Chapter 2 : Solutions to the exercises

Exercise 2.1 Solution

a) The modulations and carrier frequency are

- Figure (a): conventional AM modulation, with carrier frequency $f_c = 100$ kHz.
- Figure (b): double sideband modulation, with carrier frequency $f_c = 102$ kHz.
- Figure (c): single sideband AM modulation. There are two possibilities:
  - Upper sideband with carrier frequency $f_c = 98$ kHz.
  - Lower sideband with carrier frequency $f_c = 102$ kHz.
- Figure (d): angle modulation, with carrier frequency $f_c = 100$ kHz.

b) The modulating signal is a sinusoid of frequency 2 kHz, that is,

$$m(t) = A_c \cos(\omega_m t + \phi)$$

where $\omega_m = 2\pi f_m$ with $f_m = 2$ kHz.

Exercise 2.2 Solution

The spectrum of the modulated signal are deltas at the frequencies

$$\omega_c + n \times \omega_m$$

for every integer n, where ω_c is the frequency of the carrier and ω_m is the frequency of the modulating signal (in both cases they are angular frequencies in rad/s). In this case $\omega_c = 2\pi f_c$ and $\omega_m = 2\pi f_m$. The amplitude of each delta is π times the coefficient $J_n(\beta)$, in this case $J_n(5)$. The number of harmonics in the effective bandwidth is $M_e = 2\lfloor\beta\rfloor + 3 = 13$ harmonics. Taking into account that

$$J_0(5) = -0.18,\ J_1(5) = -0.33,\ J_2(5) = 0.05$$

$$J_3(5) = 0.37,\ J_4(5) = 0.39,\ J_5(5) = 0.26,\ J_6(5) = 0.13\cdots$$

and that $J_{-m}(\beta) = (-1)^m J_m(\beta)$, the Fourier transform of the modulated signal will have the form of Figure 2.1 (Only the $M_e = 2\lfloor\beta\rfloor + 3 = 13$ central frequencies are represented, which contain the effective bandwidth).

Exercise 2.3 Solution

In this case, it is defined for all sections $A_M = 1$.

a) The bandwidth is $2B$ Hz. The power spectral density, for a generic modulation index a, is that of the figure

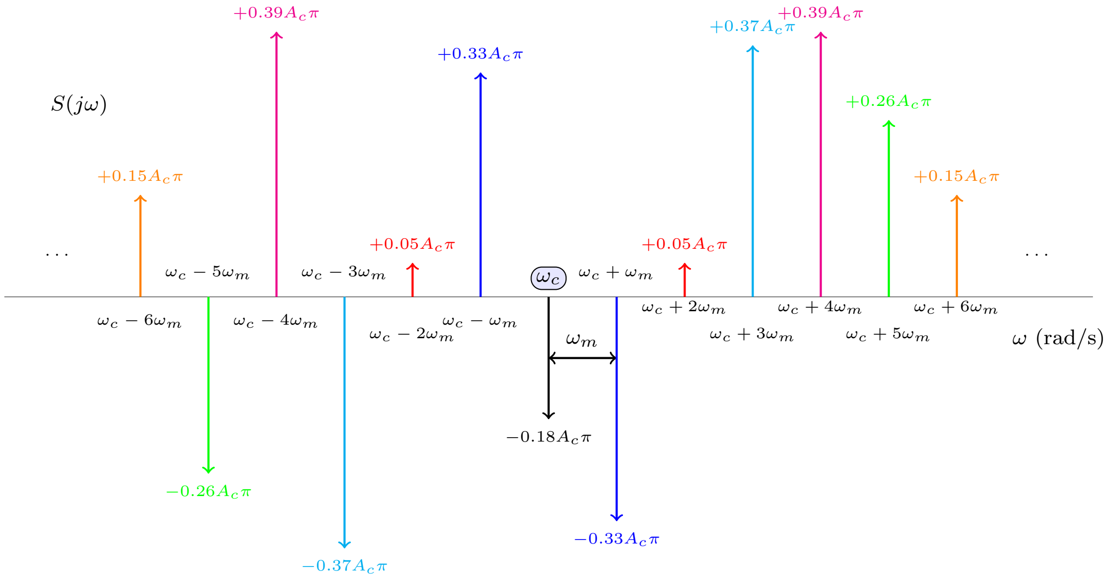
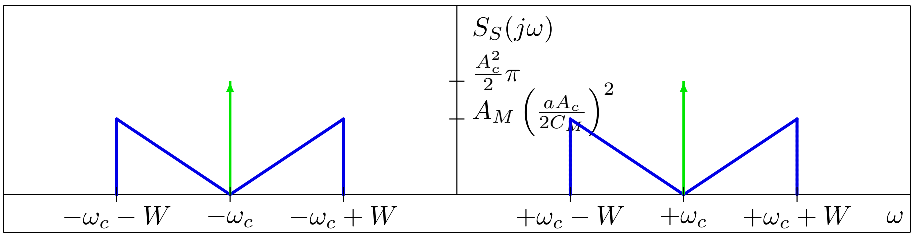
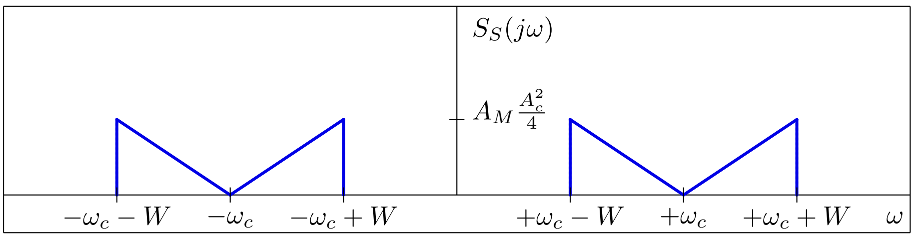


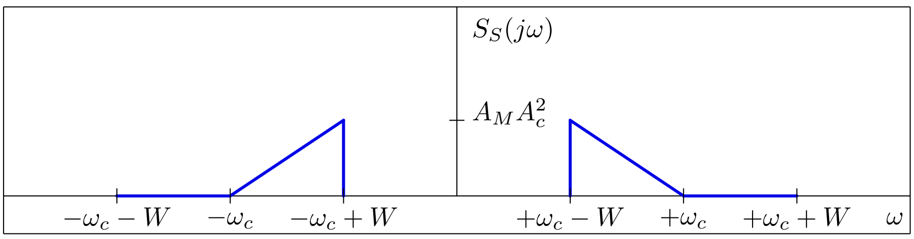
Figure 2.1: Frequency response of the modulated signal for Exercise 2.2.



b) The bandwidth is $2B$ Hz. The power spectral density is that of the figure



c) The bandwidth is B Hz. The power spectral density is that of the figure

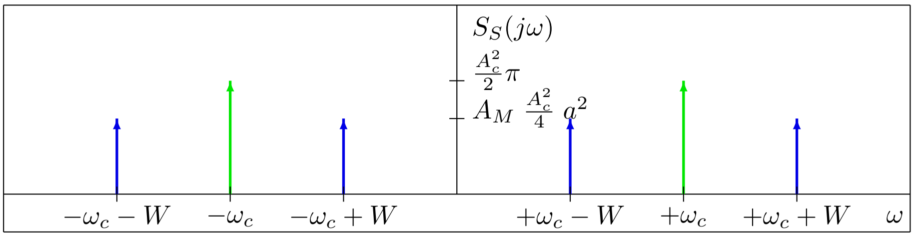


d) The approximated bandwidth is 12 B Hz.

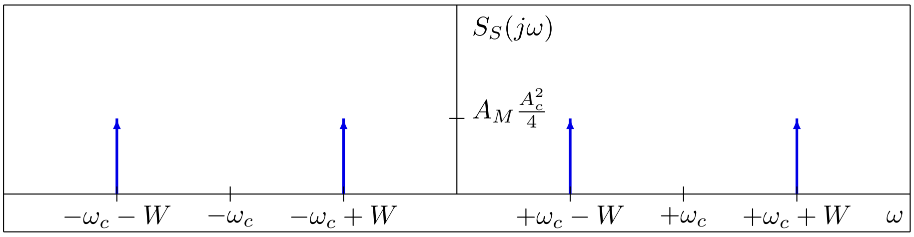
Exercise 2.4 Solution

In all cases $P_M = \frac{1}{2}$ is considered, which is the power of the modulating signal, $C_M = 1$ the maximum value in module of the signal $m(t)$, and $W = 2\pi f_m$.

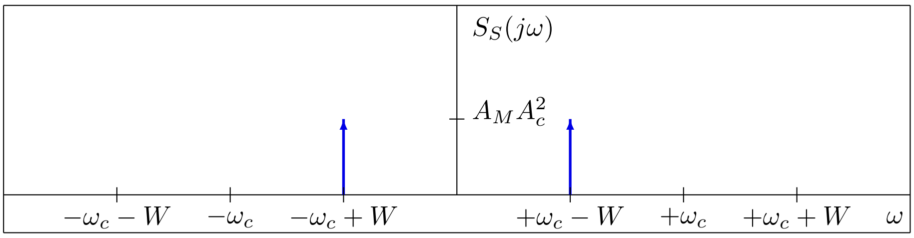
a) The power of the modulated signal is $P_S = \frac{A_c^2}{2}\left[1 + \frac{P_M}{4}\right]$. The power spectral density, for a generic modulation index a, is that of the figure, where $A_M = \frac{\pi}{2}$



b) The power of the modulated signal is $P_S = \frac{A_c^2}{2} P_M$. The power spectral density is that of the figure, where $A_M = \frac{\pi}{2}$



c) The power of the modulated signal is $P_S = A_c^2 P_M$. The power spectral density is that of the figure, where $A_M = \frac{\pi}{2}$



Exercise 2.5 Solution

a) The bandwidth is $2B$ Hz, and the signal-to-noise ratio is worse than that obtained in a baseband transmission

$$\left(\frac{S}{N}\right)_{AM} = \eta_{AM}\left(\frac{S}{N}\right)_b$$

with

$$\eta_{AM} = \frac{P_{M_a}}{1 + P_{M_a}} = \frac{\frac{a^2}{C_M^2} PM_M}{1 + \frac{a^2}{C_M^2} PM_M} = \frac{PM_M}{\frac{C_M^2}{a^2} + PM_M}$$

b) The bandwidth is $2B$ Hz, and the signal-to-noise ratio is the same as that obtained in a baseband transmission

$$\left(\frac{S}{N}\right)_{DBL} = \eta\left(\frac{S}{N}\right)_b$$

c) The bandwidth is B Hz, and the signal-to-noise ratio is the same as that obtained in a baseband transmission

$$\left(\frac{S}{N}\right)_{BLU} = \eta\left(\frac{S}{N}\right)_b$$

d) The bandwidth is $B+\Delta_B$ Hz, where Δ_B is the excess of bandwidth due to the vestige, and the signal-to-noise ratio is slightly lower than that obtained in a baseband transmission

$$\left(\frac{S}{N}\right)_{BLV} = \eta_{BLV}\left(\frac{S}{N}\right)_b$$

with

$$\eta_{BLV} = \frac{B}{B+\Delta_B}$$

e) The bandwidth is approximately $8B$ Hz, and the signal-to-noise ratio is better than that obtained in a baseband transmission, the gain being proportional to the square of the modulation index

$$\left(\frac{S}{N}\right)_{FM} \approx 3\frac{P_M}{C_M^2}\beta_f^2\left(\frac{S}{N}\right)_b$$

Exercise 2.6 Solution

a) Double Sideband Amplitude Modulation (DBL).

b) Phase modulation (PM).

c) Conventional AM amplitude modulation (or double sideband with carrier).

d) Single sideband (SSB) modulation, upper sideband.

e) Angle modulation, in particular frequency modulation (FM).

f) Single sideband (SSB) modulation, lower sideband.

Exercise 2.7 Solution

a) Single sideband (SSB) modulation, lower sideband.

b) Double Sideband Amplitude Modulation (DSB).

c) Conventional AM amplitude modulation (or double sideband with carrier).

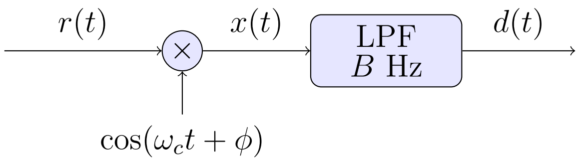
d) Single sideband (SSB) modulation, upper sideband.

e) Angle, phase (PM) or frequency (FM) modulation, when the modulating signal is a sinusoid.

f) Conventional AM amplitude modulation (or double sideband with carrier).

Exercise 2.8 Solution

a) A coherent receiver is a receiver composed of a multiplier by a sinusoid of the carrier frequency (at sometimes called a demodulator, due to its function of returning the spectrum signal to baseband) followed by a low pass filter with the bandwidth of the modulating signal, B Hz, as shown in the figure

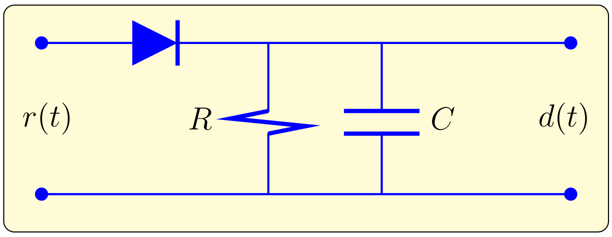


In a coherent receiver, the phase of the carrier at the receiver must coincide with the phase of the carrier that was used in the modulator to generate the signal, $c(t) = A_c \cos(\omega_c t + \phi_c)$, that is, it must be true that $\phi = \phi_c$. Normally, to achieve this synchronization of the phases, a pilot signal can be transmitted at the frequency and phase of the carrier, or a phase-locked loop, or PLL, will be used, in the receiver.

b) The variants of AM modulations that require a coherent receiver are

- Double Sideband AM Modulation (DSB)
- Single Sideband AM Modulation (SSB)
- Vestigial sideband AM modulation (VSB)

Conventional AM modulation, which can also make use of a rocket receiver, can use a simpler receiver, in this case an envelope detector, which can be implemented with a rectifier diode and a low-pass RC filter, as shown in the figure.



c) In the case of double sideband modulation, the effect of having a different phase in the carriers of the transmitter (ϕ_c) and the receiver (ϕ), will be that the ideally demodulated signal (without noise or distortions) will have the expression

$$d(t) = \frac{A_c}{2}\, m(t)\, \cos(\phi_c - \phi),$$

with which the phase difference between carriers will generate an attenuation in the received signal depending on the phase difference. It can be verified, for example, that a phase shift of 90º would be critical, since in this case the received signal would be completely cancelled.

In the case of single sideband modulation (and vestigial sideband, which is very similar), the problem is even greater, since the expression of the signal at the output of the demodulator is

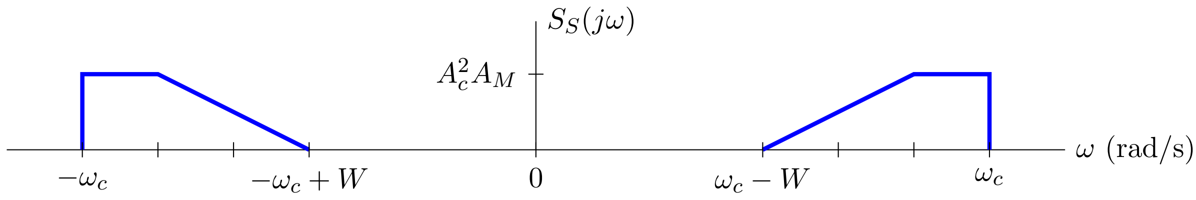
$$d(t) = \frac{A_c}{2}\, m(t)\, \cos(\phi_c - \phi) \pm \frac{A_c}{2}\, \hat{m}(t)\, \sin(\phi_c - \phi),$$

whereby a phase difference will not only attenuate the term associated with the signal $m(t)$, but will make the term not null proportional to the Hilbert transform of the modulating signal, $\hat{m}(t)$, which is an interference on the wanted signal $m(t)$.

Exercise 2.9 Solution

a)

$$S_S(j\omega) = \begin{cases} 0, & |\omega| > \omega_c \\ A_c^2\, [S_M(j\omega - j\omega_c) + S_M(j\omega + j\omega_c)], & |\omega| < \omega_c \end{cases}$$



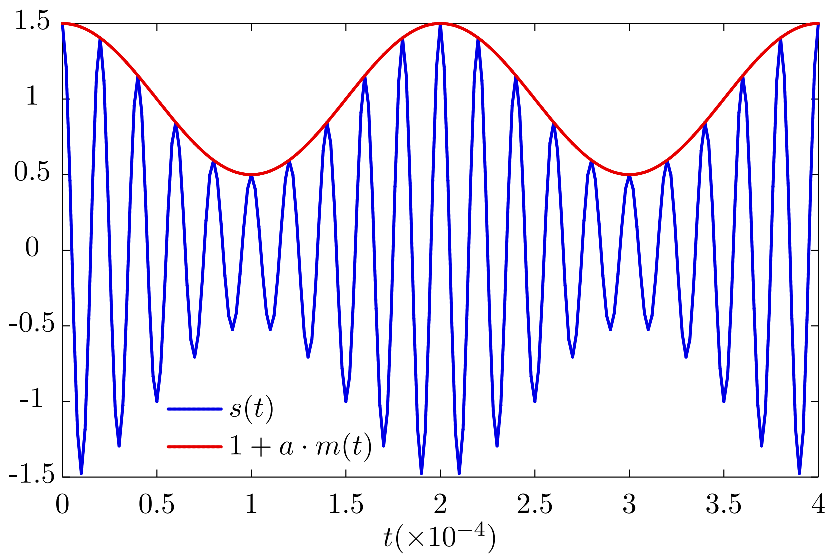
b) In terms of spectral efficiency, the SSB modulation is the most efficient of all, since the bandwidth required to transmit a modulating signal of bandwidth B Hz is B Hz, while both the conventional AM and the double side band AM require twice this bandwidth, $2B$ Hz, while the FM modulation requires a larger bandwidth, dependent on the modulation index, according to the modulation index, according to the rule, according to $2(\beta + 1)B$ Hz.

Exercise 2.10 Solution

a) The bandwidth is

$$BW_{AM} = 2B = 10 \text{ kHz}.$$

The modulated signal in the time domain will have the form



b) Amplitude modulations that do not include carrier transmission are

- Double sideband modulation (DSB).
- Single Sideband (SSB) modulation.
- Vestigial sideband modulation (VSB).

The bandwidth of the DSB modulation is the same as that of conventional AM, while that of SSB is half, and that of VSB is approximately half (neglecting the vestige of the removed sideband).

$$B_{DSB} = 2B = 10 \text{ kHz},\ B_{SSB} = B = 5 \text{ kHz},\ B_{VSB} = B + \Delta_B \approx B = 5 \text{ kHz}.$$

c) The theoretical bandwidth is infinite, since the Fourier transform can be written as an infinite sum of sinusoids at frequencies $\omega_c \pm n\ \omega_m$ rad/s. The effective bandwidth is the bandwidth that contains 98% of the signal power, and in this case it is

$$B_{FM} = 2\ (\beta_f + 1)\ f_m = 80 \text{ kHz}.$$

d) The main advantage of angular modulations is their greater signal-to-noise ratio (noise immunity), which increases with respect to the ratio in baseband proportionally to the square of the modulation index. The main drawback is that the width band increases proportionally to twice the rate of modulation plus one, which is significantly higher than that required by linear modulations.

Exercise 2.11 Solution

a) Amplitude modulations differ from angle modulations in that in the former what varies with the information or modulating signal is the amplitude of the signal, while in the latter the amplitude remains constant, but the angular information (instantaneous frequency or phase term) changes. Therefore, the amplitude modulations are those of figures B and C. The analytical expressions for conventional AM modulation and double sideband modulation are, respectively

$$s_{AM}(t) = [1 + a \times m_a(t)] \times c(t),\ s_{DBL}(t) = m(t) \times c(t).$$

In the first, the envelope of the signal is $[1 + a \times m_a(t)]$, which for $a < 1$, as is the case, is always positive, while in the second, the modulating signal is simply multiplied by the carrier, which makes the signal invert relative to the carrier for negative values of $m(t)$. It is trivial to see that

- Signal B: conventional AM modulation
- Signal C: double sideband modulation.

Regarding the angular modulations, the PM and FM modulations have the following analytical expression

$$s(t) = A_c \cos(\omega_c t + \phi(t)) \text{ with } \phi(t) = \begin{cases} k_p \times m(t), & \text{for PM} \\ 2\pi k_f \times \int_{-\infty}^{t} m(\tau)\, d\tau, & \text{for FM} \end{cases}.$$

Therefore, the phase term $\phi(t)$ is proportional in one case to the modulating signal, and in another case to its integral. In any case, when $\phi(t)$ takes positive values, the sinusoid argument is increased, which means that the modulated signal "*lead*" relative to the carrier, while when it takes negative values, the sinusoid argument is decreased, which means that the modulated signal "*lag*" relative to the carrier. Therefore, in a PM modulation, the signal will lead the carrier for positive values of $m(t)$, and it will lag for negative values, which happens in signal A. In an FM modulation, whether it leads or lags the carrier is related to the sign of the integral of $m(t)$, which happens in figure D. Therefore

- Signal A: PM modulation.
- Signal D: FM modulation.

b) The most convenient modulation for each case is:

   i) In this case it is best to use an angular modulation with a high modulation index, since these have a better signal-to-noise ratio than angular modulations, a ratio that is proportional to β^2. Among the two variants, for the same value of β (same bandwidth), FM modulation has a better signal-to-noise ratio (3 times that of PM).

   ii) The simplest receiver that can be used in analog modulation is an envelope detector, which can be implemented with a diode, and an RC filter. The modulation that this receiver can use is conventional AM modulation. All other amplitude modulations must use a coherent receiver, and angular modulations use more complex receivers.

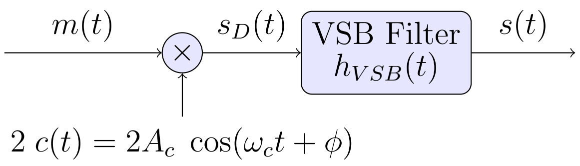
iii) In this case, the smaller the bandwidth of the modulated signal, the more signals can be multiplexed, and the modulation that has a smaller bandwidth is single-sideband modulation, which requires the same bandwidth as the modulating signal. The rest of the modulations use a greater bandwidth.

Exercise 2.12 Solution

a) The modulated signal of a vestigial sideband AM modulation is generated as follows

- A double sideband signal is generated (double amplitude, in the notation that was followed in the subject, but this amplitude factor is not really relevant) multiplying the modulator by the carrier signal.
- This double-sideband signal is filtered with a vestigial sideband filter, a filter that has to meet certain conditions (next section).

The figure shows the block diagram of a vestigial sideband AM transmitter.



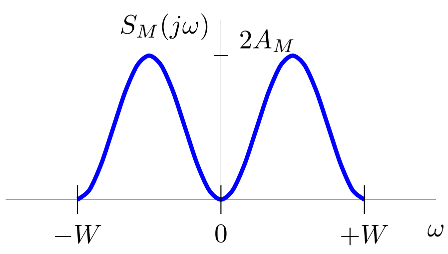
b) The condition that a vestigial sideband filter must satisfy is that its frequency response has an odd symmetry with respect to the carrier frequency, ω_c, so that the contribution of the filter response shifted ω_c to the left plus the same response shifted ω_c to the right (effect of a demodulator) is constant in the bandwidth of the signal

$$|H_{VSB}(j\omega - j\omega_c) + H_{VSB}(j\omega + j\omega_c)| = C, \text{ in } |\omega| \leq W = 2\pi B.$$

The only filter that satisfies the condition is filter A. In this case, it would be a filter for an upper sideband vestigial sideband modulation, since the frequency band above the carrier frequency (plus the corresponding lower sideband vestige) is allowed to pass.

Exercise 2.13 Solution

a) The power spectral density of the modulating signal is plotted below



The signal to noise ratio in dB is

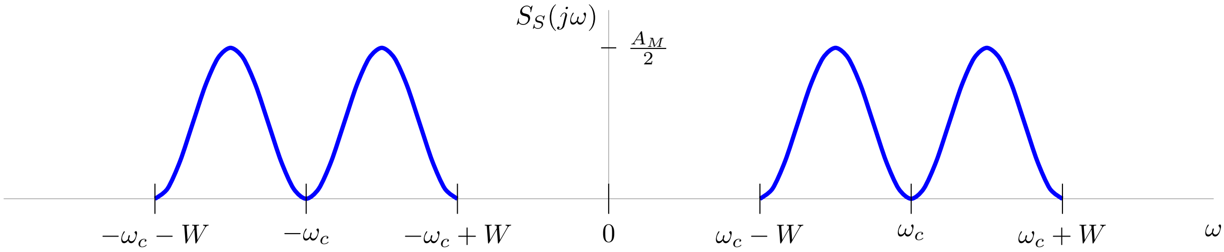
$$\frac{S}{N}(\text{dB}) = 10\log_{10}\frac{P_X}{P_Z} = 10\log_{10}\frac{2\times 10^{-14}}{2\times 10^{-17}} = 10\log_{10}1000 = 30\text{ dB}$$

b) In the case of double sideband modulation

i) The power spectral density for this modulation is

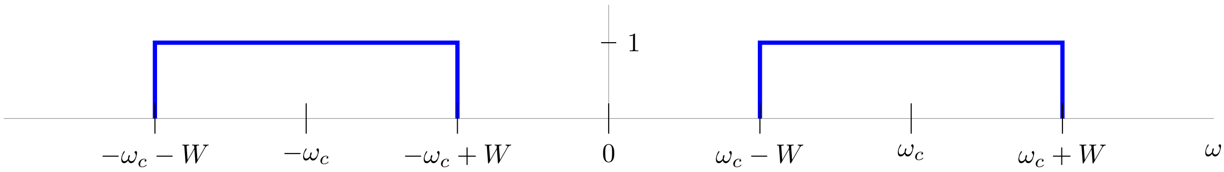
$$S_S(j\omega) = \frac{A_c^2}{4}\left[S_M(j\omega - j\omega_c) + S_M(j\omega + j\omega_c)\right],$$

that is, two replicates of the power spectral density shifted $\pm\omega_c$, with a scale factor in the amplitude $\frac{A_c^2}{4}$, giving rise to the density shown in the figure



The bandwidth is $2B$ Hz or $2W$ rad/s. In this case, it is 10 kHz.

ii) The noise filter is an ideal bandpass filter, with a bandwidth of 10 kHz and centered on the carrier frequency.



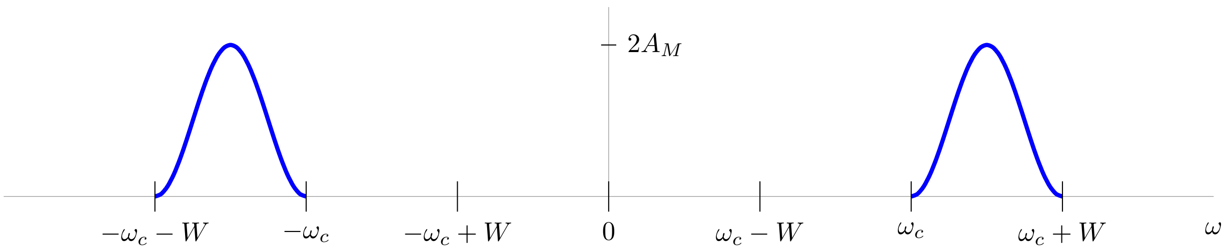
The signal-to-noise ratio using this type of modulation is equal to that obtained in a baseband transmission, which was calculated in the previous section, that is, 30 dB.

c) In the case of a higher band single sideband modulation

i) The power spectral density for this modulation is

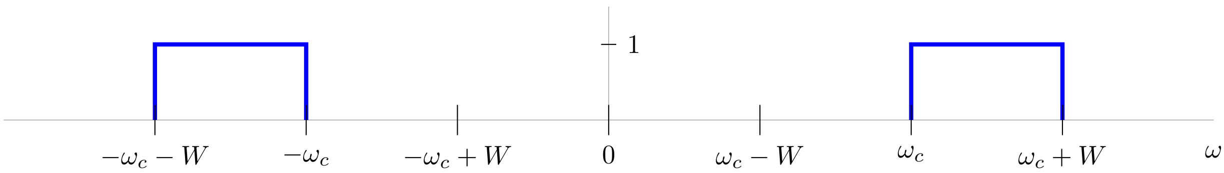
$$S_{S_{sup}}(j\omega) = \begin{cases} A_c^2\left[S_M(j\omega - j\omega_c) + S_M(j\omega + j\omega_c)\right], & |\omega| \geq \omega_c \\ 0, & |\omega| < \omega_c \end{cases}$$

that is, the part of the corresponding sideband of two replicates power spectral density shifted $\pm\omega_c$ as shown in the figure



Bandwidth is B Hz or W rad/s. In this case, it is 5 kHz.

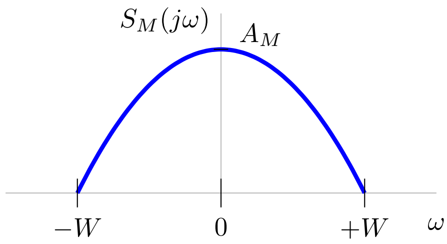
ii) The noise filter is an ideal band-pass filter, with a bandwidth of 5 kHz and whose pass band begins at the carrier frequency, as shown in the figure



The signal to noise ratio using this type of modulation is equal to that obtained in a baseband transmission, which was calculated in the first section, 30 dB.

Exercise 2.14 Solution

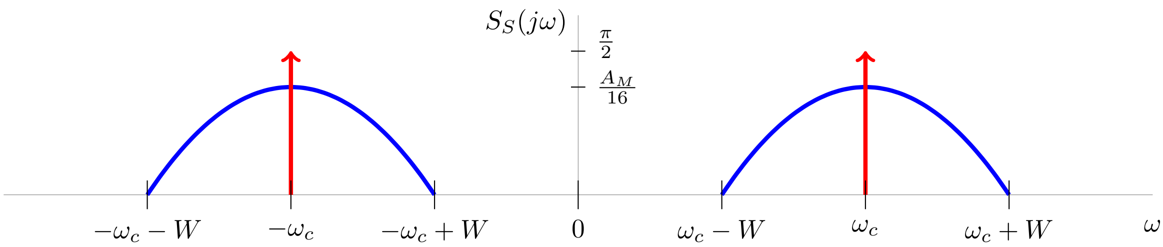
The power spectral density of the modulating signal is that of the figure



a) For a conventional AM modulation, with modulation index a

$$S_S(j\omega) = \frac{A_c^2}{2}\pi\left[\delta(\omega-\omega_c)+\delta(\omega-\omega_c)\right] + \frac{A_c^2}{4}\frac{a^2}{C_M^2}\left[S_M(j\omega - j\omega_c) + S_M(j\omega + j\omega_c)\right]$$

where given the range of the modulating signal, in this case, $C_M = 1$. That is, the two deltas of the spectrum of the carrier appear, plus the replicas in $\pm\omega_c$ of the spectrum of the modulator. This density is shown below.

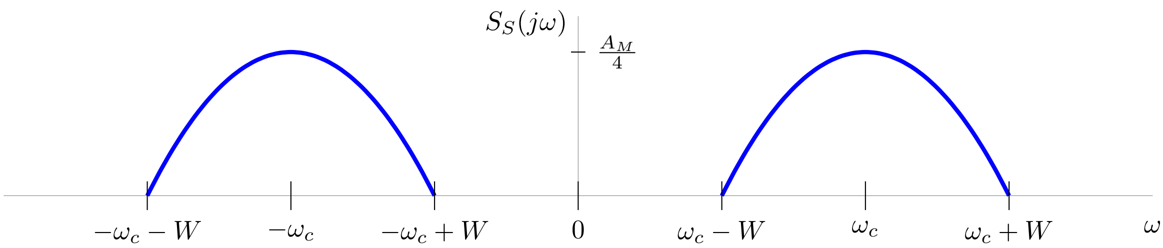


The bandwidth is $2B$ Hz or $2W$ rad/s. In this case, it is 10 kHz.

b) In the case of double sideband modulation, the power spectral density for this modulation is

$$S_S(j\omega) = \frac{A_c^2}{4}\left[S_M(j\omega - j\omega_c) + S_M(j\omega + j\omega_c)\right],$$

that is, two replicates of the power spectral density shifted $\pm\omega_c$, with a scale factor in the amplitude $\frac{A_c^2}{4}$, giving rise to the density shown in the figure

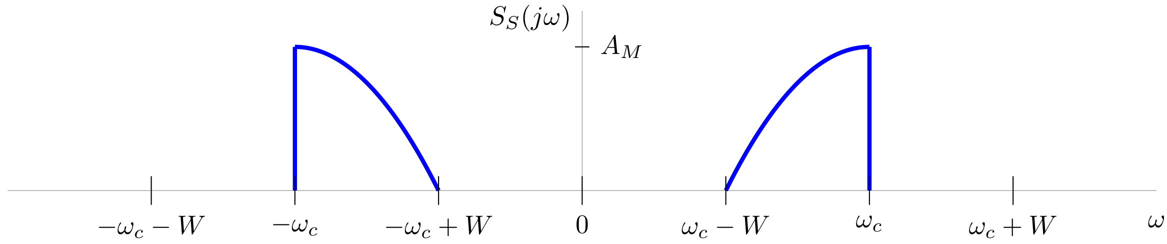


The bandwidth is $2B$ Hz or $2W$ rad/s. In this case, it is 10 kHz.

c) In the case of a higher-band single-sideband modulation, the power spectral density for this modulation is

$$S_{S_{sup}}(j\omega) = \begin{cases} A_c^2\left[S_M(j\omega - j\omega_c) + S_M(j\omega + j\omega_c)\right], & |\omega \geq \omega_c| \\ 0, & |\omega| < \omega_c \end{cases}$$

that is, the part of the corresponding sideband of two replicates power spectral density shifted $\pm\omega_c$ as shown in the figure



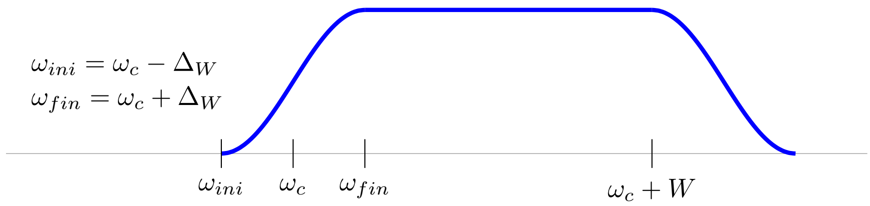
Bandwidth is B Hz or W rad/s. In this case, it is 5 kHz.

d) In this case we have a vestigial sideband modulation

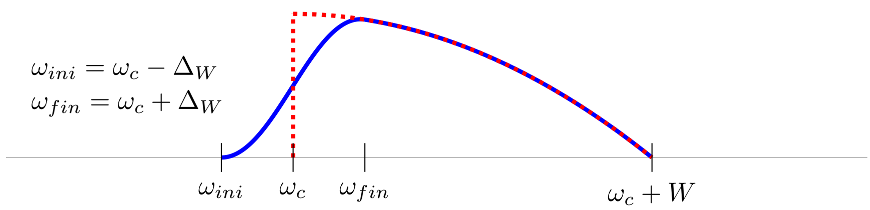
i) The frequency response must have an odd symmetry with respect to the carrier frequency, which mathematically translates into that

$$H_{BLV}(j\omega - j\omega_c) + H_{BLV}(j\omega + j\omega_c) = 1 \text{ en } |\omega| \leq W.$$

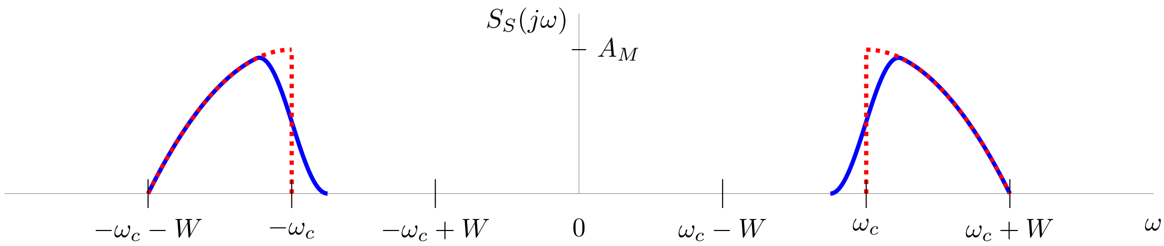
An example of this type of response would be that of the figure (positive frequencies)



ii) The power spectral density of the modulated signal, using the example filter, would be that of the figure. The dashed line superimposes the shape of the power spectral density for a single sideband modulation. Only positive frequencies are represented.



Representing both positive and negative frequencies



iii) The bandwidth of the modulated signal is

$$B_{BLV} = B + \Delta_B = 5.5 \text{ kHz}.$$

e) For angle modulations

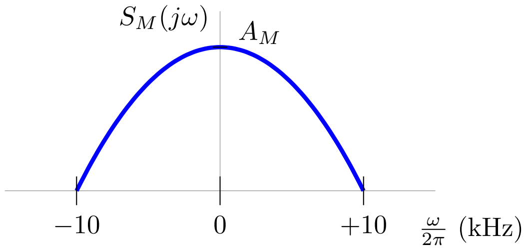
i) For narrow band modulations, the power spectral density is approximately equal to that of a conventional AM modulation but with modulation index $a = 1$, which would be the same as part (a) but with an amplitude $\frac{A_M}{4}$ for the replicas of the spectrum.

ii) In this case the bandwidth is approximated by *Carson's Rule*, which says that

$$B_{FM} \approx 2(\beta + 1)B = 2(5 + 1)B = 12B = 60 \text{ kHz}.$$

Exercise 2.15 Solution

a) The power spectral density of the modulating signal is that of the figure



so as you can see the bandwidth is 10 kHz.

The 4 modulations of the figures are identified below:

A: Single-sideband, lower-sideband, amplitude modulation with carrier frequency $f_c = 110$ kHz.

- The lower sideband (corresponding to negative carrier frequencies) appears in the spectrum, below the carrier frequency. Analytically

$$S_{S_{inf}}(j\omega) = \begin{cases} A_c^2 \left[S_M(j\omega - j\omega_c) + S_M(j\omega + j\omega_c)\right], & |\omega| \leq \omega_c \\ 0, & |\omega| > \omega_c \end{cases}$$

B: Double sideband amplitude modulation (without carrier) with carrier frequency $f_c = 100$ kHz.

- The form of the D.E.P. of the modulator centered on the carrier frequency. Analytically

$$S_S(j\omega) = \frac{A_c^2}{4} \left[S_M(j\omega - j\omega_c) + S_M(j\omega + j\omega_c)\right],$$

C: Conventional amplitude modulation (double sideband with carrier), with carrier frequency $f_c = 90$ kHz.

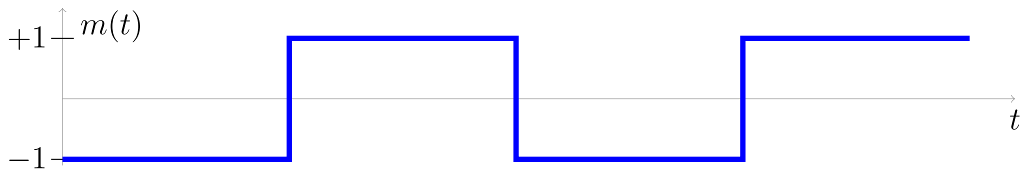
- The shape of the baseband power spectral density appears, scaled taking into account the modulation index a, centered on the carrier frequency, where the deltas corresponding to the spectrum of the carrier itself appear. Analytically

$$S_S(j\omega) = \frac{A_c^2}{2}\pi \left[\delta(\omega - \omega_c) + \delta(\omega - \omega_c)\right] + \frac{A_c^2}{4} \frac{a^2}{C_M^2} \left[S_M(j\omega - j\omega_c) + S_M(j\omega + j\omega_c)\right]$$

D: Upper sideband vestigial sideband amplitude modulation with carrier frequency $f_c = 100$ kHz.

- The upper sideband of the D.E.P. appears. of the modulator (corresponding to the positive frequencies) centered on the carrier frequency, without a trace in its initial part, and a trace of the lower sideband remains, so that the spectrum has an odd symmetry around the carrier frequency (complementing the trace that remains of the lower sideband with the one that remains in the upper sideband).

b) The modulated signal is the one shown in the figure



The 3 modulations of the figures are the following:

A: Double-sideband amplitude modulation (no carrier).

- The modulated signal is the product of the modulator with the carrier. Analytically

$$s(t) = m(t) \times c(t) = m(t) \times A_c \cos(\omega_c t).$$

B: Phase angular modulation (PM).

- It is an angular modulation, since the amplitude is constant and the modulated signal leads or lags with respect to the carrier signal. As the lead or lag depends on the sign of the modulating signal, it is a phase modulation (leads the carrier when $m(t) > 0$ and lags the carrier when $m(t) < 0$). Analytically

$$s(t) = A_c \cos(\omega_c t + \phi(t)), \text{ with } \phi(t) = k_p \times m(t).$$

C: Conventional AM modulation (double sideband, no carrier).

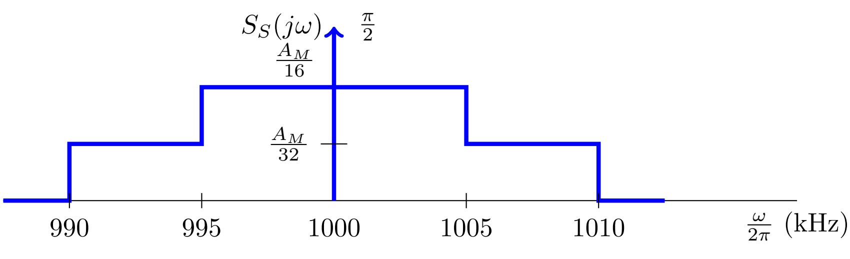
- The envelope of the carrier matches the shape of the modulating (scaled) signal. Analytically

$$s(t) = A_c[1 + a\, m_n(t)] \cos(\omega_c t), \text{ with } m_n(t) = \frac{m(t)}{c_M}$$

where c_M is the dynamic range of the modulating signal ($c_M = \max(|m(t)|)$).

Exercise 2.16 Solution

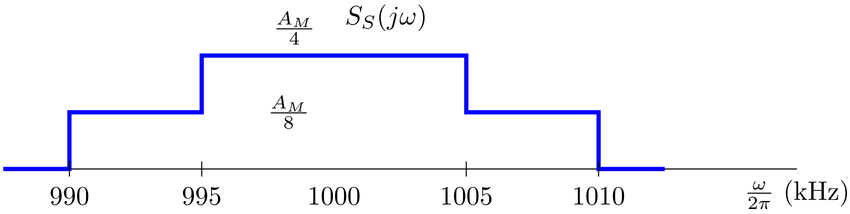
a) The power spectral density is as shown below



The bandwidth is $BW_{AM} = 20$ kHz, and the signal-to-noise ratio is

$$\left(\frac{S}{N}\right)_{AM} (\text{dB}) = 13.01 \text{ dB}.$$

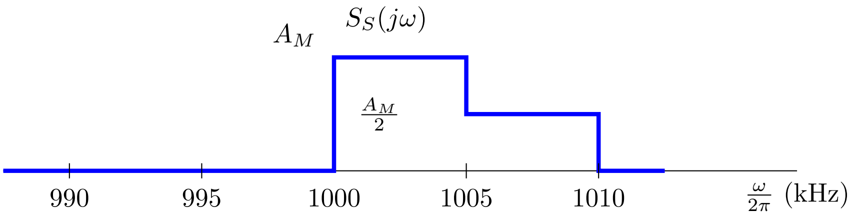
b) The power spectral density of double sideband modulation is as shown in the figure



The bandwidth is $BW_{DBL} = 20$ kHz, and the signal-to-noise ratio is

$$\left(\frac{S}{N}\right)_{DBL} (\text{dB}) = 20 \text{ dB}.$$

c) The power spectral density of the upper sideband modulation is



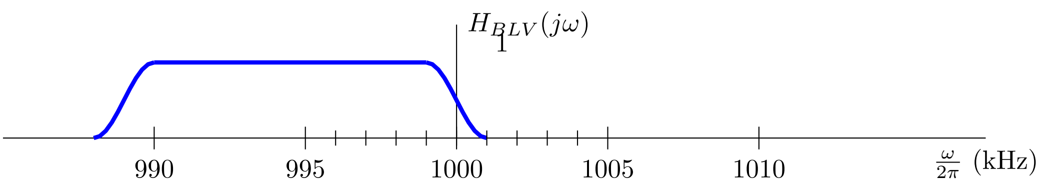
The bandwidth is $BW_{SSB} = 10$ kHz, and the signal-to-noise ratio is

$$\left(\frac{S}{N}\right)_{BLU} (\text{dB}) = 20 \text{ dB}.$$

d) The vestigial sideband filter has to have odd symmetry about the carrier frequency, to satisfy that

$$|H_{BLV}(j\omega - j\omega_c) + H_{BLV}(j\omega + j\omega)| = \text{C, in } |\omega| \leq 2\pi B$$

In this case, it must have odd symmetry with respect to $f_c = 1$ MHz, and considering that it is a lower sideband, the response must be constant between 990 and 999 kHz, with the transition from 1 to 0 with odd symmetry from 999 to 1001 kHz. A possible example would be



The bandwidth is $B_{BLV} = 11$ kHz, and the signal-to-noise ratio is

$$\left(\frac{S}{N}\right)_{BLU} (\text{dB}) = 19.586 \text{ dB}.$$

e) With the angle modulations

i) FM modulation with modulation index $\beta = 3$

$$B_{FM} \approx= 80 \text{ kHz}.$$

$$\left(\frac{S}{N}\right)_{FM} (\text{dB}) = 34.31 \text{ dB}.$$

ii) PM modulation with modulation index $\beta = 5$

$$B_{PM} \approx 120 \text{ kHz}.$$

$$\left(\frac{S}{N}\right)_{PM} (\text{dB}) = 33.98 \text{ dB}.$$