c \cos(\omega_c t + \phi_c)}_{\text{Carrier } c(t)} + \underbrace{m(t) \times A_c \cos(\omega_c t + \phi_c)}_{\text{Double Sideband (DSB): } m(t)\times c(t)}$$

$$s(t) = A_c \, [1 + m(t)] \, \cos(\omega_c t + \phi_c)$$

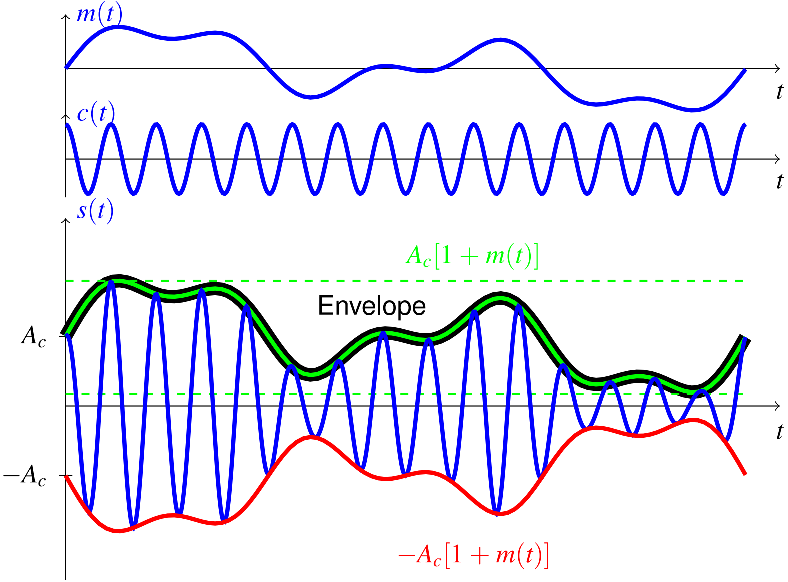
  - Desirable situation: envelope proportional to $m(t)$

    If $A_c \, [1 + m(t)] \geq 0 \; \forall t \;\Rightarrow\;$ Envelope $\equiv A_c \, [1 + m(t)]$

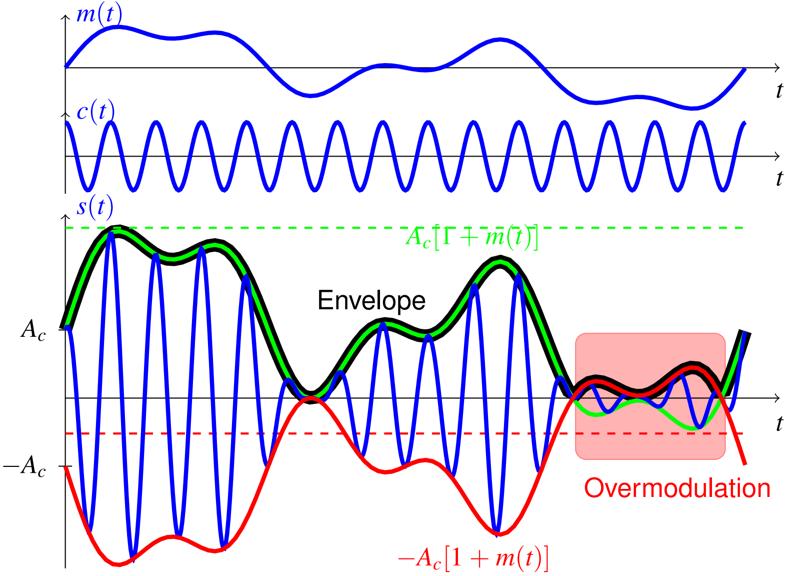
- Overmodulation: occurs when $A_c \, [1 + m(t)] < 0$ at some instants
  - Cause:
    - ★ Occurs when $m(t) < -1$
  - Effect:
    - ★ The carrier is inverted at those instants
    - ★ The envelope becomes $-A_c \, [1 + m(t)]$
  - Solution:

$$\text{Ensure that } |m(t)| \leq 1 \begin{cases} \text{Message normalization: } m_n(t) \\ \text{Introduction of the MODULATION INDEX: } a \end{cases}$$

AM modulation without overmodulation



AM modulation with overmodulation



Conventional AM Modulation - Modulation Index

- Normalized modulating signal (message) $m_n(t)$

$$m_n(t) = \frac{m(t)}{\max|m(t)|} = \frac{m(t)}{C_M}$$

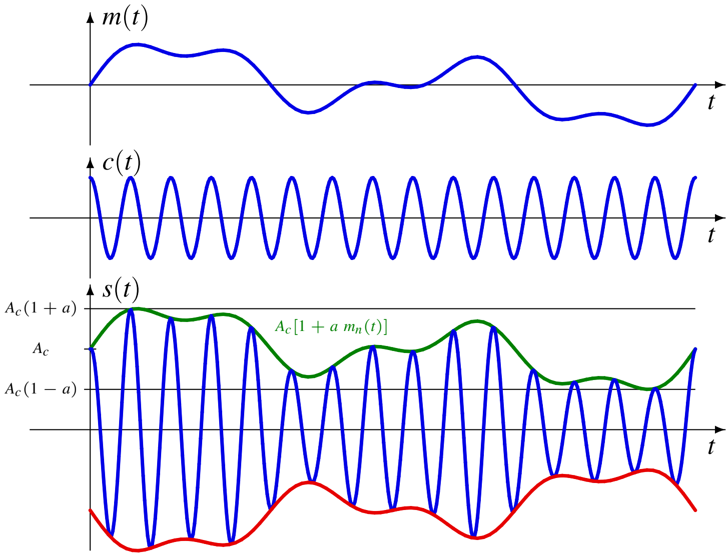
  - $C_M = \max|m(t)|$: Range of $m(t)$: $-C_M \leq m(t) \leq +C_M$
- Modulation index (a)
  - $m(t)$ is replaced by modulating signal with modulation index a

$$m_a(t) = a \times m_n(t) = \frac{a}{C_M} \times m(t)$$

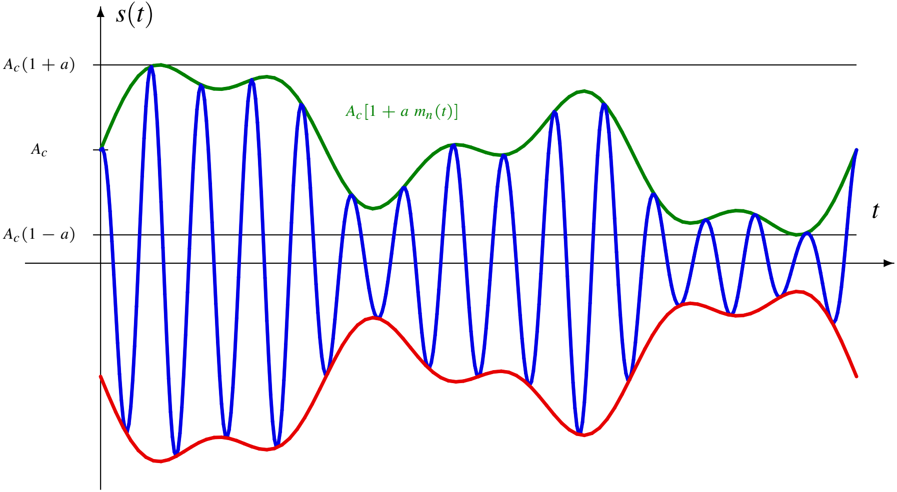
    Range of $m_a(t)$: $-a \leq m_a(t) \leq +a$
    - ★ To avoid overmodulation: $0 < a \leq 1$
- Modulated signal with modulation index a

$$s(t) = A_c \left[1 + \underbrace{a\, m_n(t)}_{m_a(t)}\right] \cos(\omega_c t + \phi_c)$$
$$= A_c \cos(\omega_c t + \phi_c) + m_a(t) \times A_c \cos(\omega_c t + \phi_c)$$

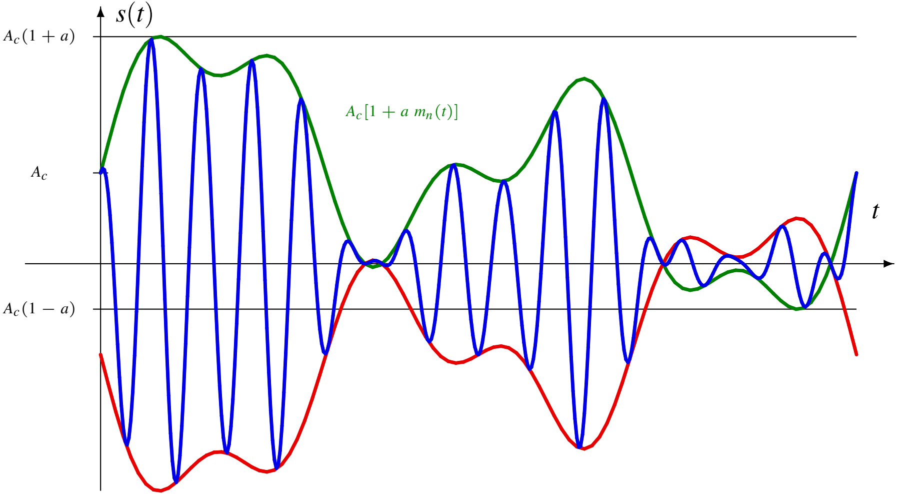
Waveform of a conventional AM modulation



Waveform of an AM modulation ($a = 0.75$)



Overmodulation ($a = 1.5$)



Spectrum of the AM signal - Deterministic

- Conventional AM signal: $s(t) = c(t) + m_a(t) \times c(t)$

$$s(t) = A_c \cos(\omega_c t + \phi_c) + \underbrace{m_a(t)}_{\frac{a}{C_M}\, m(t)} \times A_c \cos(\omega_c t + \phi_c)$$

- Deterministic signal $m(t)$ with FT $M(j\omega)$, $M(j\omega) = 0$ for $|\omega| > 2\pi B$ rad/s
  - Spectrum of $m_a(t) = a\, m_n(t)$: $M_a(j\omega) = a\, M_n(j\omega) = \frac{a}{C_M}\, M(j\omega)$
- Conventional AM signal spectrum

$$\begin{aligned} S(j\omega) =& \mathcal{FT}\{A_c \cos(\omega_c t + \phi_c)\} + \frac{1}{2\pi}\, \mathcal{FT}\{m_a(t)\} * \mathcal{FT}\{A_c \cos(\omega_c t + \phi_c)\} \\ =& A_c\, \pi \left[\delta(\omega - \omega_c)\, e^{j\phi_c} + \delta(\omega + \omega_c)\, e^{-j\phi_c}\right] \\ &+ \frac{A_c}{2} \left[\underbrace{M_a(j\omega - j\omega_c)}_{\frac{a}{C_M}\, M(j\omega - j\omega_c)}\, e^{j\phi_c} + \underbrace{M_a(j\omega + j\omega_c)}_{\frac{a}{C_M}\, M(j\omega + j\omega_c)}\, e^{-j\phi_c}\right] \end{aligned}$$

Spectrum of the AM signal - Analysis

- Modulus of the Fourier transform $S(j\omega)$
  - Two deltas, in $-\omega_c$ and in $+\omega_c$
    - Amplitude $A_c\pi$
  - Replicas of the form of $M(j\omega)$ shifted to $-\omega_c$ and $+\omega_c$
    - Scale factor $\frac{A_c}{2}\frac{a}{C_M}$
- Phase of the Fourier transform
  - The carrier phase introduces the terms $e^{\pm j\phi_c}$
    - Linear phase terms
- Bandwidth of the modulated signal

$$W_{AM} = 2\ W \text{ rad/s}$$

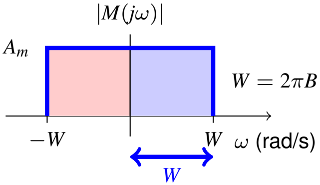
$$B_{AM} = 2\ B \text{ Hz}$$

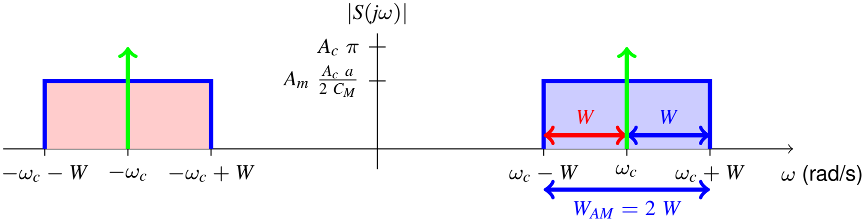
  - Twice the bandwidth of the modulating signal $m(t)$

Spectrum of conventional AM signal - Representation

$$S(j\omega) = A_c\ \pi\ [\delta(\omega-\omega_c)\ e^{j\phi_c} + \delta(\omega+\omega_c)\ e^{-j\phi_c}] + \frac{A_c}{2}\ [\underbrace{M_a(j\omega-j\omega_c)}_{\frac{a}{C_M}\ M(j\omega-j\omega_c)}\ e^{j\phi_c} + \underbrace{M_a(j\omega+j\omega_c)}_{\frac{a}{C_M}\ M(j\omega+j\omega_c)}\ e^{-j\phi_c}]$$

An example: $m(t)$ with a given shape for $M(j\omega) = \mathcal{FT}\{m(t)\}$

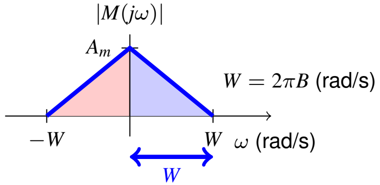


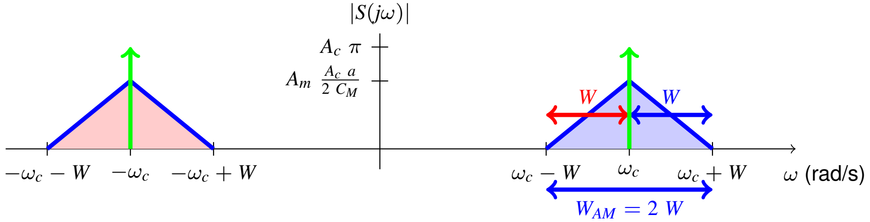


Spectrum of the conventional AM signal - Representation (II)

$$S(j\omega) = A_c\ \pi\ [\delta(\omega-\omega_c)\ e^{j\phi_c} + \delta(\omega+\omega_c)\ e^{-j\phi_c}] + \frac{A_c}{2}\ [\underbrace{M_a(j\omega-j\omega_c)}_{\frac{a}{C_M}\ M(j\omega-j\omega_c)}\ e^{j\phi_c} + \underbrace{M_a(j\omega+j\omega_c)}_{\frac{a}{C_M}\ M(j\omega+j\omega_c)}\ e^{-j\phi_c}]$$

Another example: $m(t)$ with another shape for $M(j\omega) = \mathcal{FT}\{m(t)\}$





Statistical analysis of conventional AM

- Modulating signal model: random process

$$M(t): \text{ stationary, with } m_M = 0,\ R_M(\tau),\ S_M(j\omega),\ \text{power } P_M \text{ Watts}$$

  - Definition of normalized and with modulation index processes

$$M_n(t) = \frac{1}{C_M} M(t) \qquad M_a(t) = a\, M_n(t) = \frac{a}{C_M}\, M(t)$$

- Model of the modulated signal: random process

$$S(t) = A_c\,[1 + M_a(t)]\cos(\omega_c t + \phi_c)$$

- Mean of the conventional AM signal

$$\begin{aligned} m_S(t) =& E[S(t)] = E\left[\underbrace{A_c[1 + M_a(t)]\cos(\omega_c t + \phi_c)}_{S(t)}\right] \\ =& A_c\,(1 + E[M_a(t)])\cos(\omega_c t + \phi_c) \end{aligned}$$

  - If $M_a(t) = a\, M_n(t) = \frac{a}{C_M}\, M(t)$, $E[M_a(t)] = \frac{a}{C_M}\, E[M(t)] = 0$

$$m_S(t) = A_c \cos(\omega_c t + \phi_c)$$

Statistical analysis of conventional AM (II)

- Model of the modulated signal: random process

$$S(t) = A_c\,[1 + M_a(t)]\cos(\omega_c t + \phi_c)$$

- Autocorrelation function of the conventional AM signal

$$\begin{aligned} R_S(t+\tau,t) =& E[S(t+\tau)\;S(t)] \\ =& E\left[\left(\underbrace{A_c[1+M_a(t+\tau)]\cos(\omega_c(t+\tau)+\phi_c)}_{S(t+\tau)}\right)\left(\underbrace{A_c[1+M_a(t)]\cos(\omega_c t+\phi_c)}_{S(t)}\right)\right] \\ =& A_c^2\;E\left[\underbrace{(1+M_a(t+\tau))(1+M_a(t))}_{1+M_a(t)+M_a(t+\tau)+M_a(t+\tau)\;M_a(t)}\right]\cos(\omega_c(t+\tau)+\phi_c)\cos(\omega_c t+\phi_c) \end{aligned}$$

$$E[M_a(t)] = E[M_a(t+\tau)] = 0, E[M_a(t+\tau)\;M_a(t)] = R_{M_a}(\tau),\; \cos(a)\cos(b) = \tfrac{1}{2}\cos(a-b) + \tfrac{1}{2}\cos(a+b)$$

$$R_S(t+\tau,t) = \frac{A_c^2}{2}\,[1 + R_{M_a}(\tau)]\,[\cos(\omega_c\tau) + \cos(\omega_c(2t+\tau) + 2\phi_c)]$$

Statistical analysis of conventional AM (III)

$$m_S(t) = A_c \cos(\omega_c t + \phi_c)$$

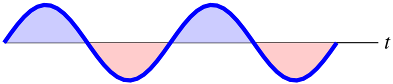
$$R_S(t+\tau,t) = \frac{A_c^2}{2}\left[1 + R_{M_a}(\tau)\right]\left[\cos(\omega_c \tau) + \cos(\omega_c(2t+\tau) + 2\phi_c)\right]$$

- Process: Cyclostationary with period $T_0 = \frac{2\pi}{\omega_c} = \frac{1}{f_c}$
  - Calculation of the PSD: FT of the time-average of the autocorrelation function

$$S_S(j\omega) = \mathcal{FT}\{\tilde{R}_S(\tau)\}, \text{ with } \tilde{R}_S(\tau) = \frac{1}{T_0}\int_{T_0} R_S(t+\tau,t)\, dt$$

    - ★ Integral of a sinusoid over an integer number of periods

$$\int_{n\times T_0} \cos\left(\frac{2\pi}{T_0}t + \theta\right) dt = \int_{n\times T_0} \sin\left(\frac{2\pi}{T_0}t + \theta\right) dt = 0$$



Statistical analysis of conventional AM (IV)

- Time average of the autocorrelation function

$$\tilde{R}_S(\tau) = \frac{1}{T_0}\int_{T_0} R_S(t+\tau,t)\;dt$$
$$= \frac{A_c^2}{2}\;[1+R_{M_a}(\tau)]\;\cos(\omega_c\tau) = \frac{A_c^2}{2}\;\left[1+\frac{a^2}{C_M^2}R_M(\tau)\right]\;\cos(\omega_c\tau)$$
$$= \frac{A_c^2}{2}\;\cos(\omega_c\tau) + \frac{A_c^2}{2}\;\frac{a^2}{C_M^2}R_M(\tau)\;\cos(\omega_c\tau)$$

If $M_a(t) = a\;M_n(t) = \frac{a}{C_M}\;M(t)$, then $m_{M_a}(t) = a\;m_{M_n}(t) = \frac{a}{C_M}\;m_M(t)$

$R_{M_a}(\tau) = a^2\;R_{M_n}(\tau) = \frac{a^2}{C_M^2}\;R_M(\tau)$, and therefore $S_{M_a}(j\omega) = \frac{a^2}{C_M^2}\;S_M(j\omega)$ and $P_{M_a} = \frac{a^2}{C_M^2}\;P_M$

- Power spectral density

$$S_S(j\omega) = \mathcal{FT}\{\tilde{R}_S(\tau)\} = \frac{A_c^2}{2}\;\pi\;[\delta(\omega-\omega_c)+\delta(\omega+\omega_c)]$$
$$+\frac{A_c^2}{4}\;[S_{M_a}(j\omega-j\omega_c)+S_{M_a}(j\omega+j\omega_c)]$$

$$S_S(j\omega) = \frac{A_c^2}{2}\;\pi\;[\delta(\omega-\omega_c)+\delta(\omega+\omega_c)] + \frac{A_c^2}{4}\;\left[\frac{a^2}{C_M^2}\;S_M(j\omega-j\omega_c)+\frac{a^2}{C_M^2}\;S_M(j\omega+j\omega_c)\right]$$

Statistical analysis of conventional AM (V)

- Power spectral density consists of
  - Two deltas, in $-\omega_c$ and in $+\omega_c$
    - Amplitude $\frac{A_c^2}{2}\pi$
  - Replicas of $S_M(j\omega)$ shifted $-\omega_c$ and $+\omega_c$
    - Scale factor $\left(\frac{A_c}{2}\frac{a}{C_M}\right)^2$
- Power of the AM modulated signal

$$P_S = \tilde{R}_S(0) = \frac{A_c^2}{2}\left[1 + R_{M_a}(0)\right] = \frac{A_c^2}{2}\left[1 + P_{M_a}\right] = \frac{A_c^2}{2}\left[1 + \frac{a^2}{C_M^2}P_M\right] \text{ Watts}$$

  - Power of the carrier: $\frac{A_c^2}{2}$ Watts
  - Power of the DSB: $\left(\frac{A_c^2}{2}\frac{a^2}{C_M^2}\right) \times P_M$ Watts

NOTE: Power can also be calculated as $P_S = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_S(j\omega)\, d\omega$

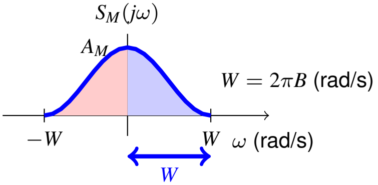
- Bandwidth of the conventional AM signal

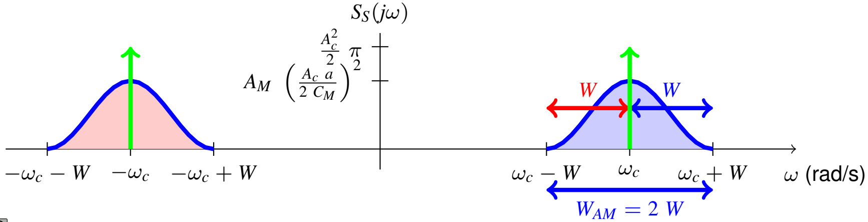
$$W_{AM} = 2\,W \text{ rad/s}, \qquad B_{AM} = 2\,B \text{ Hz}$$

PSD of the conventional AM signal - Representation

$$S_S(j\omega) = \frac{A_c^2}{2}\,\pi\,\left[\delta(\omega-\omega_c)+\delta(\omega+\omega_c)\right] + \frac{A_c^2}{4}\left[\frac{a^2}{C_M^2}\,S_M(j\omega-j\omega_c) + \frac{a^2}{C_M^2}\,S_M(j\omega+j\omega_c)\right]$$

An example: process $M(t)$ with the following $S_M(j\omega)$

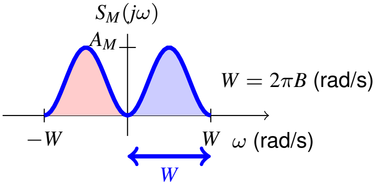


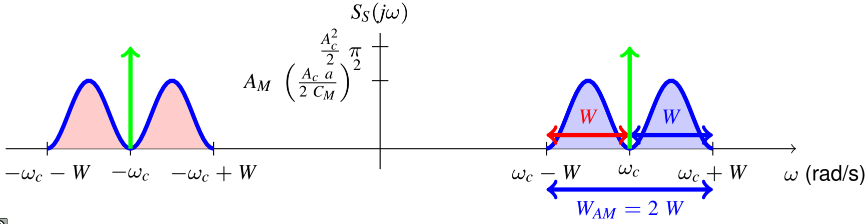


PSD of the conventional AM signal - Representation (II)

$$S_S(j\omega) = \frac{A_c^2}{2}\,\pi\,\left[\delta(\omega-\omega_c)+\delta(\omega+\omega_c)\right] + \frac{A_c^2}{4}\left[\frac{a^2}{C_M^2}\,S_M(j\omega-j\omega_c)+\frac{a^2}{C_M^2}\,S_M(j\omega+j\omega_c)\right]$$

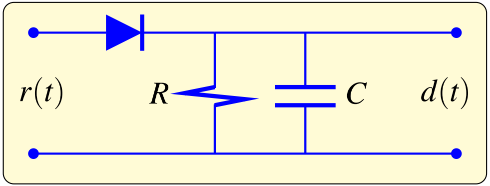
Another example: process $M(t)$ with the following $S_M(j\omega)$





Summary of characteristics of conventional AM modulation

- Disadvantages of the conventional AM modulation:
  - Low power efficiency
    - ★ Power is “wasted” in the transmission of the carrier (which does not contain information)
  - Low spectral efficiency
    - ★ The bandwidth of the modulated signal is twice that of the modulating signal
- Fundamental advantage of the conventional AM modulation
  - If $a \leq 1$, there is no overmodulation and the signal envelope is proportional to $1 + m_a(t) \geq 0$
    - ★ Recovery of $m(t)$: mean substraction and scaling
  - Simple receiver: envelope detector



    - ★ A synchronous demodulator is not needed (although it can be used as well and is actually the optimal receiver)

Double Sideband (DSB) modulation (no carrier)

- The carrier of the conventional AM modulation is suppressed
  - Eliminates the power efficiency drawback of conventional AM

$$s(t) = m(t) \times c(t) = m(t) \times A_c \cos(\omega_c t + \phi_c)$$

$$\text{Conv-AM: } s(t) = c(t) + \frac{a}{C_M} \times m(t) \times c(t)$$

- Frequency response (deterministic signal $m(t)$ with $M(j\omega) = \mathcal{FT}\{m(t)\}$)

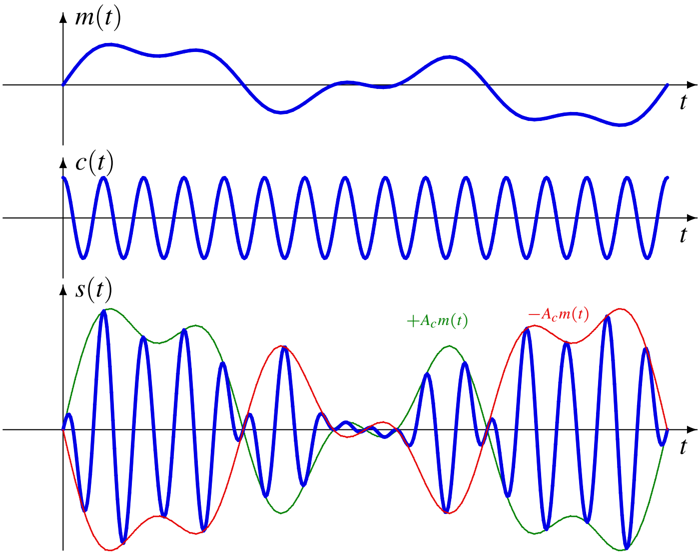
$$\begin{aligned} S(j\omega) &= \frac{1}{2\pi} \mathcal{FT}\{m(t)\} * \mathcal{FT}\{A_c \ \cos(\omega_c t + \phi_c)\} \\ &= \frac{A_c}{2} \left[M(j\omega - j\omega_c)\ e^{j\phi_c} + M(j\omega + j\omega_c)\ e^{-j\phi_c}\right] \end{aligned}$$

  - The deltas of conventional AM modulation disappear
  - Replicas of $M(j\omega)$ shifted $\pm\omega_c$ rad/s
    - The scaling of the replicas is simpler (since there is no normalization)
    - Name: two sidebands, lower ($|w| < w_c$) and upper ($|w| > w_c$)
- Bandwidth

$$W_{DSB} = 2\ W \text{ rad/s},\ B_{DSB} = 2\ B \text{ Hz}$$

It is still twice that of the modulating signal being transmitted

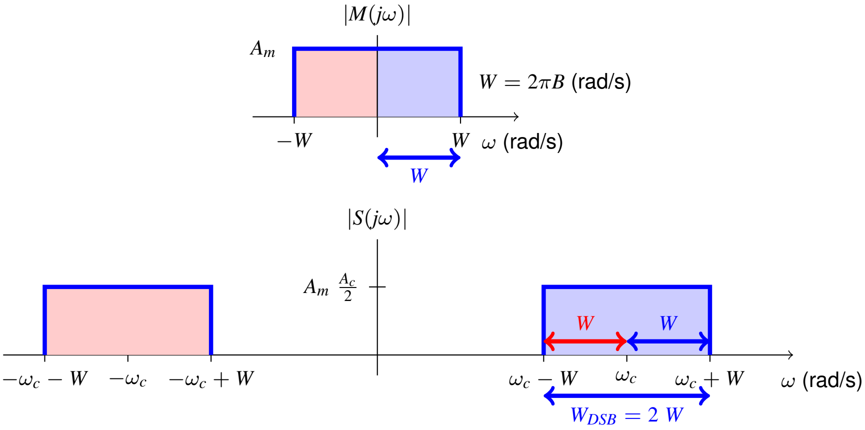
Waveform of a DSB modulation



Spectrum of the DSB signal - Representation

$$S(j\omega) = \frac{A_c}{2}\left[M(j\omega - j\omega_c)\ e^{j\phi_c} + M(j\omega + j\omega_c)\ e^{-j\phi_c}\right]$$

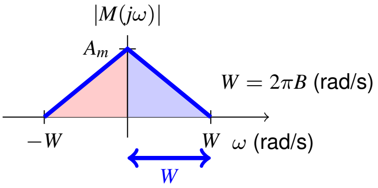
An example: $m(t)$ with a given shape for $M(j\omega) = \mathcal{FT}\{m(t)\}$

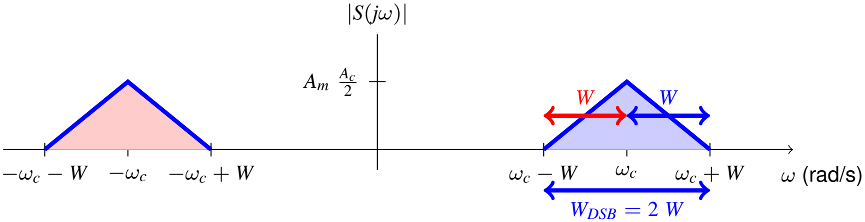


Spectrum of the DSB signal - Representation (II)

$$S(j\omega) = \frac{A_c}{2}\left[M(j\omega - j\omega_c)\ e^{j\phi_c} + M(j\omega + j\omega_c)\ e^{-j\phi_c}\right]$$

Another example: $m(t)$ with another shape for $M(j\omega) = \mathcal{FT}\{m(t)\}$





Statistical analysis of the DSB modulation

- Modulating signal model: random process

$M(t)$: stationary, with $m_M = 0,\ R_M(\tau),\ S_M(j\omega)$ and power P_M Watts

- Model of the modulated signal: random process

$$S(t) = M(t) \times c(t) = A_c M(t) \cos(\omega_c t + \phi_c)$$

- Mean of the DSB modulated signal

$$m_S(t) = E[S(t)] = A_c\ E[M(t)]\ \cos(\omega_c t + \phi_c) = 0$$

- Autocorrelation function of the DSB modulated signal

$$\begin{aligned} R_S(t+\tau, t) =& E[S(t+\tau)S(t)] \\ =& A_c^2\ E[M(t+\tau)\ M(t)]\ \cos(\omega_c(t+\tau)+\phi_c)\ \cos(\omega_c t + \phi_c) \\ =& \frac{A_c^2}{2}\ R_M(\tau)\ [\cos(\omega_c \tau) + \cos(\omega_c(2t+\tau)+2\phi_c)] \end{aligned}$$

$$\cos(a)\cos(b) = \frac{1}{2}\cos(a-b) + \frac{1}{2}\cos(a+b)$$

- Process: Cyclostationary with period $T_0 = \frac{2\pi}{2\omega_c} = \frac{1}{2f_c}$

Statistical analysis of the DSB modulation (II)

- Time average of the autocorrelation function

$$\tilde{R}_S(\tau) = \frac{1}{T}\int_{T_0} R_S(t+\tau,t)\;dt = \frac{A_c^2}{2}\;R_M(\tau)\;\cos(\omega_c\tau)$$

- Power spectral density

$$S_S(j\omega) = \mathcal{FT}\{\tilde{R}_S(\tau)\} = \frac{A_c^2}{4}\;[S_M(j\omega - j\omega_c) + S_M(j\omega + j\omega_c)]$$

- DSB signal power

$$P_S = \tilde{R}_S(0) = \frac{A_c^2}{2}\;R_M(0) = \frac{A_c^2}{2}\;P_M$$

  - Power efficient
    - Carrierless : no power is "*wasted*" on terms that contain no information
- DSB modulation bandwidth

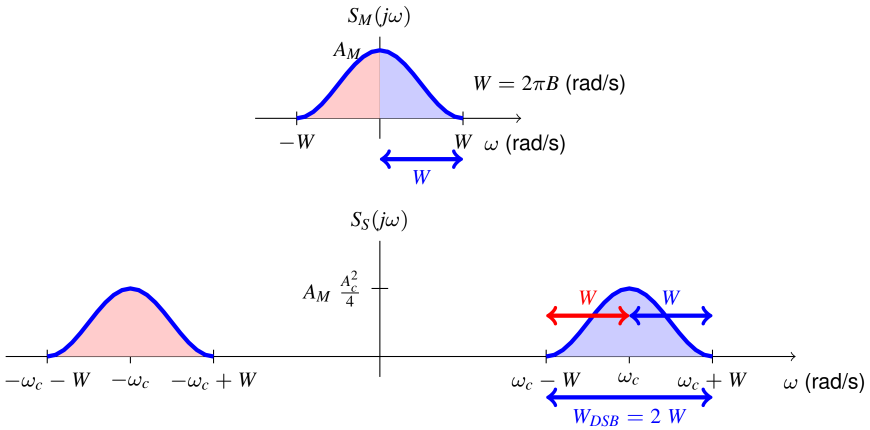
$$W_{DSB} = 2\;W\;\text{rad/s},\;\;B_{DSB} = 2\;B\;\text{Hz}$$

  - It is still twice that of the modulating signal

PSD of the DSB signal - Representation

$$S_S(j\omega) = \frac{A_c^2}{4}\ [S_M(j\omega - j\omega_c) + S_M(j\omega + j\omega_c)]$$

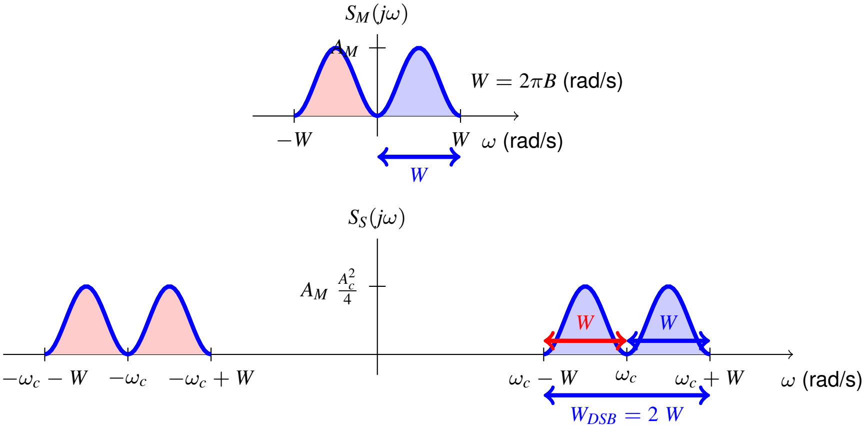
An example: process $M(t)$ with the following $S_M(j\omega)$



PSD of the DSB signal - Representation (II)

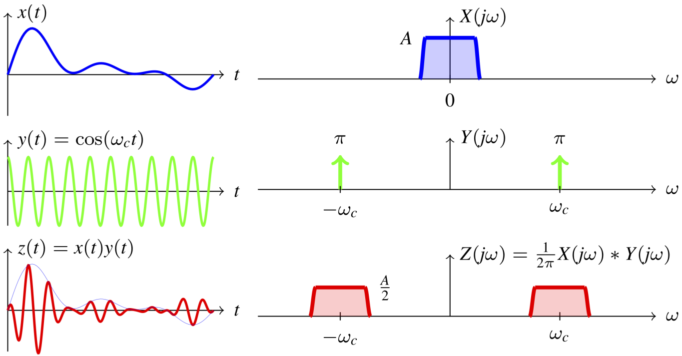
$$S_S(j\omega) = \frac{A_c^2}{4}\;[S_M(j\omega - j\omega_c) + S_M(j\omega + j\omega_c)]$$

Another example: process $M(t)$ with the following $S_M(j\omega)$



Revision : Product by a sinusoid - Effect in the frequency domain

- Sinusoid of frequency ω_c: two replicas of the spectrum of the modulated signal, shifted $\pm\omega_c$ rad/s



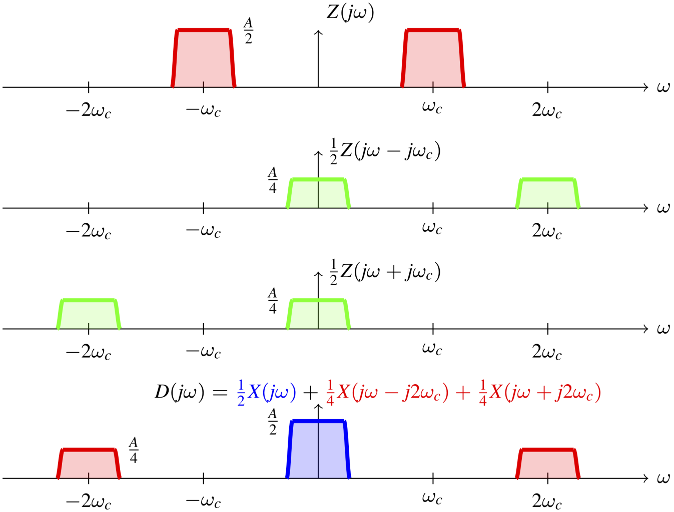
$$z(t) = x(t)\cos(\omega_c t) \stackrel{\mathcal{FT}}{\leftrightarrow} Z(j\omega) = \frac{1}{2}X(j\omega - j\omega_c) + \frac{1}{2}X(j\omega + j\omega_c)$$

- Power spectral density (for random signals)

$$S_Z(j\omega) = \frac{1}{4}S_X(j\omega - j\omega_c) + \frac{1}{4}S_X(j\omega + j\omega_c)$$

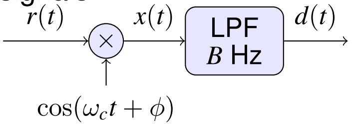
Product by a sinusoid - Effect on frequency (II)

$$d(t) = z(t)\cos(\omega_c t) \overset{\mathcal{FT}}{\leftrightarrow} D(j\omega) = \frac{1}{2}Z(j\omega - j\omega_c) + \frac{1}{2}Z(j\omega + j\omega_c)$$



DSB demodulation: Synchronous or coherent receiver

- Receiver for DSB modulated signals



LPF: low pass filter (with bandwidth B Hz)

  - Demodulator (product with the carrier $\cos(\omega_c t + \phi)$)
  - Low pass filter (bandwidth given by the signal, B Hz)
- Optimum performance with a synchronous or coherent receiver
  - Same phase in the receiver carrier as in the transmitter carrier

$$\phi = \phi_c$$

- Effect of a non-synchronous receiver ($\phi \neq \phi_c$)
  - Attenuation of the term related to the signal $m(t)$
  - Loss of signal to noise ratio (performance)
    - ★ The value of the phase ϕ does not vary the power due to the noise term

Effect of a non-coherent receiver

- Analysis of the signal term Assumption $r(t) = s(t) = A_c m(t) \cos(\omega_c t + \phi_c)$
- Unfiltered demodulated signal

$$\begin{aligned} x(t) =& r(t) \times \cos(\omega_c t + \phi) \\ =& A_c m(t) \cos(\omega_c t + \phi_c) \cos(\omega_c t + \phi) \\ =& \frac{A_c}{2}\, m(t)\, \left[\cos(\phi - \phi_c) + \cos(2\omega_c t + \phi_c + \phi)\right] \end{aligned}$$

- Filtered demodulated signal
  - Terms with spectrum in $\pm 2\omega_c$ are eliminated

$$d(t) = \frac{A_c}{2}\, m(t)\, \cos(\phi - \phi_c)$$

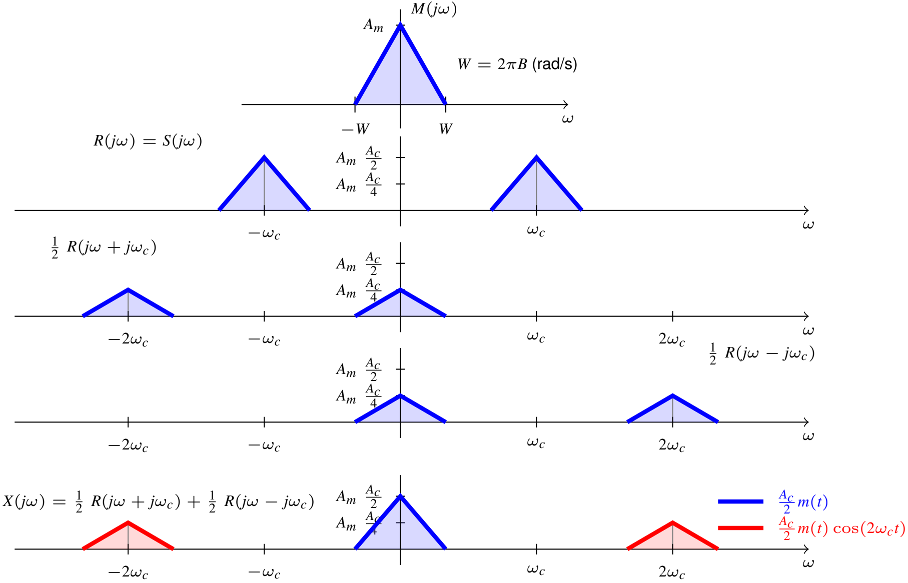
  - Ideal value, with coherent receiver ($\phi = \phi_c$)

$$d(t) = \frac{A_c}{2}\, m(t)$$

  - Effect of using a non-coherent receiver ($\phi \neq \phi_c$)
    - ★ Attenuation Term

$$\cos(\phi - \phi_c)$$

Synchronous Demodulation of DSB - Frequency Interpretation

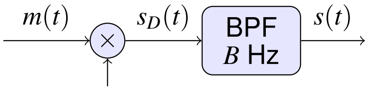


Coherent Receiver - Possible Options

- The receiver must identify the phase of the carrier with which the signal was modulated $\phi = \phi_c$
- Frequent options
  - Transmission of a pilot (carrier of reduced amplitude)
    - ⋆ Power inefficiency
  - Use of a phase-locked loop (PLL: *Phase Looked Loop*)
    - ⋆ Increases the cost of the receiver

Single SideBand (SSB) modulation

- Spectral efficiency: $B_{SSB} = B$ Hz
  - One sideband is removed
- Signal generation by direct filtering
  - A double sideband signal is generated (with double amplitude)
  - One of the two sidebands is removed by filtering
    - SSB Upper SideBand (USB): frequencies $|\omega| < \omega_c$ are removed
    - SSB Lower SideBand (LSB): frequencies $|\omega| > \omega_c$ are removed



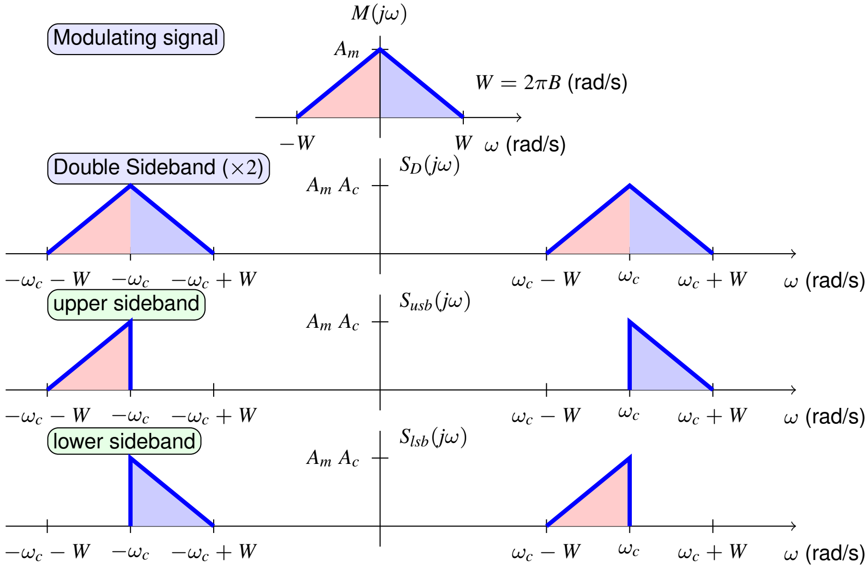
$$2 \times c(t) = 2A_c \cos(\omega_c t + \phi_c)$$

- Analytical expression of the resulting SSB signal

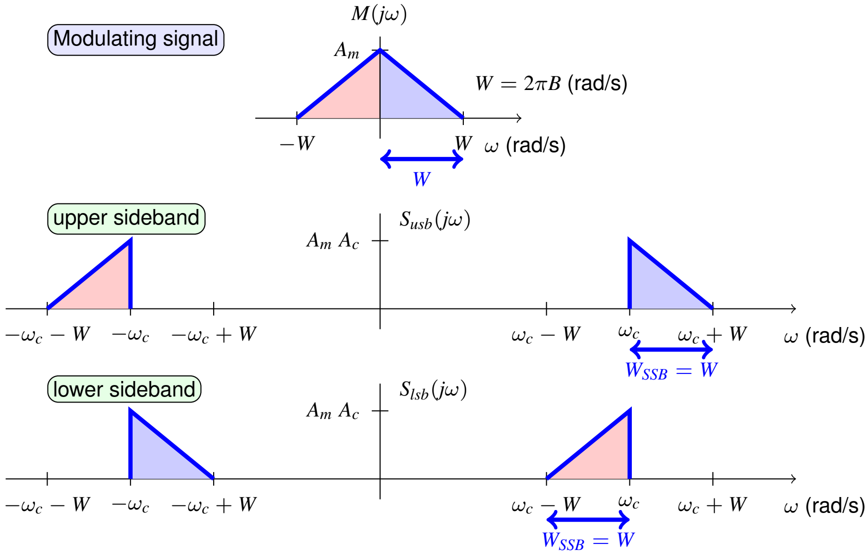
$$s(t) = A_c\, m(t)\, \cos(\omega_c t + \phi_c) \mp A_c\, \hat{m}(t)\, \sin(\omega_c t + \phi_c)$$

$\hat{m}(t)$: Hilbert transform of the modulating signal $m(t)$
  - Upper sideband (USB): $-$ sign
  - Lower sideband (LSB): $+$ sign

Spectrum of the SSB signal



SSB AM signal spectrum - Bandwidth



Hilbert transform

- Signal generated by filtering with a Hilbert transformer

$$\hat{m}(t) = m(t) * h_{Hilbert}(t)$$

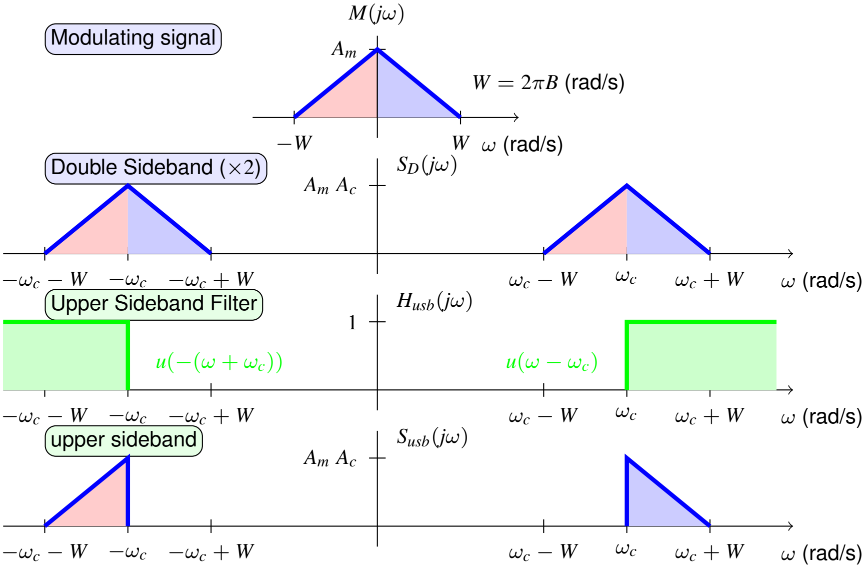
- Hilbert transformer:
  - Impulse response

$$h_{Hilbert}(t) = \frac{1}{\pi t}$$

  - Frequency response

$$H_{Hilbert}(j\omega) = \begin{cases} -j, & \omega > 0 \\ +j, & \omega < 0 \\ 0, & \omega = 0 \end{cases}$$

Upper Sideband SSB Generation



Analytic expression of $s(t)$ - Upper sideband

- Frequency response of the upper sideband filter:

$$H_{usb}(j\omega) = u(\omega - \omega_c) + u(-(\omega + \omega_c)) \text{ with } u(x) \text{ step function}$$

- Spectrum of the DBL signal with double amplitude, $s_D(t)$

$$S_D(j\omega) = A_c\left[M(j\omega - j\omega_c) + M(j\omega + j\omega_c)\right]$$

- SSB signal spectrum with upper sideband

$$\begin{aligned} S(j\omega) =& S_D(j\omega)\; H_{usb}(j\omega) \\ =& A_c M(j\omega)\; u(\omega)|_{\omega=\omega-\omega_c} + A_c M(j\omega)\; u(-\omega)|_{\omega=\omega+\omega_c} \end{aligned}$$

The following properties of the Fourier transform and Euler's formulas for sinusoids are used

$$\mathcal{FT}\left\{\frac{1}{2}\delta(t) + \frac{j}{2\pi t}\right\} = u(\omega),\;\; \mathcal{FT}\left\{\frac{1}{2}\delta(t) - \frac{j}{2\pi t}\right\} = u(-\omega),\;\; \mathcal{FT}\{x(t)\; e^{j\omega_c t}\} = X(j\omega - j\omega_c)$$

$$\cos(\omega_c t) = \frac{e^{+j\omega_c t} + e^{-j\omega_c t}}{2},\;\; \sin(\omega_c t) = \frac{e^{+j\omega_c t} - e^{-j\omega_c t}}{2j} = j\frac{e^{-j\omega_c t} - e^{+j\omega_c t}}{2}$$

- Upper sideband SSB signal

$$\begin{aligned} s_{usb}(t) =& A_c\; m(t) * \left[\frac{1}{2}\delta(t) + \frac{j}{2\pi t}\right]\; e^{j\omega_c t} + A_c\; m(t) * \left[\frac{1}{2}\delta(t) - \frac{j}{2\pi t}\right]\; e^{-j\omega_c t} \\ =& \frac{A_c}{2}\;[m(t) + j\hat{m}(t)]\; e^{j\omega_c t} + \frac{A_c}{2}\;[m(t) - j\hat{m}(t)]\; e^{-j\omega_c t} \\ =& A_c\; m(t)\;\; \cos(\omega_c t) - A_c\; \hat{m}(t)\;\; \sin(\omega_c t) \end{aligned}$$

Analytic expression of $s(t)$ - Lower sideband

- Upper sideband SSB signal

$$s_{usb}(t) = A_c\, m(t)\; \cos(\omega_c t) - A_c\, \hat{m}(t)\; \sin(\omega_c t)$$

- Relations of the two SSB signals and signal $s_D(t)$

$$s_D(t) = 2\, A_c\, m(t)\; \cos(\omega_c t) = s_{usb}(t) + s_{lsb}(t)$$

- Lower sideband SSB signal

$$s_{lsb}(t) = s_D(t) - s_{usb}(t)$$

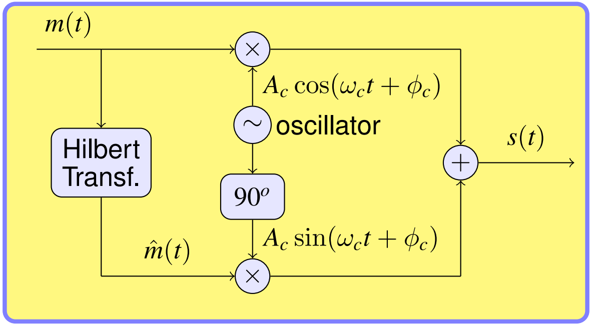
$$s_{lsb}(t) = A_c\, m(t)\; \cos(\omega_c t) + A_c\, \hat{m}(t)\; \sin(\omega_c t)$$

Alternative SSB Generation - Hartley Modulator

- Implementation based on the analytic expression from the Hilbert transform

$$s(t) = A_c\, m(t)\, \cos(\omega_c t + \phi_c) \mp A_c\, \hat{m}(t)\, \sin(\omega_c t + \phi_c)$$

REMARK: for ease of notation, we had previously considered $\phi_c = 0$, now an arbitrary value is used



Bandwidth and power of the SSB signal

- Power spectral density
  - upper sideband

$$S_{S_{usb}}(j\omega)=\begin{cases}A_c^2\,[S_M(j\omega-j\omega_c)+S_M(j\omega+j\omega_c)], & |\omega|>\omega_c\\ 0, & |\omega|<\omega_c\end{cases}$$

  - lower sideband

$$S_{S_{lsb}}(j\omega)=\begin{cases}0, & |\omega|>\omega_c\\ A_c^2\,[S_M(j\omega-j\omega_c)+S_M(j\omega+j\omega_c)], & |\omega|<\omega_c\end{cases}$$

- Signal power

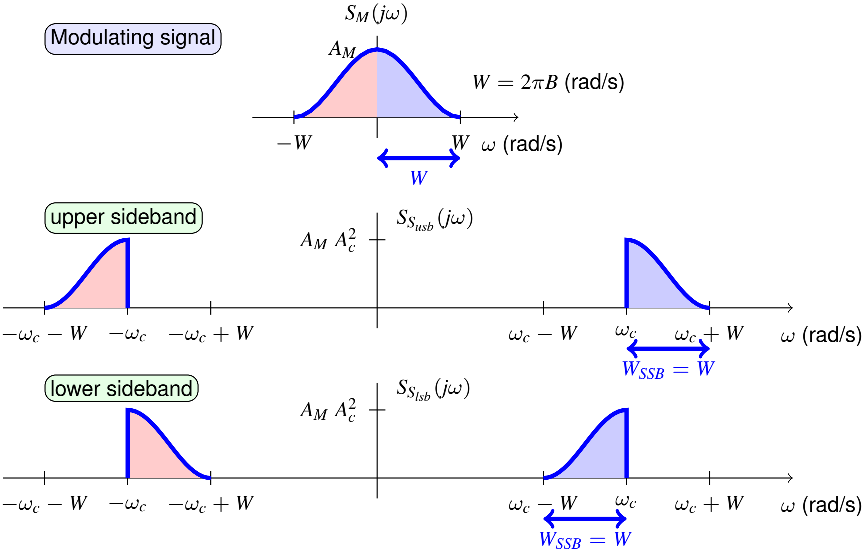
$$P_S=\frac{1}{2\pi}\int_{-\infty}^{\infty}S_S(j\omega)\,d\omega=A_c^2\,P_M \text{ Watts}$$

- Bandwidth

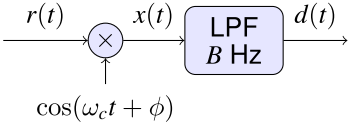
$$W_{SSB}=W\text{ rad/s},\;\; B_{SSB}=B\text{ Hz}$$

Same bandwidth as the transmitted modulating signal

PSD of the SSB signal - Representation



Coherent (synchronous) demodulation of SSB signals



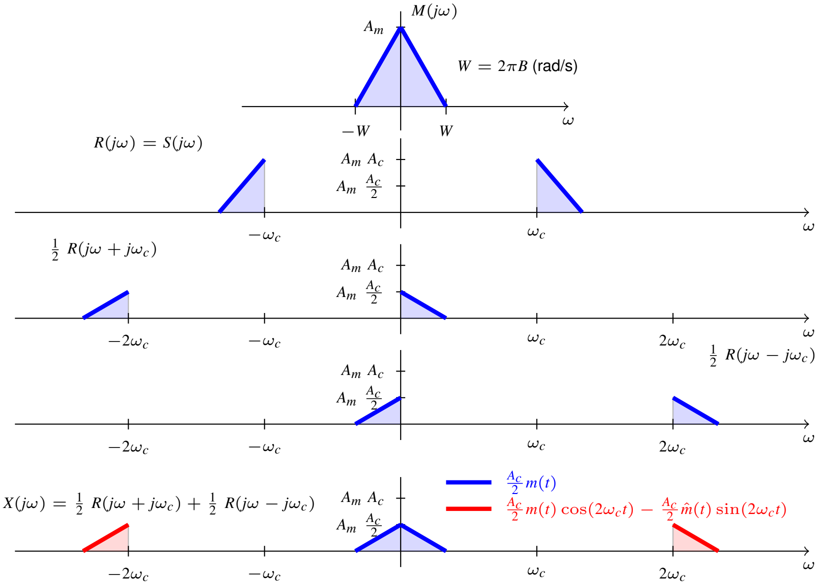
- Received signal $r(t) = s(t) = A_c\ m(t)\ \cos(\omega_c t + \phi_c) \mp A_c\ \hat{m}(t)\ \sin(\omega_c t + \phi_c)$
- Unfiltered demodulated signal $x(t)$

$$
\begin{aligned}
x(t) =& r(t) \times \cos(\omega_c t + \phi) \\
=& \left[A_c\ m(t)\ \cos(\omega_c t + \phi_c) \mp A_c\ \hat{m}(t)\ \sin(\omega_c t + \phi_c)\right] \times \cos(\omega_c t + \phi) \\
=& \frac{1}{2}\ A_c\ m(t)\ \cos(\phi - \phi_c) \pm \frac{1}{2}\ A_c\ \hat{m}(t)\ \sin(\phi - \phi_c) \\
& + \frac{1}{2}\ A_c\ m(t)\ \cos(2\omega_c t + \phi + \phi_c) \mp \frac{1}{2}\ A_c\ \hat{m}(t)\ \sin(2\omega_c t + \phi + \phi_c)
\end{aligned}
$$

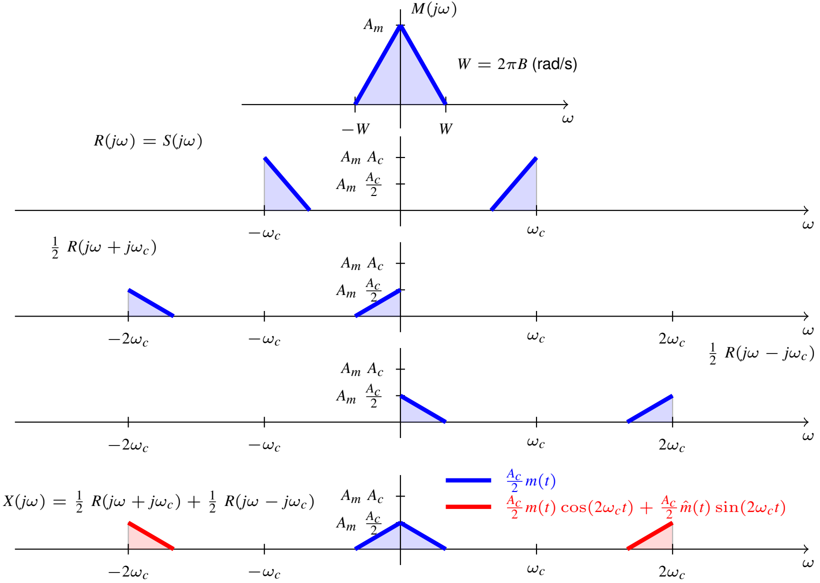
- Filtered demodulated signal

$$
d(t) = \frac{1}{2}\ A_c\ m(t)\ \cos(\phi - \phi_c) \pm \frac{1}{2}\ A_c\ \hat{m}(t)\ \sin(\phi - \phi_c)
$$

Synchronous Demodulation of SSB (USB) - Frequency Interpretation



Synchronous Demodulation of SSB (LSB) - Frequency Interpretation



Demodulation of SSB signals (II)

- Filtered demodulated signal (with arbitrary non-zero ϕ_c phase)

$$d(t) = \frac{1}{2} A_c \, m(t) \; \cos(\phi - \phi_c) \pm \frac{1}{2} A_c \, \hat{m}(t) \; \sin(\phi - \phi_c)$$

- Negative effects present with non-coherent demodulators
  - Attenuation of the received signal term due to $m(t)$

$$\frac{A_c}{2} \, m(t) \; \cos(\phi - \phi_c), \text{ attenuation term } \cos(\phi - \phi_c)$$

    - ★ Same as for double sideband modulation
  - Additional distortion term

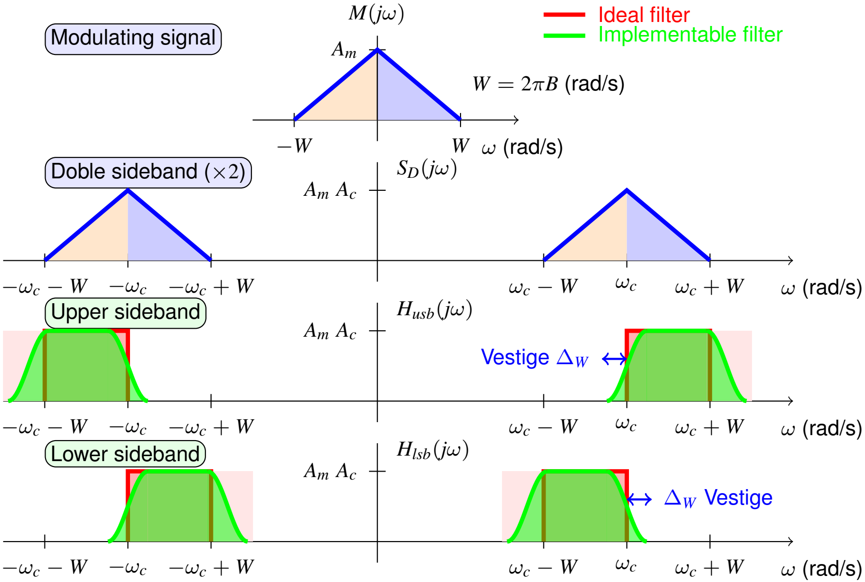
$$\pm \frac{A_c}{2} \, \hat{m}(t) \; \sin(\phi - \phi_c), \text{ gain term } \sin(\phi - \phi_c)$$

    - ★ Situation worse than for double sideband modulation
- Need for a synchronous or coherent demodulator

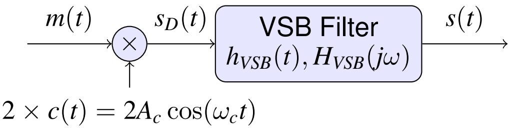
Characteristics of single sideband modulation

- SSB modulation overcomes the two main drawbacks of conventional AM modulation
  - Spectral efficiency
    - Same bandwidth as the information signal that is transmitted
  - Power efficiency
    - All the power of the signal is associated with the component that contains the information (no energy is used to transmit a carrier)
- Disadvantage of SSB modulation
  - Implementation using direct filtering: requires ideal filters to eliminate one of the sidebands
    - The implementation with real filters can generate distortion in the transmitted signal
  - Implementation with Hartley modulator: requires a Hilbert transformer
    - Ideal response of the transformer (not achievable without error)

SSB to VSB : Form ideal filters to implmentable filters



Vestigial SideBand (VSB) modulation



- Same modulation scheme as SSB
  - The VSB "*ideal*" filter is replaced by an implementable filter
    - ★ Vestigial sideband filter (which must meet certain conditions)
- VSB modulated signal
  - A double amplitude double sideband signal $s_D(t)$ is filtered with a VSB filter

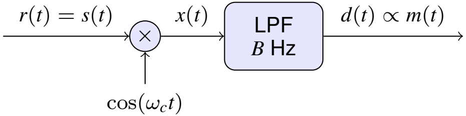
$$s(t) = \Bigg[\underbrace{m(t) \times 2A_c \cos(\omega_c t)}_{s_D(t)}\Bigg] * h_{VSB}(t)$$

- VSB signal in the frequency domain

$$S(j\omega) = A_c \;\left[M(j\omega - j\omega_c) + M(j\omega + \omega_c)\right]\; H_{VSB}(j\omega)$$

Characteristics of the VSB filter

- The received signal and its dependence on the filter will be analyzed.
  - Obtaining the conditions that must be met



- Received signal (same as transmitted) in the frequency domain

$$R(j\omega) = S(j\omega) = A_c\ [M(j\omega - j\omega_c) + M(j\omega + \omega_c)]\ H_{VSB}(j\omega)$$

- Demodulated (unfiltered) signal in the frequency domain

$$x(t) = r(t)\ \cos(\omega_c t) \rightarrow X(j\omega) = \frac{1}{2}\ [R(j\omega - j\omega_c)) + R(j\omega + j\omega_c))]$$

$$X(j\omega) = \frac{A_c}{2}\ [M(j\omega - j2\omega_c) + M(j\omega)]\ H_{VSB}(j\omega - j\omega_c) + \frac{A_c}{2}\ [M(j\omega) + M(j\omega + j2\omega_c)]\ H_{VSB}(j\omega + j\omega_c)$$

- Filtered demodulated signal in the frequency domain

$$D(j\omega) = \frac{A_c}{2}\ M(j\omega)\ [H_{VSB}(j\omega - j\omega_c) + H_{VSB}(j\omega + j\omega_c)]$$

Characteristics of the VSB filter (II)

- Filtered demodulated signal in the frequency domain

$$D(j\omega) = \frac{A_c}{2}\, M(j\omega)\, [H_{VSB}(j\omega - j\omega_c) + H_{VSB}(j\omega + j\omega_c)]$$

  - Interpretation: the modulating signal is filtered with the equivalent filter

$$H_{EQ}(j\omega) = H_{VSB}(j\omega - j\omega_c) + H_{VSB}(j\omega + j\omega_c)$$

- The filter must have a response that in $|\omega| \leq 2\pi B$ rad/s meets
  - Constant module
  - Linear phase
- Therefore, the conditions that the VSB filter must meet are

$$|H_{VSB}(j\omega - j\omega_c) + H_{VSB}(j\omega + j\omega)| = \text{C, in } |\omega| \leq 2\pi B \text{ rad/s}$$

Odd symmetry around ω_c in $\omega_c - \Delta_W < \omega < \omega_c + \Delta_W$ rad/s

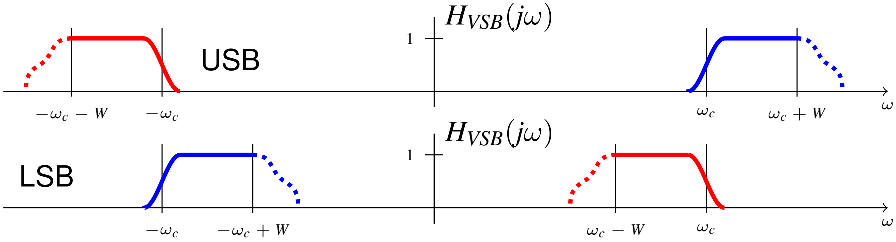
Δ_W: bandwidth excess (vestige) in rad/s

- Bandwidth of the modulated signals

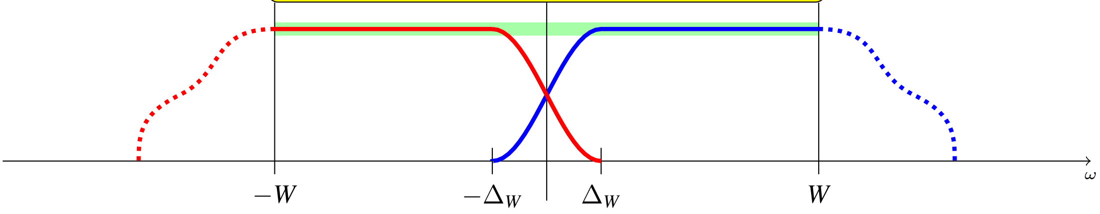
$$W_{VSB} = W + \Delta_W \text{ rad/s, or } B_{VSB} = B + \Delta_B \text{ Hz, with } \Delta_B = \frac{\Delta_W}{2\pi}$$

Δ_B: bandwidth excess of VSB in Hz (typically $\Delta_B << B$)

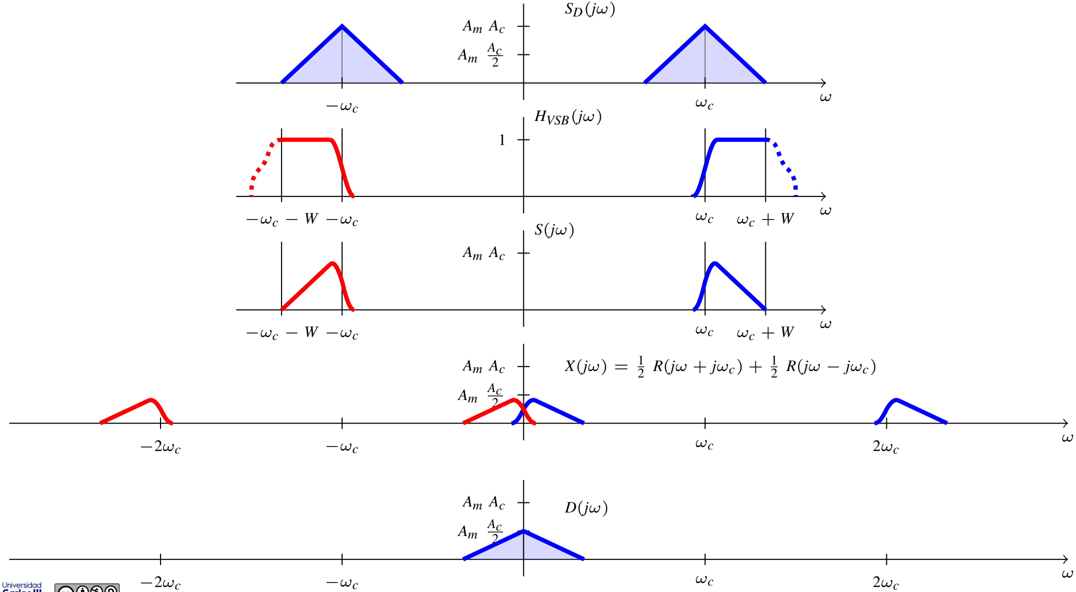
VSB Filter



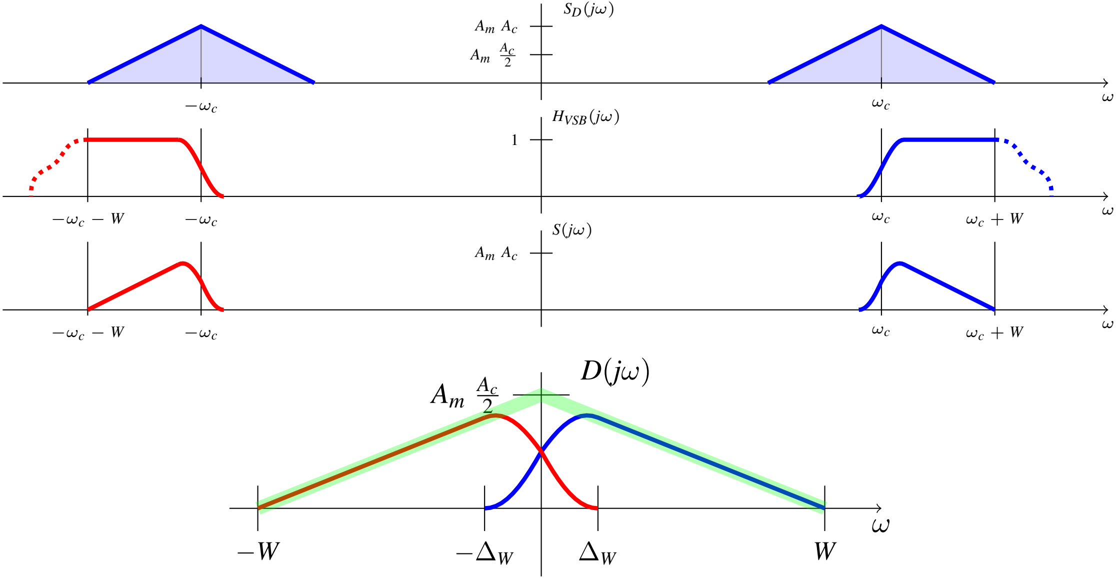
$H_{VSB}(j\omega - j\omega_c) + H_{VSB}(j\omega + j\omega_c)$ in $|\omega| \leq W$



Synchronous demodulation of the VSB (USB) - Frequency Interpretation



Synchronous demodulation of the VSB (USB) - Frequency Interpretation



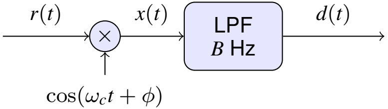
Amplitude modulations - Summary

Modulation	BW (Hz)	P_S	$P_S(m(t))$	$d(t)$	$P_d(m(t))$
Conventional AM	$2B$	$\frac{A_c^2}{2}\left[1+P_{M_a}\right]$	$\frac{A_c^2}{2}P_{M_a}$	$\frac{A_c}{2}\left[1+m_a(t)\right]$	$\frac{A_c^2}{4}P_{M_a}$
DSB	$2B$	$\frac{A_c^2}{2}P_M$	$\frac{A_c^2}{2}P_M$	$\frac{A_c}{2}m(t)$	$\frac{A_c^2}{4}P_M$
SSB	B	$A_c^2P_M$	$A_c^2P_M$	$\frac{A_c}{2}m(t)$	$\frac{A_c^2}{4}P_M$
VSB	$B+\Delta_B$	$A_c^2P_M$	$A_c^2P_M$	$\frac{A_c}{2}m(t)$	$\frac{A_c^2}{4}P_M$

BW (Hz) : bandwidth of the modulated signal in Hz
P_S : power of the modulated signal
$P_S(m(t))$: power of the modulated signal relative to $m(t)$
$d(t)$: signal recovered with a synchronous or coherent receiver
$P_d(m(t))$: power of the demodulated signal relative to $m(t)$

- Power efficiency
  - All signal power is related to $m(t)$
    - DSB, SSB y VSB
- Spectral efficiency
  - Minimum transmission bandwidth (same bandwidth as the modulating signal, B Hz)
    - SSB and VSB (in this case with a vestigial increment Δ_B)

Synchronous demodulation of conventional AM modulation



- Received signal: Conv. AM $r(t) = s(t) = A_c\,[1+m_a(t)]\,\cos(\omega_c t+\phi_c)$
- Unfiltered demodulated signal $x(t)$

$$\begin{aligned} x(t) =& r(t) \times \cos(\omega_c t+\phi) \\ =& A_c\,[1+m_a(t)]\,\cos(\omega_c t+\phi_c) \times \cos(\omega_c t+\phi) \\ =& \frac{A_c}{2}\,[1+m_a(t)]\,\cos(\phi_c-\phi) + \frac{A_c}{2}\,[1+m_a(t)]\,\cos(2\omega_c t+\phi_c+\phi) \end{aligned}$$

- Filtered demodulated signal

$$d(t) = \frac{A_c}{2}\,[1+m_a(t)]\,\cos(\phi_c-\phi)$$

- Need for a synchronous or coherent demodulator: $\phi = \phi_c$

$$d(t) = \frac{A_c}{2}\,[1+m_a(t)]$$

Angle Modulations

PM and FM

Angle modulations

- The information is in the argument of a sinusoidal carrier

$$c(t) = A_c \cos(2\pi f_c t + \phi_c)$$

  It is printed in the phase or frequency of the carrier
  - Phase Modulation (PM)

$$\phi_c \rightarrow \phi_c(t) = f(m(t))$$

  - Frequency modulation (FM)

$$f_i(t) = f_c \rightarrow f_i(t) = f(m(t))$$

  $f_i(t)$: instantaneous frequency of the carrier signal
- Common representation of FM and PM signals

$$s(t) = A_c \ \cos(\theta(t))$$

The information is in $\theta(t) = f(m(t))$

Instantaneous frequency

- Common representation of PM and FM signals

$$s(t) = A_c \cos(\theta(t))$$

  - The information is in the phase term

$$\theta(t) = \omega_c t + \phi(t) \text{ rad/s}$$

- Definition of instantaneous frequency of a sinusoid

$$\omega_i(t) = \frac{d}{dt}\theta(t) \text{ rad/s}, \quad f_i(t) = \frac{1}{2\pi}\frac{d}{dt}\theta(t) \text{ Hz}$$

- Modulated signal $s(t)$ and instantaneous frequency

$$s(t) = A_c \cos(\underbrace{\omega_c t}_{2\pi f_c t} + \phi(t)), \quad f_i(t) = f_c + \frac{1}{2\pi}\frac{d}{dt}\phi(t) \text{ Hz}$$

Phase (PM) and Frequency (FM) modulations

- Modulated signal $s(t)$ and instantaneous frequency

$$s(t) = A_c \cos(\underbrace{\omega_c t}_{2\pi f_c t} + \phi(t)), \quad f_i(t) = f_c + \frac{1}{2\pi}\frac{d}{dt}\phi(t) \text{ Hz}$$

- If $m(t)$ is the modulating signal (message)
  - Phase Modulation (PM)

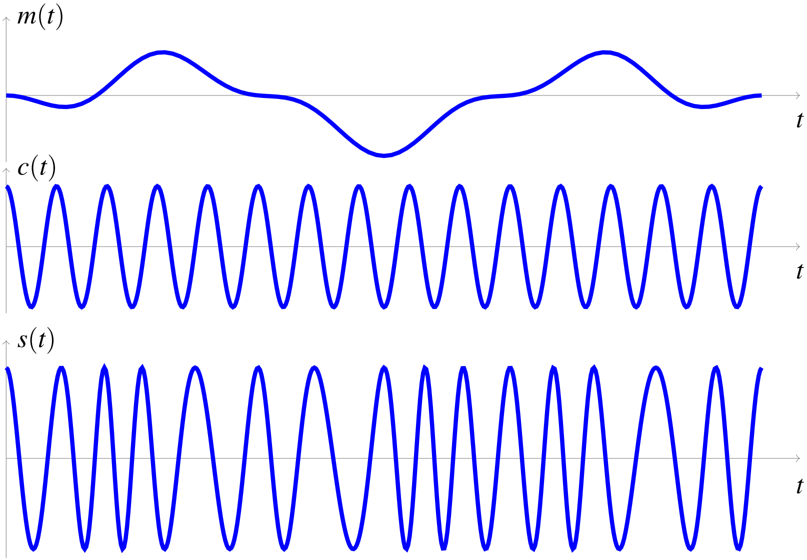
$$\phi(t) = k_p\, m(t) \text{ rad/s}$$

    - k_p: phase deviation constant
  - Frequency Modulation (FM)

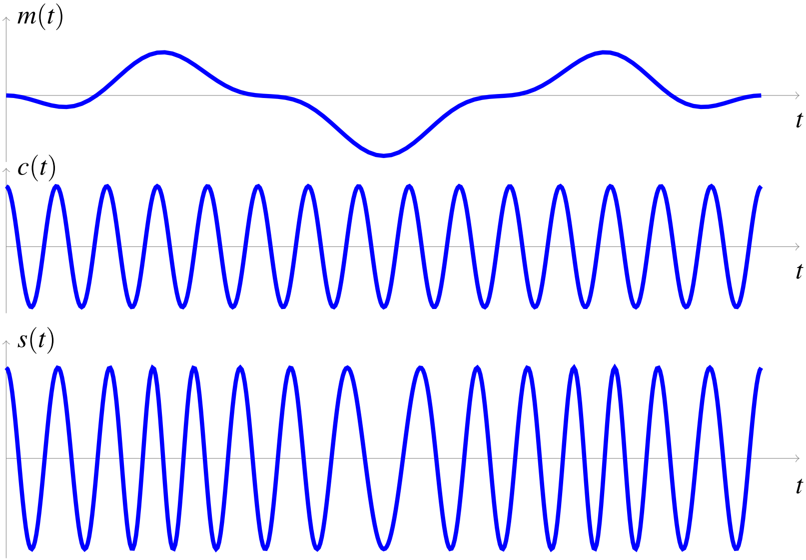
$$\Delta f_i(t) = f_i(t) - f_c = \frac{1}{2\pi}\frac{d}{dt}\phi(t) = k_f\, m(t) \text{ Hz}$$

    - k_f: frequency deviation constant

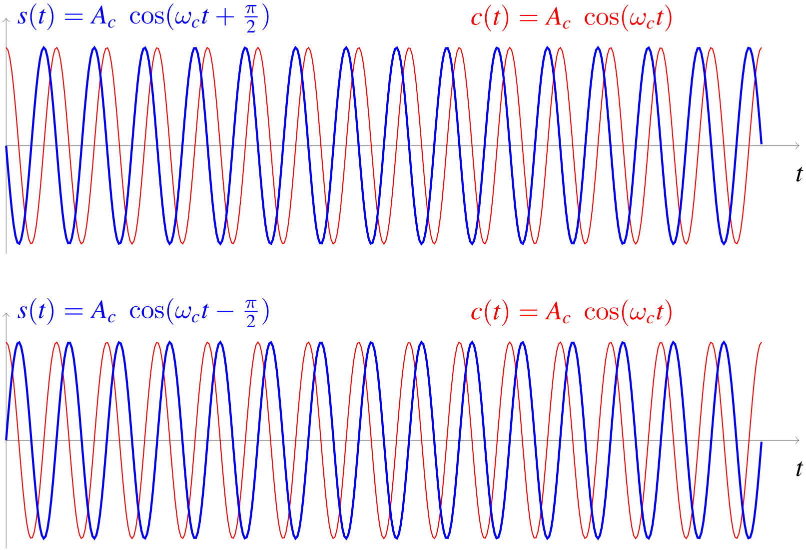
Waveform of a PM Modulation



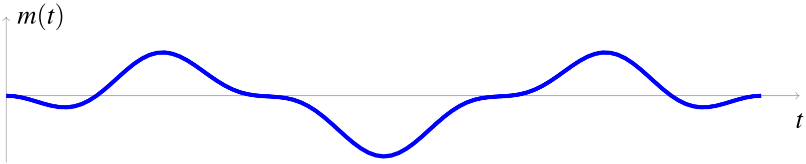
Waveform of an FM Modulation

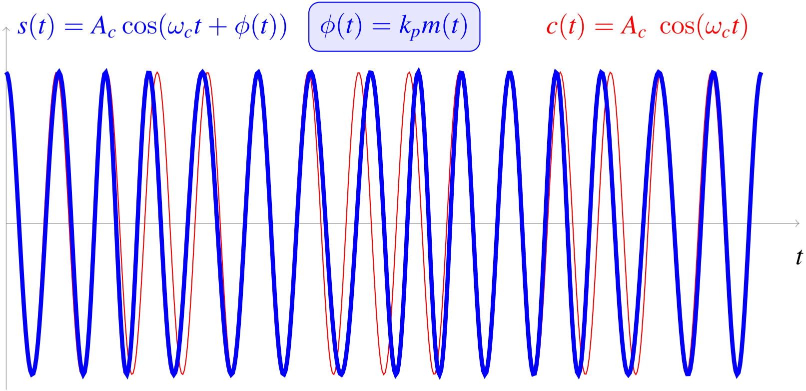


Time offset: shift of a sinusoid (ahead or behind)

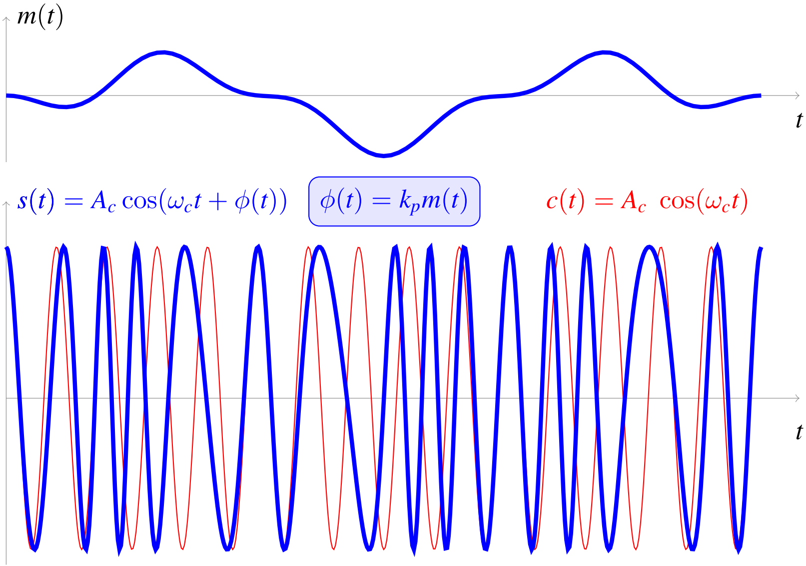


PM Modulation - Signal for $k_p = 2\pi \times \frac{1}{4}$

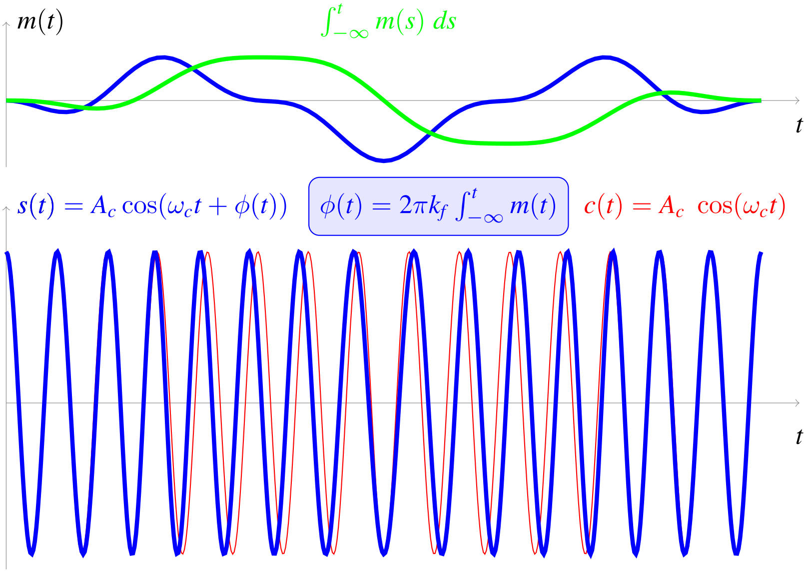




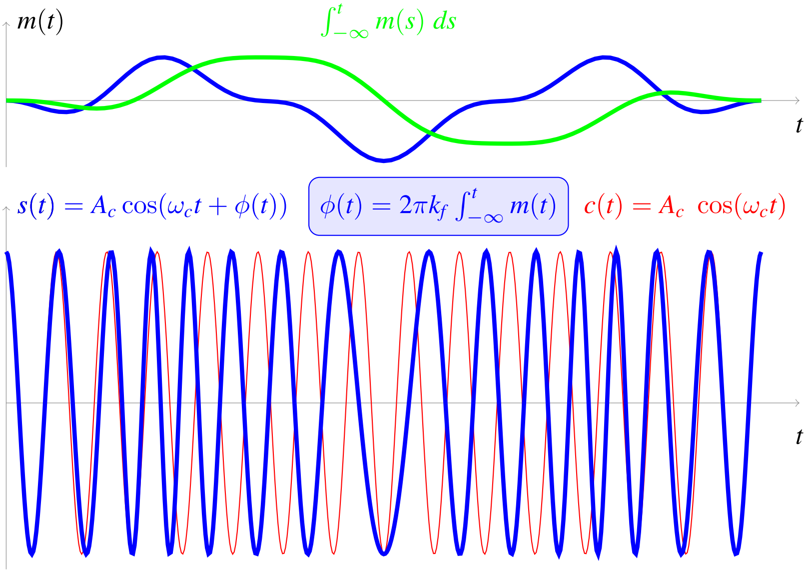
PM Modulation - Signal for $k_p = 2\pi \times \frac{3}{4}$



FM Modulation - Signal for $k_f = 2\pi \times \frac{1}{4}$



FM Modulation - Signal for $k_f = 2\pi \times \frac{3}{4}$



PM / FM relationship

- Phase modulation:

$$\phi(t) = k_p\ m(t)$$

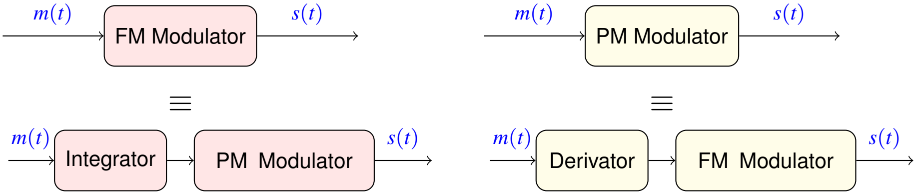
- Frequency modulation:

$$\Delta f_i(t) = \frac{1}{2\pi}\frac{d}{dt}\phi(t) = k_f\ m(t)$$

- Expressions of $\phi(t)$ and $\frac{d}{dt}\phi(t)$ in PM and FM

$$\phi(t) = \begin{cases} k_p\ m(t) & \text{PM} \\ 2\pi\ k_f\ \int_{-\infty}^{t} m(\tau)\ d\tau & \text{FM} \end{cases}$$

$$\frac{d}{dt}\phi(t) = \begin{cases} k_p\ \frac{d}{dt}m(t) & \text{PM} \\ 2\pi\ k_f\ m(t) & \text{FM} \end{cases}$$



Modulation index β

- It defines the main characteristics of an angle modulation
  - Bandwidth
  - Signal to noise ratio (SNR)
- Maximum deviations
  - PM: maximum phase deviation

$$\Delta\phi_{\max} = k_p \ \max(|m(t)|) = k_p \ C_M$$

  - FM: maximum frequency deviation

$$\Delta f_{\max} = k_f \ \max(|m(t)|) = k_f \ C_M$$

- Modulation indexes of PM and FM modulations

$$\beta_p = \Delta\phi_{\max} = k_p \ \max(|m(t)|) = k_p \ C_M$$

$$\beta_f = \frac{\Delta f_{\max}}{B} = \frac{k_f \ \max(|m(t)|)}{B} = \frac{k_f \ C_M}{B}$$

  - B: bandwidth in Hz of the modulating signal $m(t)$

Spectral analysis of angle modulations

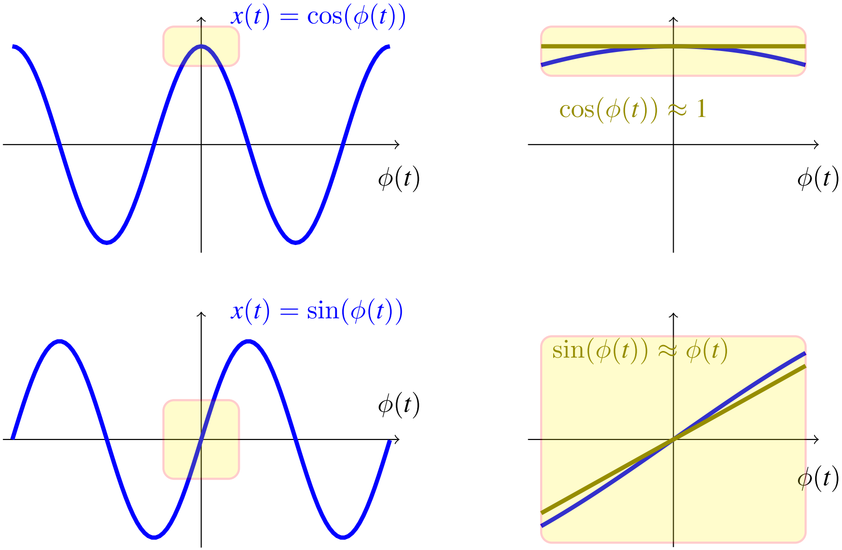
- A general analysis is difficult
  - Non-linear relationship between $m(t)$ and $s(t)$

$$s(t) = A_c \cos(\omega_c t + \phi(t))$$

$$\phi(t) = \begin{cases} k_p\, m(t) & \text{PM} \\ 2\pi\, k_f \int_{-\infty}^{t} m(\tau)\, d\tau & \text{FM} \end{cases}$$

- Calculation (exact or approximate) of the spectral representation
  - Narrowband angle modulations
  - Angle modulations with sinusoidal modulating signal
  - Angle modulations with periodic modulating signal
- Heuristic rule for calculating bandwidth
  - Angle modulations with generic modulating signal
    - ⋆ Carson's rule: approximation of the bandwidth

Approximations of cosine and sine (small argument)



Narrow band angle modulation

- Narrowband angle modulation: $\phi(t) << 1$
  - Constants k_p or k_f are small (β is small)
- Generic expression of the modulated signal

$$s(t) = A_c \, \cos(\omega_c t + \phi(t))$$

  - Trigonometric relation:

$$\cos(A \pm B) = \cos(A) \, \cos(B) \mp \sin(A) \, \sin(B)$$

  - Approximations considered (for small $\phi(t)$)

$$\cos(\phi(t)) \approx 1, \; \sin(\phi(t)) \approx \phi(t)$$

- Alternative expression for the modulated signal

$$\begin{aligned} s(t) &= A_c \, \cos(\omega_c t) \, \cos\phi(t) - A_c \, \sin(\omega_c t) \, \sin\phi(t) \\ &\approx A_c \, \cos(\omega_c t) - A_c \, \phi(t) \, \sin(\omega_c t) \end{aligned}$$

  - Expression similar to that of a conventional AM modulation

AM: $s(t) = A_c \, \cos(\omega_c t) + A_c \, m_a(t) \, \cos(\omega_c t)$

Remember that $\phi(t) = \begin{cases} k_p \, m(t) & \text{PM} \\ 2\pi \, k_f \, \int_{-\infty}^{t} m(\tau) \, d\tau & \text{FM} \end{cases}$

Narrow band angle modulation - Analysis

- Approximate expression for the modulated signal

$$s(t) \approx A_c \, \cos(\omega_c t) - A_c \, \phi(t) \, \sin(\omega_c t)$$

  - Expression similar to that of a conventional AM modulation

$$s(t) = A_c \, \cos(\omega_c t) + A_c \, m_a(t) \, \cos(\omega_c t)$$

- Spectrum of the modulated signal (considering the approximation)
  - Two deltas, located at $\pm\omega_c$ (carrier spectrum)
  - Replicas of the spectrum of $\phi(t)$ located at $\omega = \pm\omega_c$
  - Spectrum shape of $\phi(t)$
    - PM: proportional to the spectrum of $m(t)$

$$\phi(t) = k_p \, m(t) \leftrightarrow \Phi(j\omega) = k_p \, M(j\omega)$$

    - FM: proportional to the spectrum of the integral of $m(t)$

$$\phi(t) = 2\pi \, k_f \int_{-\infty}^{t} m(\tau) \, d\tau \leftrightarrow \Phi(j\omega) = 2\pi k_f \frac{M(j\omega)}{j\omega}$$

- Bandwidth (similar to conventional AM)

$$W_{NB} \approx 2 \, W \text{ rad/s}, \quad B_{NB} \approx 2 \, B \text{ Hz}$$

Narrow band angle modulation - Summary

- Signal in time domain

$$s(t) \approx A_c \cos(\omega_c t) - A_c\, \phi(t) \sin(\omega_c t)$$

$$\phi(t) = \begin{cases} k_p\ m(t) & \text{PM} \\ 2\pi\ k_f\ \int_{-\infty}^{t} m(\tau)\ d\tau & \text{FM} \end{cases}$$

- Fourier transform

$$S(j\omega) \approx A_c \pi \left[\delta(\omega - \omega_c) + \delta(\omega + \omega_c)\right] - \frac{A_c}{2j} \left[\Phi(j\omega - j\omega_c) - \Phi(j\omega + j\omega_c)\right]$$

$$\Phi(j\omega) = \begin{cases} k_p\ M(j\omega) & \text{PM} \\ 2\pi\ k_f\ \frac{M(j\omega)}{j\omega} & \text{FM} \end{cases}$$

- Power spectral density

$$S_S(j\omega) \approx \frac{A_c^2}{2} \pi \left[\delta(\omega - \omega_c) + \delta(\omega + \omega_c)\right] + \frac{A_c^2}{4} \left[S_\Phi(j\omega - j\omega_c) + S_\Phi(j\omega + j\omega_c)\right]$$

$$S_\Phi(j\omega) = \begin{cases} k_p^2\ S_M(j\omega) & \text{PM} \\ (2\pi\ k_f)^2\ \frac{S_M(j\omega)}{\omega^2} & \text{FM} \end{cases}$$

Modulation by a sinusoidal signal

- Sinusoidal modulating signal of amplitude a and frequency ω_m rad/s

$$m(t) = \begin{cases} a\ \sin(\omega_m t) & \text{PM} \\ a\ \cos(\omega_m t) & \text{FM} \end{cases}$$

- Modulation indices of a PM and FM modulation

$$\beta_p = \Delta\phi_{\max} = k_p\ \max(|m(t)|) = k_p\ C_M = k_p\ a$$

$$\beta_f = \frac{\Delta f_{\max}}{B} = \frac{k_f\ \max(|m(t)|)}{B} = \frac{k_f\ C_M}{B} = k_f\ a\ \frac{2\pi}{\omega_m}$$

- Expressions of the phase term $\phi(t)$
  - Expressions of $\phi(t)$ for PM

$$\phi(t) = k_p\ m(t) = k_p\ a\ \sin(\omega_m t) = \beta_p \sin(\omega_m t)$$

  - Expressions of $\phi(t)$ for FM

$$\phi(t) = 2\pi\ k_f \int_{-\infty}^{t} m(\tau)\ d\tau = 2\pi\ k_f\ a\ \frac{1}{\omega_m}\ \sin(\omega_m t) = \beta_f \sin(\omega_m t)$$

- Modulated signal: common expression for PM and FM

$$s(t) = A_c \cos(\omega_c t + \phi(t)) = A_c\ \cos(\omega_c t + \beta\ \sin(\omega_m t))$$

Modulation by a sinusoidal signal (II)

- Sinusoidal modulating signal $m(t) = \begin{cases} a \ \sin(\omega_m t) & \text{PM} \\ a \ \cos(\omega_m t) & \text{FM} \end{cases}$
- Modulated signal

$$s(t) = A_c \ \cos(\omega_c t + \beta \ \sin(\omega_m t)) = \mathcal{R}e \left(A_c \ e^{j\omega_c t} \ e^{j\beta \ \sin(\omega_m t)} \right)$$

The function $e^{j\beta \ \sin(\omega_m t)}$ is periodic with frequency $f_m = \frac{\omega_m}{2\pi}$ Hz

$$\text{Fourier series expansion: } e^{j\beta \ \sin(\omega_m t)} = \sum_{n=-\infty}^{\infty} J_n(\beta) \ e^{j(n \ \omega_m)t}$$

Index coefficient n of the series expansion: $J_n(\beta)$

$J_n(\beta)$: Bessel function of the first kind of order n and argument β

- Alternative expression of the modulated signal

$$s(t) = \mathcal{R}e \left(A_c \ e^{j\omega_c t} \sum_{n=-\infty}^{\infty} J_n(\beta) \ e^{j(n \ \omega_m)t} \right) = \mathcal{R}e \left(\sum_{n=-\infty}^{\infty} A_c \ J_n(\beta) \ \underbrace{e^{j\omega_c t} \ e^{j(n \ \omega_m)t}}_{e^{j(\omega_c + n \ \omega_m)t}} \right)$$

$$s(t) = \sum_{n=-\infty}^{\infty} A_c \ J_n(\beta) \ \cos\left((\omega_c + n \ \omega_m) \ t\right)$$

Modulation by a sinusoidal signal - Analysis

- The modulated signal contains sinusoids with the frequencies

$$\text{Frequencies (Hz)}: f_c + n\, f_m, \quad \text{for } n = 0, \pm 1, \pm 2, \cdots$$

$$\text{Ang. Freq. (rad/s)}: \omega_c + n\, \omega_m, \quad \text{for } n = 0, \pm 1, \pm 2, \cdots$$

$$\text{Amplitudes}: A_c\, J_n(\beta)$$

- Effective Bandwidth (contains 98% of the power):

$$B_e = 2\,(\beta + 1)\, f_m \text{ Hz}$$

$$B_e = 2\,(\beta+1)\,f_m = \begin{cases} 2(k_p a + 1) f_m & \text{PM} \\ 2\left(\frac{k_f a}{f_m} + 1\right) f_m & \text{FM} \end{cases} = \begin{cases} 2(k_p a + 1) f_m & \text{PM} \\ 2\,(k_f a + f_m) & \text{FM} \end{cases}$$

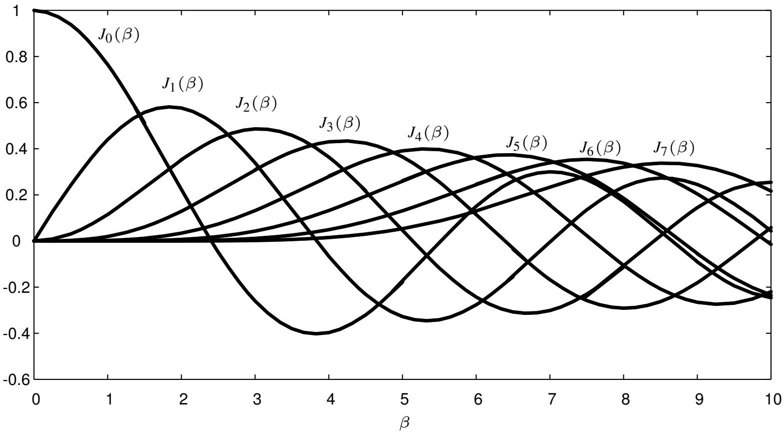
- Total number of harmonics in the effective bandwidth B_e

$$M_e = 2\lfloor \beta \rfloor + 3 = \begin{cases} 2\lfloor k_p a \rfloor + 3 & \text{PM} \\ 2\left\lfloor \frac{k_f a}{f_m} \right\rfloor + 3 & \text{FM} \end{cases}$$

Bessel functions $J_n(\beta)$

$$J_n(\beta) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{\beta}{2}\right)^{n+2k}}{k!(k+n)!}, \quad \text{For } \beta \downarrow J_n(\beta) \approx \frac{\beta^n}{2^n n!}, \quad J_{-n}(\beta) = \begin{cases} J_n(\beta), & n \text{ even} \\ -J_n(\beta), & n \text{ odd} \end{cases}$$

It is usually found in tables or figures



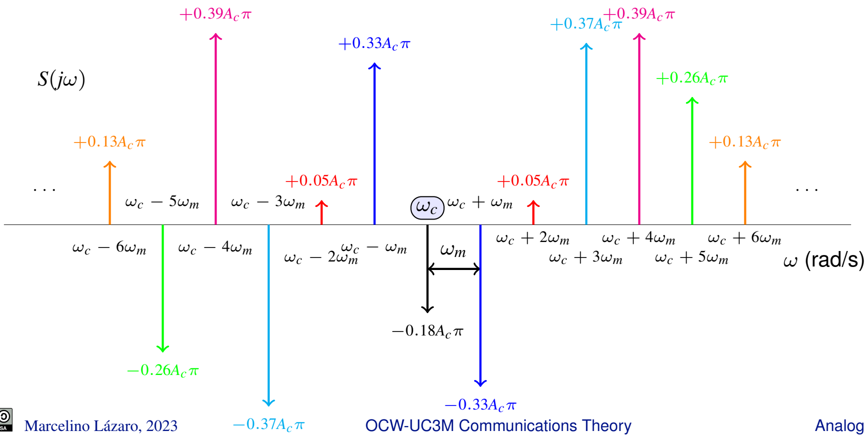
Bessel functions $J_n(\beta)$

n	$\beta = 0.1$	$\beta = 0.2$	$\beta = 0.5$	$\beta = 1$	$\beta = 2$	$\beta = 5$	$\beta = 8$	$\beta = 10$
0	0.9975	0.9900	0.9385	0.7652	0.2239	−0.1776	0.1717	−0.2459
1	0.0499	0.0995	0.2423	0.4401	0.5767	−0.3276	0.2346	0.0435
2	0.0012	0.0050	0.0306	0.1149	0.3528	0.0466	−0.1130	0.2546
3		0.0002	0.0026	0.0196	0.1289	0.3648	−0.2911	0.0584
4			0.0002	0.0025	0.0340	0.3912	−0.1054	−0.2196
5				0.0002	0.0070	0.2611	0.1858	−0.2341
6					0.0012	0.1310	0.3376	−0.0145
7					0.0002	0.0534	0.3206	0.2167
8						0.0184	0.2235	0.3179
9						0.0055	0.1263	0.2919
10						0.0015	0.0608	0.2075
11						0.0004	0.0256	0.1231
12						0.0001	0.0096	0.0634
13							0.0033	0.0290
14							0.0010	0.0120
15							0.0003	0.0045
16							0.0001	0.0016

Spectrum Shape - Example - Modulation with $\beta = 5$

- Sum of sinusoids of amplitude $A_c\ J_n(\beta)$ and frequencies $\omega_c + n\ \omega_m$ rad/s
  - Spectrum: sum of deltas
    - Frequencies: $\omega_c + n\ \omega_m$
    - Amplitudes: $A_c\ J_n(\beta)\ \pi$
    - Number of harmonics in effective bandwidth: $M_e = 2\lfloor\beta\rfloor + 3 = 13$

$$J_0(5) = -0.18,\ J_1(5) = -0.33,\ J_2(5) = 0.05,\ J_3(5) = 0.37,\ J_4(5) = 0.39,\ J_5(5) = 0.26,\ J_6(5) = 0.13\cdots$$



Other types of modulators

- Modulation by means of a periodic signal
  - Supports a Fourier series expansion
    - Sum of sinusoids of multiple frequencies of the one that defines the period
      - Frequencies in the spectrum of the signal

$$f_c \pm n\, f_m \ \text{ or } (\omega_c \pm n\, \omega_m)$$

      - Amplitudes of each frequency: sum of the contributions of each harmonic
- Modulation by means of a non-periodic deterministic signal
  - Complicated analysis due to non-linearity
  - The Carson's Rule (heuristic rule) is applied

  For modulating signals with bandwidth B Hz

$$B_{Carson} \approx 2\,(\beta+1)\,B \text{ Hz}$$

Bandwidth dependent on the modulation index β

Noise Effect

at

Analog Modulations

Effect of noise on amplitude modulations

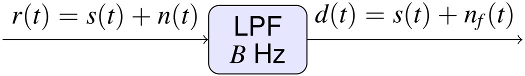
- Premises
  - Modulating signal $m(t)$
    - Bandwidth B Hz
    - Power P_M Watts
  - Received signal
    - Transmission on Gaussian channel
      Ideal transmission without attenuation, without distortion, only with thermal noise

$$r(t) = s(t) + n(t)$$

    - P_S : Power of the signal term (at the input of the receiver)
  - Thermal noise: usual statistical model
    - Random process $n(t)$ stationary, ergodic, white, Gaussian, with power spectral density $S_n(j\omega) = \frac{N_0}{2}$
  - Amplitude modulations: Coherent receiver
    - Filters will be introduced to limit the effect of noise
    - The filters will be considered as ideal (limit performance)
- The signal-to-noise ratio of the demodulated signal is analyzed for the different types of modulation and will be compared with the signal-to-noise ratio of the signal in a baseband transmission $\left(\frac{S}{N}\right)_b$

Reference - Baseband Transmission

- The unmodulated signal is transmitted: $s(t) = m(t) \rightarrow P_S = P_M$
- Signal at receiver: $r(t) = s(t) + n(t)$
  - There is usually attenuation during transmission
    - In this analysis it is omitted for simplicity
    - It is trivial to include it in development: $s(t) \rightarrow \alpha\, s(t),\, P_S \rightarrow \alpha^2\, P_S$
- Filtering in the receiver to minimize the effect of noise
  - Ideal low-pass filter of bandwidth B Hz



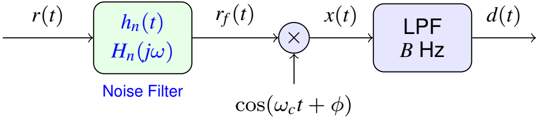
  - Noise power at the filter output

$$P_{n_f} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{n_f}(j\omega)\, d\omega = \frac{1}{2\pi}\int_{-2\pi B}^{+2\pi B} \frac{N_0}{2}\, d\omega = N_0\, B \text{ Watts}$$

- Baseband signal-to-noise ratio

$$\left(\frac{S}{N}\right)_b = \frac{P_S}{N_0\, B}, \qquad \left(\frac{S}{N}\right)_b \text{(dB)} = 10\log_{10}\frac{P_S}{N_0\, B} \text{ dB}$$

Coherent Receiver + Noise Filtering - Notation



- Noise filter $h_n(t) \stackrel{\mathcal{FT}}{\leftrightarrow} H_n(j\omega)$
  - Ideal band-pass filter: depends on the modulation that is used
    - Passband and bandwidth: same as modulated signal $s(t)$
- Coherent Receiver: $\phi = \phi_c$ (for simplicity, $\phi_c = 0$)
- Received signal: $r(t) = s(n) + n(t)$
- Filtered signal - filter suitable for modulation: $s(t) * h_n(t) = s(t)$

$$r_f(t) = s(n) + n_f(t), \text{ with } n_f(t) = n(t) * h_n(t)$$

- Demodulated signal

$$x(t) = r_f(t) \times \cos(\omega_c t) = s(t)\ \cos(\omega_c t) + n_f(t)\ \cos(\omega_c t) = x_S(t) + x_n(t)$$

- Filtered demodulated signal

$$d(t) = x(t) * h_{LPF}(t) = x_S(t) * h_{LPF}(t) + x_n(t) * h_{LPF}(t) = d_S(t) + d_n(t)$$

Coherent Receiver + Noise Filtering - Analysis

- Receiver output: Signal term $d_S(t)$
  - Not affected by noise filter
  - For amplitude modulations, it was calculated previously

Modulation	P_S (Watts)	$d_S(t)$	P_{d_S} (Watts)
Conventional AM	$\frac{A_c^2}{2}\left[1+P_{M_a}\right]$	$\frac{A_c}{2}\left[1+m_a(t)\right]$	$\frac{A_c^2}{4}P_{M_a}$
DSB	$\frac{A_c^2}{2}P_M$	$\frac{A_c}{2}m(t)$	$\frac{A_c^2}{4}P_M$
SSB	$A_c^2 P_M$	$\frac{A_c}{2}m(t)$	$\frac{A_c^2}{4}P_M$
VSB	$A_c^2 P_M$	$\frac{A_c}{2}m(t)$	$\frac{A_c^2}{4}P_M$

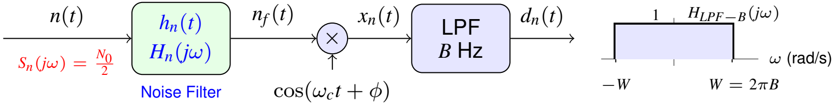
P_S : power of the modulated signal at the input of the receiver

P_{d_S}: power in $d_S(t)$ relative to $m(t)$ - P_{M_a}: power of $m_a(t)$, $P_{M_a}=\frac{a^2}{C_M^2}P_M$

- Receiver output: Noise term $d_n(t)$
  - Depends on the noise filter used
    - ★ Depends on the type of modulation
    - ★ Power: P_{d_n}

- Signal-to-noise ratio after demodulation: $\left(\frac{S}{N}\right)_d=\frac{P_{d_S}}{P_{d_n}}$
  - It will be compared with the baseband signal-to-noise ratio: $\left(\frac{S}{N}\right)_b=\frac{P_S}{N_0\,B}$

Noise power after demodulation - General



- Power spectral density of the filtered noise $n_f(t)$

$$S_{n_f}(j\omega) = S_n(j\omega)\ |H_n(j\omega)|^2 = \frac{N_0}{2}|H_n(j\omega)|^2$$

- Power spectral density of the demodulated noise $x_n(t)$

$$S_{x_n}(j\omega) = \frac{1}{4}S_{n_f}(j\omega - j\omega_c) + \frac{1}{4}S_{n_f}(j\omega + j\omega_c) = \frac{N_0}{8}\left[|H_n(j\omega - j\omega_c)|^2 + |H_n(j\omega + j\omega_c)|^2\right]$$

- Power spectral density after low-pass filtering $d_n(t)$

$$S_{d_n}(j\omega) = S_{x_n}(j\omega)\ |H_{LPF-B}(j\omega)|^2 = \begin{cases} S_{x_n}(j\omega), & \text{if } |\omega| \leq W = 2\pi B \\ 0, & \text{if } |\omega| > W = 2\pi B \end{cases}$$

- Power after low pass filtering

$$P_{d_n} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{d_n}(j\omega)\ d\omega = \frac{1}{2\pi}\int_{-2\pi B}^{2\pi B} S_{x_n}(j\omega)\ d\omega$$
$$= \frac{N_0}{8}\left[\frac{1}{2\pi}\int_{-2\pi B}^{2\pi B} |H_n(j\omega - j\omega_c)|^2\ d\omega + \frac{1}{2\pi}\int_{-2\pi B}^{2\pi B} |H_n(j\omega + j\omega_c)|^2\ d\omega\right]$$

Noise power after demodulation - General (II)

The noise power at the output of a synchronous AM receiver is

$$P_{d_n} = \frac{N_0}{8} \times F_0 \text{ Watts}$$ with $$F_0 = \frac{1}{2\pi}\int_{-W}^{W} |H_n(j\omega - j\omega_c)|^2 \, d\omega + \frac{1}{2\pi}\int_{-W}^{W} |H_n(j\omega + j\omega_c)|^2 \, d\omega$$

Making simple changes of variables ($\omega' = \omega - \omega_c$ and $\omega' = \omega + \omega_c$)

$$F_0 = \frac{1}{2\pi}\int_{-\omega_c-W}^{-\omega_c+W} |H_n(j\omega')|^2 \, d\omega' + \frac{1}{2\pi}\int_{\omega_c-W}^{\omega_c+W} |H_n(j\omega')|^2 \, d\omega'$$

Considering the even symmetry of $|H_n(j\omega)|$

$$F_0 = 2 \times F, \text{ with } F = \frac{1}{2\pi}\int_{\omega_c-W}^{\omega_c+W} |H_n(j\omega)|^2 \, d\omega$$

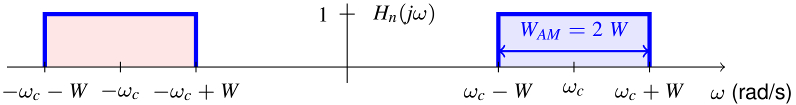
$$P_{d_n} = \frac{N_0}{4} \times F$$

Calculation of noise power - conventional AM and DSB

- For both modulations the noise filter is identical

$$H_n(j\omega) = \begin{cases} 1, & \text{if } \omega_c - W \leq |\omega| \leq \omega_c + W \\ 0, & \text{otherwise} \end{cases}$$

W: bandwidth in rad/s ($W = 2\pi B$)

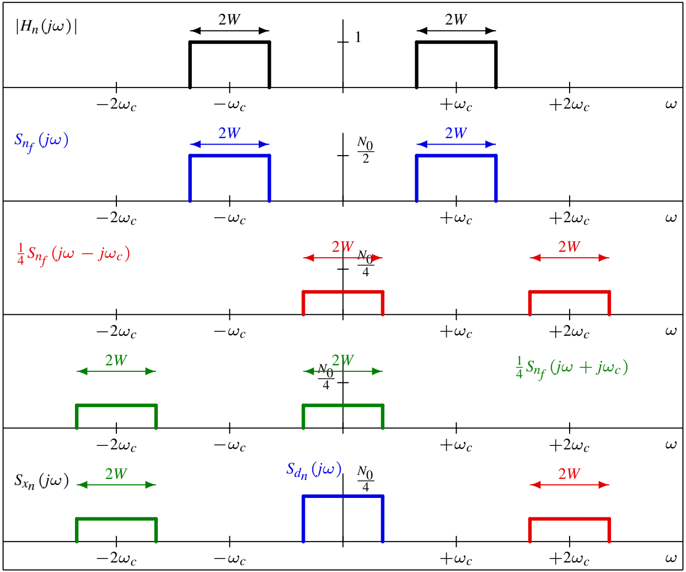


- Calculation of noise power

$$F = \frac{1}{2\pi}\int_{\omega_c-W}^{\omega_c+W} |H_n(j\omega)|^2\, d\omega = \frac{1}{2\pi}\int_{\omega_c-W}^{\omega_c+W} 1\, d\omega$$
$$= \frac{1}{2\pi}\, 2W = 2B$$

$$P_{d_n} = \frac{N_0}{4} \times F = \frac{1}{2} N_0 B \text{ Watts}$$

Noise in conventional AM and DSB - Frequency interpretation

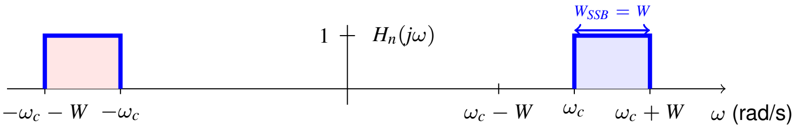


Calculation of noise power - SSB

- The case of upper sideband is presented

$$H_n(j\omega) = \begin{cases} 1, & \text{if } \omega_c \leq |\omega| \leq \omega_c + W \\ 0, & \text{otherwise} \end{cases}$$

W: bandwidth in rad/s ($W = 2\pi B$)

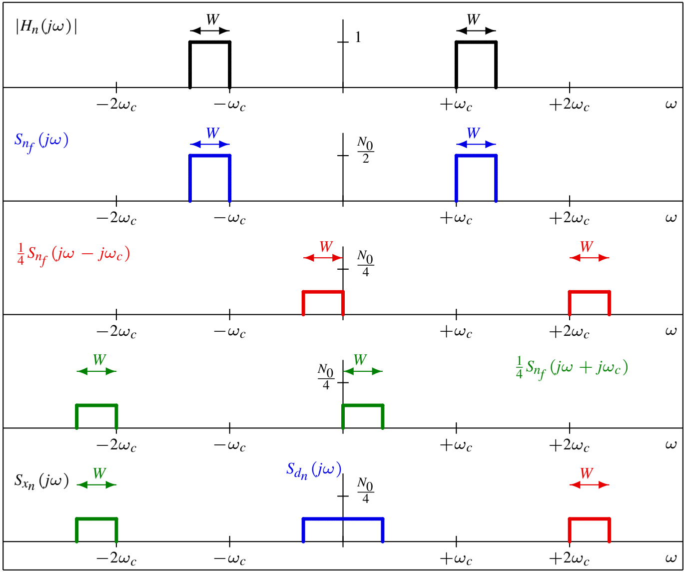


- Calculation of noise power

$$F = \frac{1}{2\pi} \int_{\omega_c - W}^{\omega_c + W} |H_n(j\omega)|^2 \, d\omega = \frac{1}{2\pi} \int_{\omega_c}^{\omega_c + W} 1 \, d\omega$$
$$= \frac{1}{2\pi} W = B$$

$$P_{d_n} = \frac{N_0}{4} \times F = \frac{1}{4} N_0 B \text{ Watts}$$

SSB Noise - Frequency Interpretation (USB)

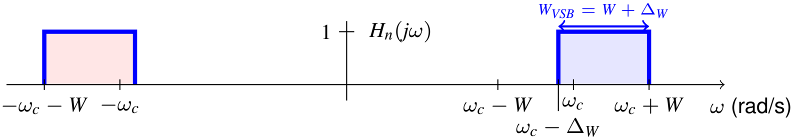


Calculation of noise power - VSB

- The case of upper sideband is presented

$$H_n(j\omega) = \begin{cases} 1, & \text{if } \omega_c - \Delta_W \leq |\omega| \leq \omega_c + W \\ 0, & \text{otherwise} \end{cases}$$

W: bandwidth in rad/s ($W = 2\pi B$)
Δ_W: bandwidth excess (vestige) in rad/s ($\Delta_W = 2\pi\Delta_B$)

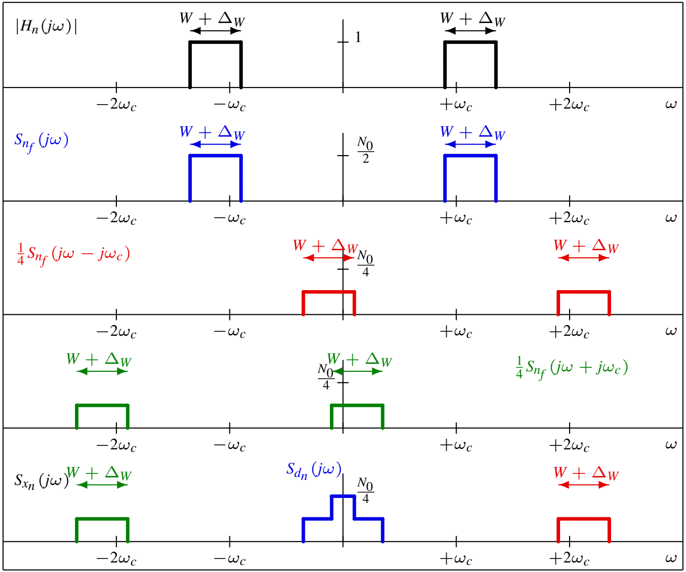


- Calculation of noise power

$$F = \frac{1}{2\pi}\int_{\omega_c - W}^{\omega_c + W} |H_n(j\omega)|^2 \, d\omega = \frac{1}{2\pi}\int_{\omega_c - \Delta_W}^{\omega_c + W} 1 \, d\omega$$
$$= \frac{1}{2\pi}\,(W + \Delta_W) = (B + \Delta_B)$$

$$P_{d_n} = \frac{N_0}{4} \times F = \frac{1}{4} N_0 \,(B + \Delta_B) \text{ Watts}$$

Noise in VSB - Frequency Interpretation (USB)



Signal to noise ratio at the output of a coherent receiver

Modulation	P_S	$d_S(t)$	P_{d_S}	P_{d_n}
Conventional AM	$\frac{A_c^2}{2}\left[1+P_{M_a}\right]$	$\frac{A_c}{2}\left[1+m_a(t)\right]$	$\frac{A_c^2}{4}P_{M_a}$	$\frac{1}{2}N_0B$
DSB	$\frac{A_c^2}{2}P_M$	$\frac{A_c}{2}m(t)$	$\frac{A_c^2}{4}P_M$	$\frac{1}{2}N_0B$
SSB	$A_c^2P_M$	$\frac{A_c}{2}m(t)$	$\frac{A_c^2}{4}P_M$	$\frac{1}{4}N_0B$
VSB	$A_c^2P_M$	$\frac{A_c}{2}m(t)$	$\frac{A_c^2}{4}P_M$	$\frac{1}{4}N_0(B+\Delta_B)$

P_S: signal power at the input of the receiver (Assumption: $r(t)=s(t)+n(t)$)

- Signal to noise ratio

$$\left(\frac{S}{N}\right)=\frac{P_{d_S}}{P_{d_n}}$$

- Signal to noise ratio compared with a baseband transmission

$$\left(\frac{S}{N}\right)=\eta\left(\frac{S}{N}\right)_b=\eta\,\frac{P_S}{N_0B}$$

Analysis of the result: $\eta \gtreqqless 1$

Calculation of signal to noise ratios - DSB and SSB

Modulation	P_S	$d_S(t)$	P_{d_S}	P_{d_n}
Conventional AM	$\frac{A_c^2}{2}\left[1+P_{M_a}\right]$	$\frac{A_c}{2}\left[1+m_a(t)\right]$	$\frac{A_c^2}{4}P_{M_a}$	$\frac{1}{2}N_0B$
DSB	$\frac{A_c^2}{2}P_M$	$\frac{A_c}{2}m(t)$	$\frac{A_c^2}{4}P_M$	$\frac{1}{2}N_0B$
SSB	$A_c^2P_M$	$\frac{A_c}{2}m(t)$	$\frac{A_c^2}{4}P_M$	$\frac{1}{4}N_0B$
VSB	$A_c^2P_M$	$\frac{A_c}{2}m(t)$	$\frac{A_c^2}{4}P_M$	$\frac{1}{4}N_0(B+\Delta_B)$

- Signal to noise ratio for DSB

$$\left(\frac{S}{N}\right)_{DSB}=\frac{P_{d_S}}{P_{d_n}}=\frac{\frac{A_c^2}{4}P_M}{\frac{1}{2}N_0B}=\frac{\frac{A_c^2}{2}P_M}{N_0B}=\frac{P_S}{N_0B}=\left(\frac{S}{N}\right)_b$$

Same signal to noise ratio as transmitting in baseband

- Signal to noise ratio for SSB

$$\left(\frac{S}{N}\right)_{SSB}=\frac{P_{d_S}}{P_{d_n}}=\frac{\frac{A_c^2}{4}P_M}{\frac{1}{4}N_0B}=\frac{A_c^2P_M}{N_0B}=\frac{P_S}{N_0B}=\left(\frac{S}{N}\right)_b$$

Same signal to noise ratio as transmitting in baseband

Calculation of signal to noise ratios - Conventional AM

Modulation	P_S	$d_S(t)$	P_{d_S}	P_{d_n}
Conventional AM	$\frac{A_c^2}{2}\,[1+P_{M_a}]$	$\frac{A_c}{2}\,[1+m_a(t)]$	$\frac{A_c^2}{4}P_{M_a}$	$\frac{1}{2}N_0B$
DSB	$\frac{A_c^2}{2}P_M$	$\frac{A_c}{2}m(t)$	$\frac{A_c^2}{4}P_M$	$\frac{1}{2}N_0B$
SSB	$A_c^2P_M$	$\frac{A_c}{2}m(t)$	$\frac{A_c^2}{4}P_M$	$\frac{1}{4}N_0B$
VSB	$A_c^2P_M$	$\frac{A_c}{2}m(t)$	$\frac{A_c^2}{4}P_M$	$\frac{1}{4}N_0(B+\Delta_B)$

- Signal to noise ratio for conventional AM

$$\left(\frac{S}{N}\right)_{AM}=\frac{P_{d_S}}{P_{d_n}}=\frac{\frac{A_c^2}{4}P_{M_a}}{\frac{1}{2}N_0B}=\frac{\frac{A_c^2}{2}P_{M_a}}{N_0B}=\frac{P_{M_a}}{1+P_{M_a}}\frac{\frac{A_c^2}{2}\,[1+P_{M_a}]}{N_0B}$$

$$=\underbrace{\frac{P_{M_a}}{1+P_{M_a}}}_{\eta_{AM}}\frac{P_S}{N_0B}=\eta_{AM}\left(\frac{S}{N}\right)_b$$

  - Worse than baseband transmission: Efficiency factor $\eta_{AM}<1$

$$\eta_{AM}=\frac{P_{M_a}}{1+P_{M_a}}=\frac{\frac{a^2}{C_M^2}P_M}{1+\frac{a^2}{C_M^2}P_M}=\frac{P_M}{\frac{C_M^2}{a^2}+P_M}$$

Depends on the modulation index: worse efficiency for lower values of a

Calculation of signal to noise ratios - VSB

Modulation	P_S	$d_S(t)$	P_{d_S}	P_{d_n}
Conventional AM	$\frac{A_c^2}{2}\left[1+P_{M_a}\right]$	$\frac{A_c}{2}\left[1+m_a(t)\right]$	$\frac{A_c^2}{4}P_{M_a}$	$\frac{1}{2}N_0B$
DSB	$\frac{A_c^2}{2}P_M$	$\frac{A_c}{2}m(t)$	$\frac{A_c^2}{4}P_M$	$\frac{1}{2}N_0B$
SSB	$A_c^2P_M$	$\frac{A_c}{2}m(t)$	$\frac{A_c^2}{4}P_M$	$\frac{1}{4}N_0B$
VSB	$A_c^2P_M$	$\frac{A_c}{2}m(t)$	$\frac{A_c^2}{4}P_M$	$\frac{1}{4}N_0(B+\Delta_B)$

- Signal to noise ratio for VSB

$$\left(\frac{S}{N}\right)_{VSB}=\frac{P_{d_S}}{P_{d_n}}=\frac{\frac{A_c^2}{4}P_M}{\frac{1}{4}N_0(B+\Delta_B)}=\frac{A_c^2P_M}{N_0(B+\Delta_B)}=\frac{B}{B+\Delta_B}\frac{A_c^2P_M}{N_0B}$$
$$=\underbrace{\frac{B}{B+\Delta_B}}_{\eta_{VSB}}\frac{P_S}{N_0B}=\eta_{VSB}\left(\frac{S}{N}\right)_b$$

  - Worse than baseband transmission: Efficiency factor $\eta_{VSB}<1$

$$\eta_{VSB}=\frac{B}{B+\Delta_B}$$

Depends on bandwidth excess Δ_B (vestige): usually $\Delta_B << B$ and in that case $\eta_{VSB}\approx 1$, i.e., signal-to-noise ratio is relatively close to $(S/N)_b$

Noise in angle modulations

- Much more complex analysis (non-linear dependency)
- Demodulator output (noisy)

$$d(t) = \begin{cases} k_p\ m(t) + Y_n(t), & \text{PM} \\ k_f\ m(t) + \frac{1}{2\pi}\ \frac{d}{dt} Y_n(t), & \text{FM} \end{cases}$$

$Y_n(t)$: noise term at the demodulator output

- Signal to noise ratio (for received signal power $P_S = \frac{A_c^2}{2}$)

$$\left(\frac{S}{N}\right)_d = \begin{cases} \frac{k_p^2 A_c^2}{2} \frac{P_M}{N_0 B} = P_M \left(\frac{\beta_p}{\max|m(t)|}\right)^2 \left(\frac{S}{N}\right)_b, & \text{PM} \\ \frac{3k_f^2 A_c^2}{2B^2} \frac{P_M}{N_0 B} = 3P_M \left(\frac{\beta_f}{\max|m(t)|}\right)^2 \left(\frac{S}{N}\right)_b, & \text{FM} \end{cases}$$

Signal-to-noise gain proportional to modulation index squared

- General expression

$$\left(\frac{S}{N}\right)_d = \alpha \left(\frac{P_M}{C_M^2}\right) \times \beta^2 \times \left(\frac{S}{N}\right)_b$$

  - The factor α depends on the modulation: $\alpha_{PM} = 1, \alpha_{FM} = 3$
  - The term $\left(\frac{P_M}{C_M^2}\right)$ is usually constant (depends on the type of signals)

Noise in angle modulations - Threshold effect

- The gain in $\left(\frac{S}{N}\right)_d$ is only obtained from a threshold signal-to-noise ratio at the receiver input

$$\left(\frac{S}{N}\right)_{threshold} = 20\,(\beta+1)$$

- This implies in practice that there is a threshold level of received power that must be reached

$$P_{R_{threshold}} = (N_0 B) \times \left(\frac{S}{N}\right)_{threshold} \rightarrow A_{c,threshold} = \sqrt{2 P_{R_{threshold}}}$$