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Communication Theory

English Grades

Chapter 3

Modulation and Detection in Gaussian Channels

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Table of Contents

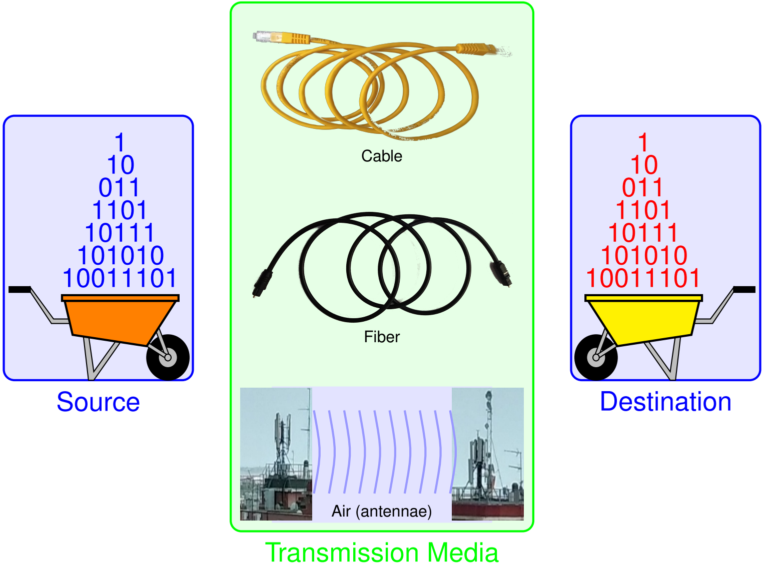
Analog vs digital communications systems

- Analog communications system
  - Designed to send the information encoded in a continuous waveform

- Digital communications system
  - Designed to send the information encoded in a sequence of symbols
    - ★ Finite alphabet: M possible values
    - ★ Most common example: Bits ($M = 2$): $\{0, 1\}$

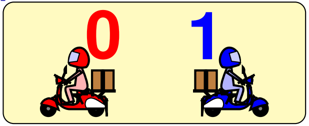
      - Information: 0110001101110011010101110010011010...
  - Transmission at a given rate (symbol rate): R_s symbols/s
    - ★ One symbol is transmitted every $T = \frac{1}{R_s}$ seconds
  - Symbols must be converted into electrical signals for transmission
    - ★ Digital Modulation: Each symbol is associated with a waveform
    - ★ Simplest case: waveforms with length $T = \frac{1}{R_s}$ seconds
- Preponderance of digital communications systems

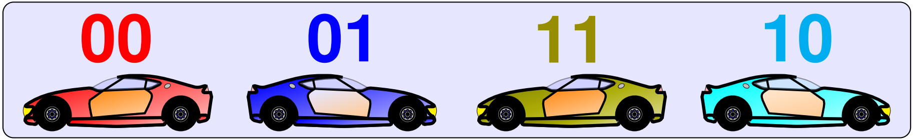
Digital modulation - Bits-to-signal conversion

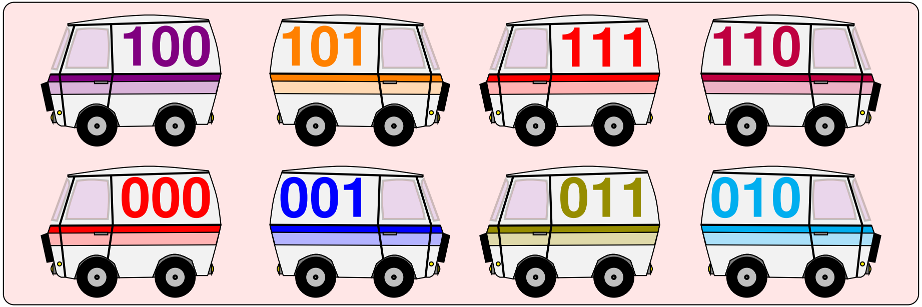


Symbols : vehicles for bit transportation

- Symbol: block of m bits
  - ⋆ M-ary systems (with $M = 2^m$)



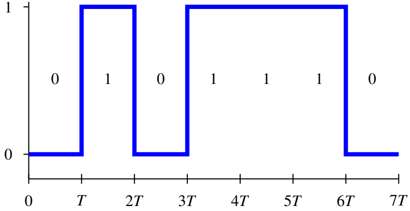




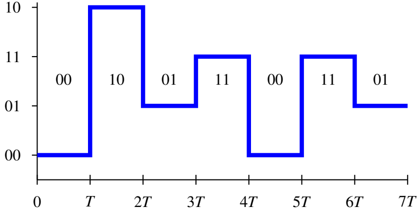
Digital modulation - Simplest example

- A block of m bits (symbol) is associated to a voltage level
  - M-ary system (with $M = 2^m$ possible symbols)

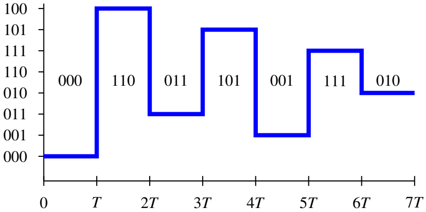
BINARY SYSTEM ($m = 1, M = 2$)



4-ARY SYSTEM ($m = 2, M = 4$)



8-ARY SYSTEM ($m = 3, M = 8$)

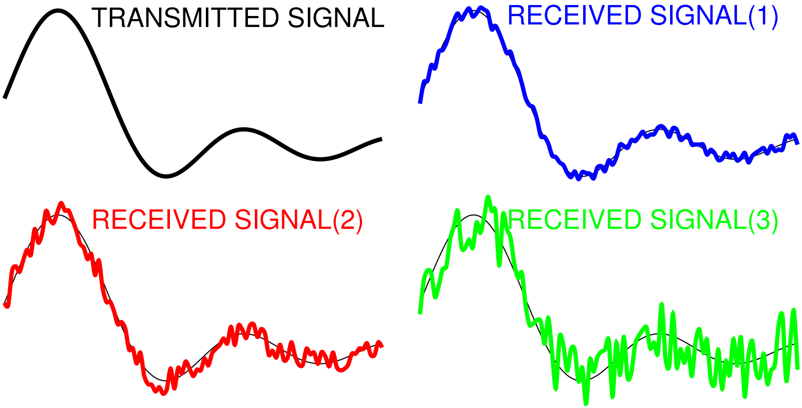


Advantages of digital systems

- **Regeneration** capability
- Existence of techniques for error detection and correction
- Channel distortion can be compensated (equalization)
  - Much easier than in analog systems
- Information can be easily encrypted
- For multiplex/medium access, CDM/CDMA and TDM/TDMA can be used (as well as FDM/FDMA)
- Information format is independent of the nature of information (voice, data, TV, etc.)
  - Nature of information: transmission rate (symbols/s, bits/s)
- In general, circuits are
  - More reliable
  - Cheaper
  - More flexible (programmable)

Distortion in analog signals

- There always exists some distortion during transmission
  - The received signal is different from the transmitted signal
  - Design: to minimize the distortion (maximum fidelity)
- Re-transmission: distortions are accumulated

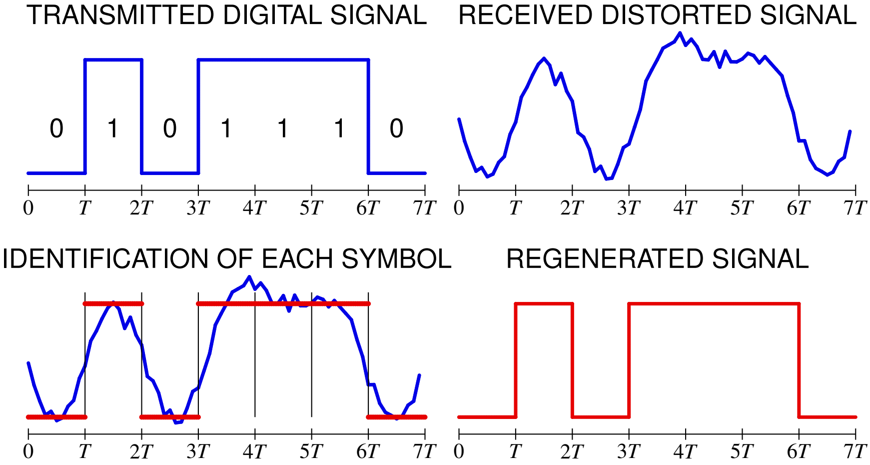


Digital regeneration

BIT ENCODING - Binary system using squared pulses

$1 \equiv$ High level

$0 \equiv$ Low level



Disadvantages of digital systems

- Need for synchronism
  - Identification of the interval for each symbol
- Higher bandwidth
  - Lower as compression techniques improve
- Many information sources are analog in nature
  - A/D conversion
    - Sampling
    - Quantization $\rightarrow$ Quantization error
  - D/A conversion
    - Interpolation
    - Low pass filtering

A/D and D/A are based on the Nyquist sampling theorem (also known as Nyquist-Shannon or Whittaker-Nyquist-Kotelnikov-Shannon theorem)

Analog to digital (A/D) conversion

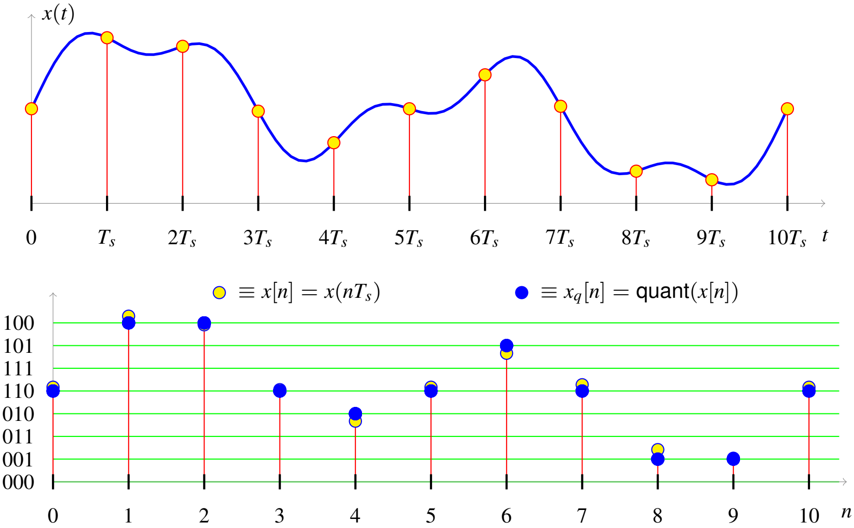
- Analog sources: continuous amplitude, continuous time
- Analog to digital (A/D) conversion:
  - Discrete time : Sampling at rate f_s samples/s
  - Discrete amplitudes: Quantization with n bits/sample
    - Quantification noise: only 2^n quantification levels
      - Difference between sampled value and quantified value
    - Lower as n increases
  - Binary rate (bits/s): R_b (bits/s) $= f_s$ (samples/s) $\times\, n$ (bits/sample)

    Digital Voice: 64 kbits/s ($B = 4$ kHz, $f_s = 2B = 8000$ samples/s, $n = 8$ bits/sample)
- Digital to analog (D/A) conversion:
  - Conversion of bits to samples (quantized)
  - Reconstruction of the signal from samples (quantized)
    - Interpolation with impulses + low pass filtering

$$x_i(t) = \sum_n x_q[n]\ \delta(t - nT_s) \rightarrow x_r(t) = x_i(t) * h_{LPF}(t)$$

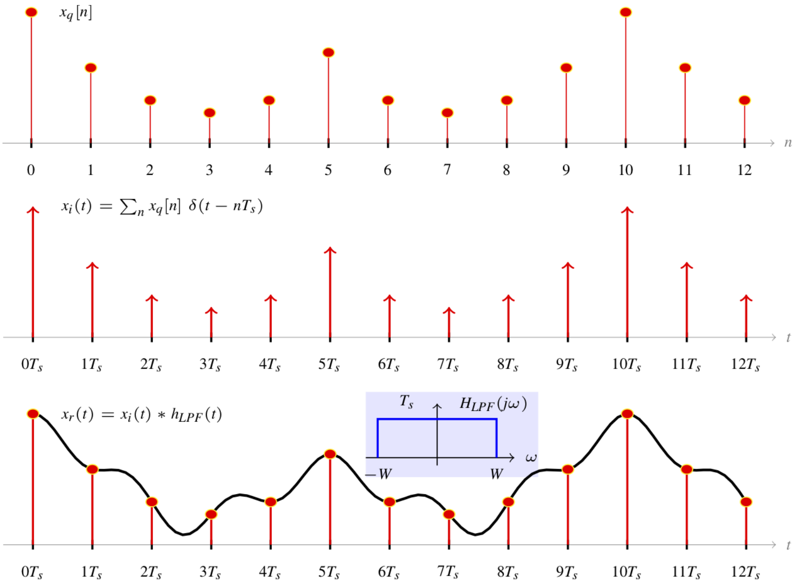
$$x_r(t) = \sum_n x_q[n]\ \mathrm{sinc}\left(\frac{t - nT_s}{T_s}\right)$$

A/D conversion: Sampling + Quantification

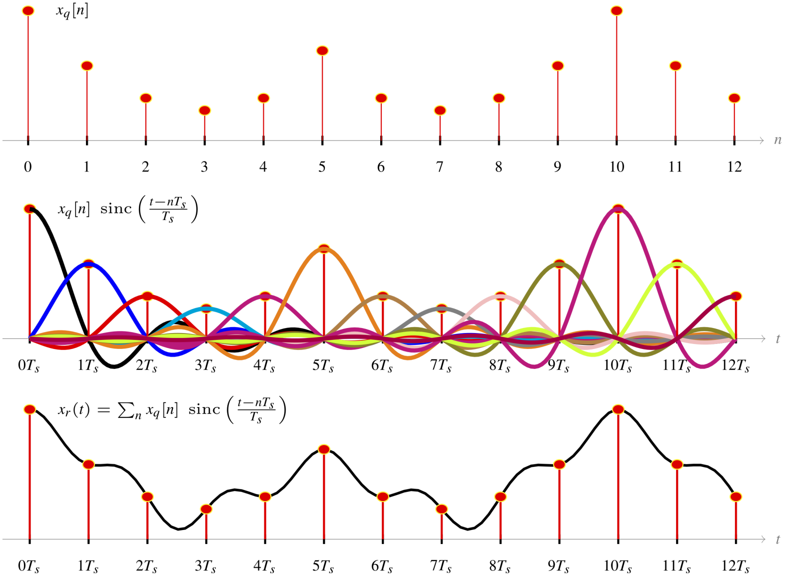


$$B_b[\ell] = 110\ 100\ 100\ 110\ 010\ 110\ 101\ 110\ 001\ 001\ 110 \cdots$$

D/A Conversion: Interpolation with impulses + Filtering



D/A Conversion: Interpolation with sincs at T_s s



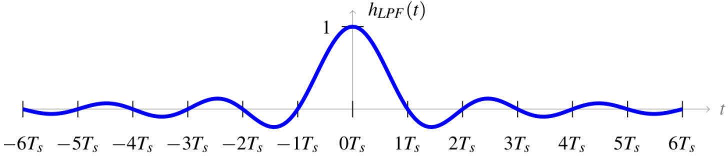
Equivalence of both options

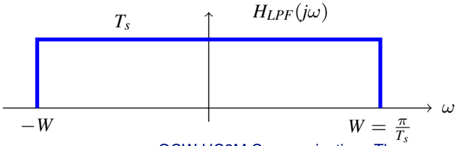
- Sampling at T_s : Sampling frequency

$$f_s = \frac{1}{T_s} = 2B \text{ Hz} \left(\omega_s = \frac{2\pi}{T_s} = 2W \text{ rad/s}\right) \text{ with } W = 2\pi B$$

$$h_{LPF}(t) = \text{sinc}\left(\frac{t}{T_s}\right) \overset{\mathcal{FT}}{\leftrightarrow} H_{LPF}(j\omega) = T_s\,\Pi\left(\frac{\omega}{\omega_s}\right) = T_s\,\Pi\left(\frac{\omega}{2W}\right)$$

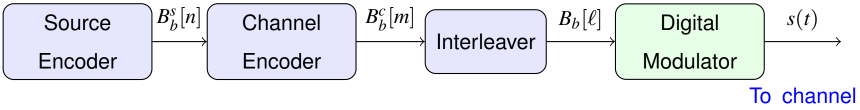
$$\text{sinc}\left(\frac{t}{T_s}\right) * \delta(t - nT_s) = \text{sinc}\left(\frac{t - nT_s}{T_s}\right)$$





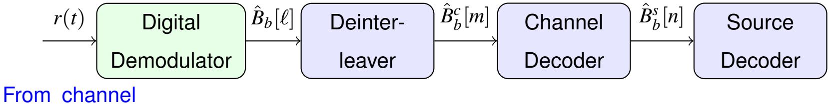
Digital Transmitter/Receiver - Basic functional blocks

- Digital transmitter



  - Digital modulator: Transmission of a sequence of symbols (typically bits, $B_b[\ell]$) through an analog communication channel (electromagnetic signal $s(t)$)

- Digital receiver



  - Digital demodulador: Recovery of the symbol sequence (bits, $\hat{B}_b[\ell]$) from signal received through channel, $r(t)$

Source and channel coding

- Source coding
  - Reduction of redundancy (compression)
  - Lower binary data rate requirements for transmission
  - Examples: MPEG or DivX (video), MP3 or OGG (audio), ZIP or RAR (files),...
- Channel coding
  - Error detection and/or correction
  - Introduction of redundancy (structured)
  - Capability of detection/correction depends on complexity
  - Simplest example: repetition code
    - ★ Repetition code of order 1: $0 \rightarrow 00$ $1 \rightarrow 11$
      - Detects 1 error over a two-bits block
    - ★ Repetition code of order 2: $0 \rightarrow 000$ $1 \rightarrow 111$
      - Detects 2 errors or corrects 1 error (correction based on majority decision) over a three-bits block

Interleaving

- Protection for burst errors
  - In combination with channel encoder
- Re-arrangement of data in a non-contiguous way
  - Goal: to transform burst error in several isolated errors
    - Channel decoder deals with relatively few errors per block
- Kinds of interleavers
  - Block interleavers
  - Convolutional interleavers

Interleaving - One example (block interleaver)

Uncoded Data

1 0 1 1 1 0

Channel Code

Repetition code (order 2)

Coded Data

111 000 111 111 111 000

Interleaved data

101 110 101 110 101 110

Transmission

101 110 101 1 10 101 1 10

Burst of errors

De-interleaved data

11 1 00 0 11 1 11 1 1 1 1 0 0 0

Isolated errors

1	0	1	1	1	0
1	0	1	1	1	0
1	0	1	1	1	0

Interleaver

$N_c \times N_b$

Block interleaver

Data input: per column

Data output: per row

1	1	1
0	0	0
1	1	1
1	1	1
1	1	1
0	0	0

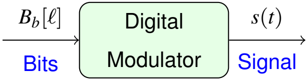
De-interleaver

$N_b \times N_c$

Design of a communication system

- Factors to consider in the design
  - Existing technologies
  - Cost
  - Quality (performance)
    - ★ Analog system: fidelity $\rightarrow$ signal-to-noise ratio (S/N)
    - ★ Digital system: bit error rate (BER)
  - Resource consumption
    - ★ Power (energy)
      - Physical limitations
      - Administrative limitations
      - Economic limitations
    - ★ Bandwidth
      - Same type of limitations
- Fundamental objective of this chapter:
  - Design of digital modulators/demodulators considering the trade-off between performance and consumption of resources

Digital modulator - Conversion of bits ($B_b[\ell]$) into a signal



- Transmission of a sequence of bits $B_b[\ell]$ at rate $R_b = \frac{1}{T_b}$ bits/s
  - Conversion into an electrical signal $s(t)$
- Block-wise bit transmission - Sequence of symbols
  - Segmentation of sequence $B_b[\ell]$ in blocks of m bits
  - Each block of m bits is a symbol
    - ★ 1 symbol $\equiv m$ bits
  - Sequence of symbols $B[n]$
    - ★ Alphabet of possible symbols: $M = 2^m$ symbols: $B[n] \in \{b_i\}_{i=0}^{M-1}$
    - ★ Symbol rate $R_s = \frac{1}{T}$ symbols/s (bauds)
    - ★ Relationship between rates R_b / R_s: $R_b = m \times R_s$ (or $T = m \times T_b$)
  - Transmission of a symbol (block of m bits) each T seg.
- Simplest conversion of symbols into a signal $s(t)$
  - Piecewise generation: "pieces" of T seconds (corresponding to 1 symbol)
    - ★ Symbol interval for $B[n]$: interval $nT \le t < (n+1)T$

Symbol-to-signal conversion - Simplest model

- Initially, the case of the first symbol is studied
  - $B \equiv B[0]$
  - Symbol interval: $0 \leq t < T$
- Symbol-to-signal conversion
  - Alphabet of M possible symbols: $B \in \{b_0, b_1, \cdots, b_{M-1}\}$
  - Definition of M waveforms of duration T seconds

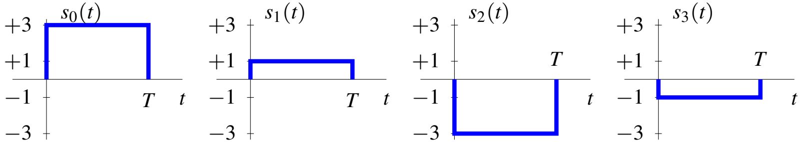
$$\{s_0(t), s_1(t), \cdots, s_{M-1}(t)\}, \text{ defined in } 0 \leq t < T$$

  - Symbol $\leftrightarrow$ waveform association: $b_i \leftrightarrow s_i(t)$
  - Generation of the signal to be transmitted
    - If $B = b_i$ then $s(t) = s_i(t)$
- Transmission of the symbol $B[n]$
  - Symbol interval: $nT \leq t < (n+1)T$
  - Symbol value: $B[n] = b_j$
    - The waveform associated to b_j is shifted to the symbol interval

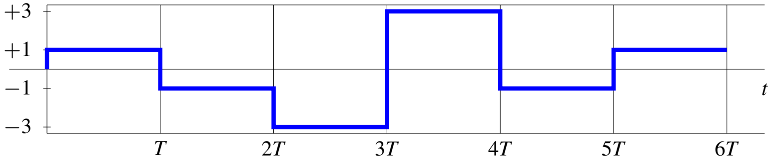
$$\text{If } B[n] = b_j \text{ then } s(t) = s_j(t - nT), \text{ in } nT \leq t < (n+1)T$$

Example A - $M = 4$

- Number of bits per symbol: $m = 2 \rightarrow M = 4$ symbols
- Symbols: $b_0 \equiv 00$, $b_1 \equiv 01$, $b_2 \equiv 10$, $b_3 \equiv 11$
- Selected signals (defined in $0 \leq t < T$)

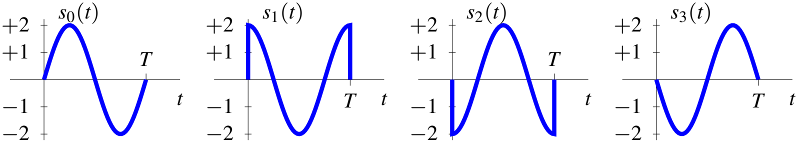


- Binary sequence: $B_b[\ell] = 011110001101 \cdots$
- Symbol sequence:
  - Segmentation of $B_b[\ell]$: $01 \mid 11 \mid 10 \mid 00 \mid 11 \mid 01 \mid \cdots$
  - Sequence $B[n] = b_1 \mid b_3 \mid b_2 \mid b_0 \mid b_3 \mid b_1 \mid \cdots$
- Transmitted signal
  - Generation by intervals: $s(t) = \{s_1(t) \mid s_3(t-T) \mid s_2(t-2T) \mid s_0(t-3T) \mid s_3(t-4T) \mid s_1(t-5T) \mid \cdots\}$

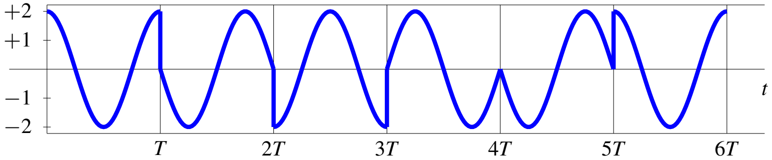


Example B - $M = 4$

- Number of bits per symbol: $m = 2 \rightarrow M = 4$ symbols
- Symbols: $b_0 \equiv 00$, $b_1 \equiv 01$, $b_2 \equiv 10$, $b_3 \equiv 11$
- Selected signals (defined in $0 \leq t < T$)

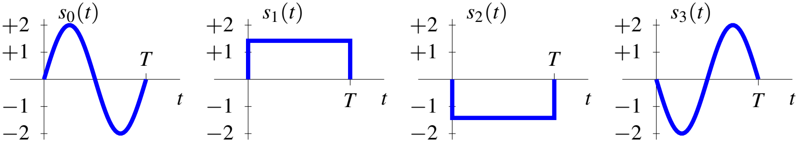


- Binary sequence: $B_b[\ell] = 011110001101\cdots$
- Symbol sequence:
  - Segmentation of $B_b[\ell]$: $01 \mid 11 \mid 10 \mid 00 \mid 11 \mid 01 \mid \cdots$
  - Sequence $B[n] = b_1 \mid b_3 \mid b_2 \mid b_0 \mid b_3 \mid b_1 \mid \cdots$
- Transmitted signal
  - Generation by intervals: $s(t) = \{s_1(t) \mid s_3(t-T) \mid s_2(t-2T) \mid s_0(t-3T) \mid s_3(t-4T) \mid s_1(t-5T) \mid \cdots\}$

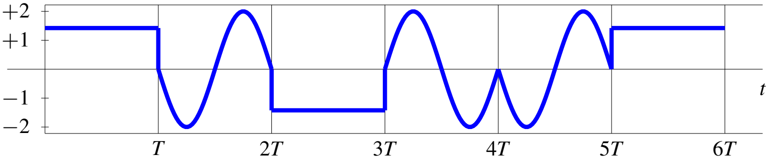


Example C - $M = 4$

- Number of bits per symbol: $m = 2 \rightarrow M = 4$ symbols
- Symbols: $b_0 \equiv 00$, $b_1 \equiv 01$, $b_2 \equiv 10$, $b_3 \equiv 11$
- Selected signals (defined in $0 \leq t < T$)

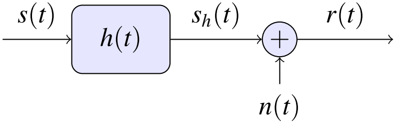


- Binary sequence: $B_b[\ell] = 011110001101 \cdots$
- Symbol sequence:
  - Segmentation of $B_b[\ell]$: $01 \mid 11 \mid 10 \mid 00 \mid 11 \mid 01 \mid \cdots$
  - Sequence $B[n] = b_1 \mid b_3 \mid b_2 \mid b_0 \mid b_3 \mid b_1 \mid \cdots$
- Transmitted signal
  - Generation by intervals: $s(t) = \{s_1(t) \mid s_3(t-T) \mid s_2(t-2T) \mid s_0(t-3T) \mid s_3(t-4T) \mid s_1(t-5T) \mid \cdots\}$



Transmission through the channel

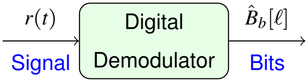
- Transmitted vs received signals (channel input/channel output): $s(t)$ vs $r(t)$
  - The signal is distorted during transmission
  - The received signal does not match the transmitted signal: $r(t) \neq s(t)$
- Channel model - Considered distortion effects
  - Linear distortion
    - Model: linear and time-invariant system: $h(t) \stackrel{\mathcal{FT}}{\leftrightarrow} H(j\omega)$
  - Thermal noise
    - Model: random process $n(t)$ stationary, ergodic, white, Gaussian, with power spectral density $S_n(t) = \frac{N_0}{2}$, where $N_0 = k \times T$(ºK)



  - Received signal

$$r(t) = s(t) * h(t) + n(t)$$

Digital demodulator



- Recovery of the bits $B_b[\ell]$ from the received signal $r(t)$
  - The signal is distorted during transmission: $r(t) \neq s(t)$
- Processing of $r(t)$ to recover the transmitted bits
  - Piecewise processing: partition into symbol intervals
  - Estimate of the symbol (m bits) transmitted in each interval
- Estimate of the n-th symbol: $\hat{B}[n]$
  - Observation of the signal in the n-th symbol interval

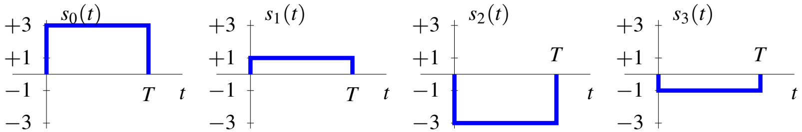
$$r(t) \text{ in } nT \leq t < (n+1)T$$

  - Comparison of the signal to the M possible waveforms

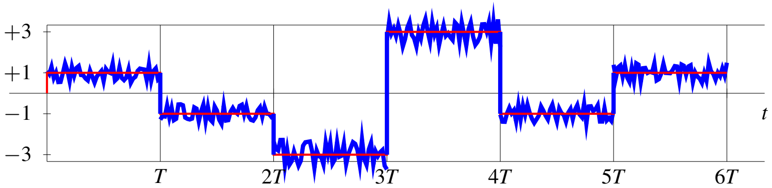
If the "*closest match*" is $s_j(t)$, then $\hat{B}[n] = b_j$

Example A - $M = 4$

- Symbols: $b_0 \equiv 00$, $b_1 \equiv 01$, $b_2 \equiv 10$, $b_3 \equiv 11$ and associated signals



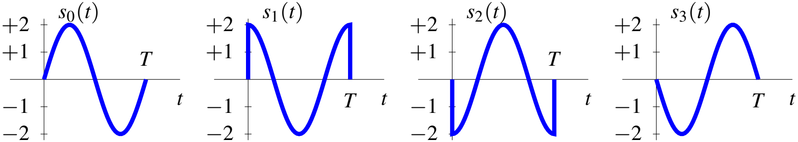
- Received signal



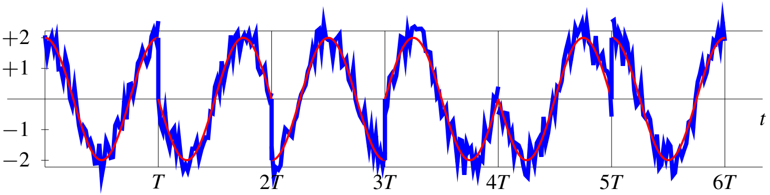
- Symbol detection
  - Segmentation of the signal in symbol intervals
    - ★ $n = 0$, interval $0 \leq t < T$ - "*most similar*" signal: $s_1(t) \rightarrow \hat{B}[0] = b_1$
    - ★ $n = 1$, interval $T \leq t < 2T$ - "*most similar*" signal: $s_3(t) \rightarrow \hat{B}[1] = b_3$
    - ★ Following the same process: $\hat{B}[2] = b_2$, $\hat{B}[3] = b_0$, $\hat{B}[4] = b_3$, $\hat{B}[5] = b_1$
- Decided sequences: $\hat{B}[n] = b_1 | b_3 | b_2 | b_0 | b_3 | b_1 | \cdots \Rightarrow \hat{B}_b[\ell]$: $01 | 11 | 10 | 00 | 11 | 01 | \cdots$

Example B - $M = 4$

- Symbols: $b_0 \equiv 00$, $b_1 \equiv 01$, $b_2 \equiv 10$, $b_3 \equiv 11$ and associated signals



- Received signal



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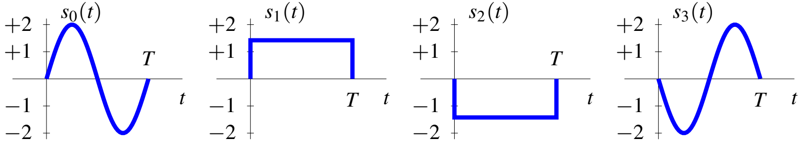
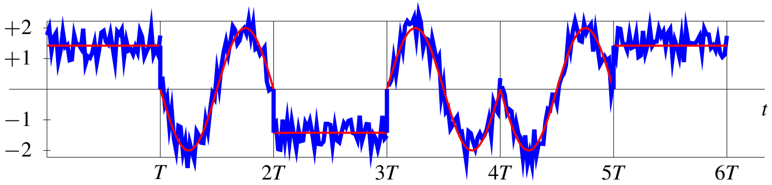
    - ★ Following the same process: $\hat{B}[2] = b_2$, $\hat{B}[3] = b_0$, $\hat{B}[4] = b_3$, $\hat{B}[5] = b_1$
- Decided sequences: $\hat{B}[n] = b_1 | b_3 | b_2 | b_0 | b_3 | b_1 | \cdots \Rightarrow \hat{B}_b[\ell]$: $01 | 11 | 10 | 00 | 11 | 01 | \cdots$

Example C - $M = 4$

- Symbols: $b_0 \equiv 00$, $b_1 \equiv 01$, $b_2 \equiv 10$, $b_3 \equiv 11$ and associated signals



- Received signal



- Symbol detection
  - Segmentation of the signal in symbol intervals
    - ★ $n = 0$, interval $0 \leq t < T$ - "*most similar*" signal: $s_1(t) \rightarrow \hat{B}[0] = b_1$
    - ★ $n = 1$, interval $T \leq t < 2T$ - "*most similar*" signal: $s_3(t) \rightarrow \hat{B}[1] = b_3$
    - ★ Following the same process: $\hat{B}[2] = b_2$, $\hat{B}[3] = b_0$, $\hat{B}[4] = b_3$, $\hat{B}[5] = b_1$
- Decided sequences: $\hat{B}[n] = b_1 | b_3 | b_2 | b_0 | b_3 | b_1 | \cdots \Rightarrow \hat{B}_b[\ell]$: $01|11|10|00|11|01|\cdots$

Selection of the M waveforms - Factors to consider

1. Performance: probability of making a mistake in the receiver (P_e)
   - Decision: most similar signal - P_e depends on the "*similarity*" between signals
   - Measure of "*similarity*" (distance): energy of the difference (square root)

$$d(s_i(t), s_k(t)) = \sqrt{\mathcal{E}\{s_i(t) - s_k(t)\}} = \sqrt{\int_{-\infty}^{\infty} |s_i(t) - s_k(t)|^2 \, dt}$$

     - ★ Reduce errors: increase the distance between signals
2. Power of the transmitted signal
   - The energy of the transmitted signal is limited in practice
   - Quantification: average energy per transmitted symbol (E_s)
     - ★ Probability of each symbol: $p_B(b_i) = P(B[n] = b_i)$
     - ★ Energy of symbol $b_i \equiv$ energy of signal $s_i(t)$
     - ★ Average energy per symbol: average of the energy of the M symbols

$$E_s = \sum_{i=0}^{M-1} p_B(b_i) \times \mathcal{E}\{s_i(t)\}, \;\; \text{where } \mathcal{E}\{s_i(t)\} = \int_{-\infty}^{\infty} |s_i(t)|^2 \, dt$$

Selection of the M waveforms - Factors to consider (II)

3. Channel adaptation (considering $h(t)$)
   - Minimize the distortion suffered by the signal in the transmission:

$$r(t) = s(t) * h(t) + n(t)$$

   - Ideal situation: linear distortion introduced by channel is null
     - Noise is the only element of distortion:

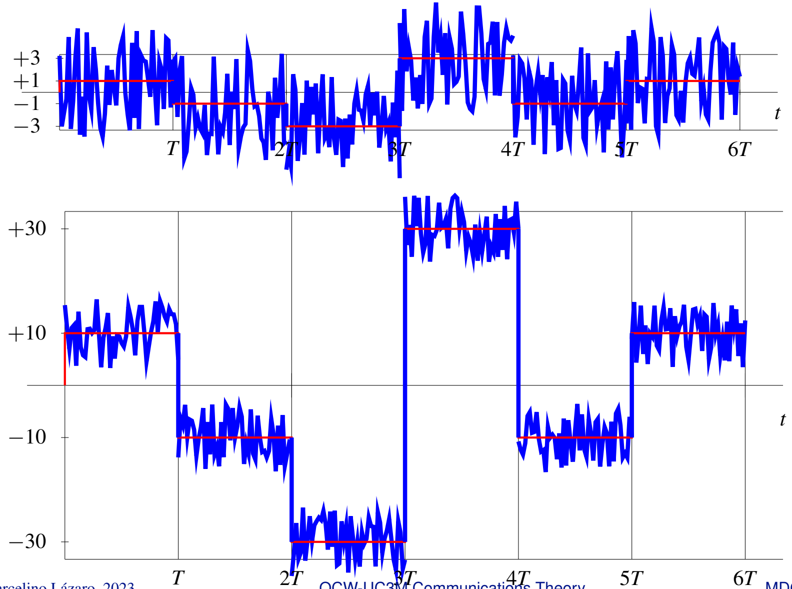
$$r(t) = s(t) + n(t)$$

     - Is achieved if:

$$s_i(t) * h(t) = s_i(t) \overset{\mathcal{FT}}{\leftrightarrow} S_i(j\omega) \times H(j\omega) = S_i(j\omega)$$

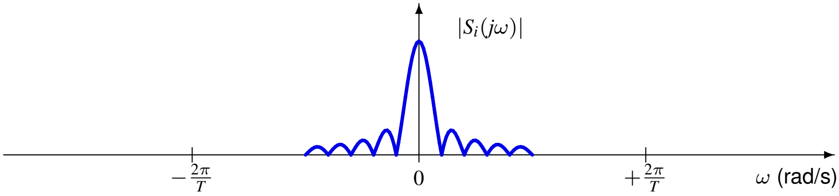
for $i = 0, 1, \cdots, M-1$

Scaling: more distance, more energy

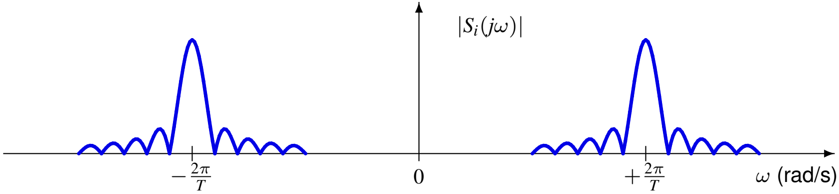


Frequency response of signals

- Rectangular pulse of duration T seconds



  - Signals suitable for transmission on channels with "*good response*" at low frequencies
- Sinusoids (sine/cosine) with $w_c = \frac{2\pi}{T}$ rad/s (one cycle in T seconds)



  - Signals suitable for transmission on channels with "*good response*" around the frequency $\frac{2\pi}{T}$ radians/s

Convenience of a vector representation of the signals

- Digital modulator design
  - Selection of the M signals that allow each block of m bits to be transmitted, considering
    - Similarity (or difference)
    - Energy
    - Channel adaptation
- Considering all together the 3 factors is a difficult problem
  - Simplified by using a vector representation of the signals
    - A signal can be represented as a vector in a Hilbert space
- Vectorial representation of the signals
  - It facilitates the calculation of the energy of each signal
  - It makes it easier to calculate the "*similarity/difference*" between signals
  - It allows to isolate these 2 factors from the adequacy of the signals to the channel

Geometric representation of signals - Vector spaces

A vector space ($\mathbb{V}$) is a set of elements (vectors) that have the following properties:

- Law of internal composition: sum ($+$)

$$\mathbf{x}, \mathbf{y}, \in \mathbb{V}, \text{ addition operation: } \mathbf{x} + \mathbf{y} \ \in \mathbb{V}$$

  which must satisfy the following properties

  a) Commutative: $\forall\, \mathbf{x}, \mathbf{y} \ \in \mathbb{V}$ $\boxed{\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}}$

  b) Associative: $\forall\, \mathbf{x}, \mathbf{y}, \mathbf{z} \ \in \mathbb{V}$ $\boxed{\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}}$

  c) Existence of neutral element (null or zero vector)

$$\exists\, \mathbf{0} \in \mathbb{V} : \forall\, \mathbf{x} \in \mathbb{V} \quad | \quad \mathbf{x} + \mathbf{0} = \mathbf{0} + \mathbf{x} = \mathbf{x}$$

  d) Existence of inverse element

$$\forall\, \mathbf{x} \in \mathbb{V}, \ \exists\, (-\mathbf{x}) \quad | \quad \mathbf{x} + (-\mathbf{x}) = \mathbf{0}$$

Vector spaces (II)

- Law of external composition: product ($\times$) with scalars ($\mathbb{C}$)

$$\alpha \in \mathbb{C},\ \mathbf{x} \in \mathbb{V},\ \text{product operation: } \alpha \times \mathbf{x} \in \mathbb{V}$$

which must satisfy the following properties

a) Associative:

$$\forall\, \alpha, \beta \in \mathbb{C}; \forall\, \mathbf{x} \in \mathbb{V}; \qquad \alpha \times (\beta \times \mathbf{x}) = (\alpha \times \beta) \times \mathbf{x}$$

b) Neutral element (existence):

$$\exists\, n_e \in \mathbb{C} : \forall\, \mathbf{x} \in \mathbb{V}; \qquad n_e \times \mathbf{x} = \mathbf{x}$$

- ★ Typically $n_e \equiv 1$

c) Distributive with respect to sum:

$$\forall\, \alpha \in \mathbb{C}; \forall\, \mathbf{x}, \mathbf{y} \in \mathbb{V}; \qquad \alpha \times (\mathbf{x} + \mathbf{y}) = \alpha \times \mathbf{x} + \alpha \times \mathbf{y}$$

d) Distributive with respect to the product with a scalar:

$$\forall\, \alpha, \beta \in \mathbb{C}; \forall\, \mathbf{x} \in \mathbb{V}; \qquad (\alpha + \beta) \times \mathbf{x} = \alpha \times \mathbf{x} + \beta \times \mathbf{x}$$

Hilbert spaces

- Hilbert space[1]
  - Vector space with an inner product (aka: dot or scalar product)

Notation: $\langle \mathbf{x}, \mathbf{y} \rangle$ - Operation $f : (\mathbb{V}, \mathbb{V}) \rightarrow \mathbb{C}$

- Properties of the inner product operation
  - **a)** $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle^*$
  - **b)** $\langle (\alpha \times \mathbf{x} + \beta \times \mathbf{y}), \mathbf{z} \rangle = \alpha \times \langle \mathbf{x}, \mathbf{z} \rangle + \beta \times \langle \mathbf{y}, \mathbf{z} \rangle$
  - **c)** $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$
  - **d)** $\langle \mathbf{x}, \mathbf{x} \rangle = 0 \Leftrightarrow \mathbf{x} = \mathbf{0}$ (zero vector)
- The inner product defines a norm for the vector space

$$||\mathbf{x}|| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$$

$$||\mathbf{x}||^2 = \langle \mathbf{x}, \mathbf{x} \rangle$$

[1] Strictly speaking, the inner product must induce a distance such that the space has the Cauchy completion property. In other case, the space is a pre-Hilbert space.

Norm for vector space

- Measure of distance between vectors

$$d(\mathbf{x},\mathbf{y}) = ||\mathbf{x}-\mathbf{y}||$$

- Angle between two vectors

$$\text{Re}\{\langle\mathbf{x},\mathbf{y}\rangle\} = ||\mathbf{x}||\ ||\mathbf{y}||\ \cos(\theta)$$

$$\theta = \cos^{-1}\left(\frac{\text{Re}\{\langle\mathbf{x},\mathbf{y}\rangle\}}{||\mathbf{x}||\ ||\mathbf{y}||}\right)$$

- Cauchy-Schwarz inequality

$$|\langle\mathbf{x},\mathbf{y}\rangle| \leq ||\mathbf{x}||\ ||\mathbf{y}|| \quad (\text{equality: } \mathbf{y} = \alpha \times \mathbf{x})$$

- The definition of the inner product is not unique
  - Each definition gives rise to a different Hilbert space

Hilbert space for continuous-time energy signals (L_2 space)

- Scalar product that defines the space L_2

$$\langle \mathbf{x}, \mathbf{y} \rangle = \int_{-\infty}^{\infty} x(t)\; y^*(t)\; dt$$

- Norm induced by this scalar product

$$||\mathbf{x}|| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \sqrt{\int_{-\infty}^{\infty} |x(t)|^2\; dt} = \sqrt{\mathcal{E}\{x(t)\}}$$

- Distance between two signals

$$d(\mathbf{x}, \mathbf{y}) = ||\mathbf{x} - \mathbf{y}|| = \sqrt{\int_{-\infty}^{\infty} |x(t) - y(t)|^2\; dt} = \sqrt{\mathcal{E}\{x(t) - y(t)\}}$$

Hilbert space for discrete-time energy signals (ℓ_2 space)

- Inner product that defines the space ℓ_2

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{n=-\infty}^{\infty} x[n]\, y^*[n]$$

- Norm induced by this scalar product

$$||\mathbf{x}|| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \sqrt{\sum_{n=-\infty}^{\infty} |x[n]|^2} = \sqrt{\mathcal{E}\{x[n]\}}$$

- Distance: Euclidean distance

$$d(\mathbf{x}, \mathbf{y}) = ||\mathbf{x} - \mathbf{y}|| = \sqrt{\sum_{n=-\infty}^{\infty} |x[n] - y[n]|^2} = \sqrt{\mathcal{E}\{x[n] - y[n]\}}$$

Cauchy-Schwarz inequality

- Cauchy-Schwarz inequality

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq ||\mathbf{x}|| \; ||\mathbf{y}||$$

- Expressions for the energy signal spaces L_2 and ℓ_2

$$\left| \int_{-\infty}^{\infty} x(t)\, y^*(t)\, dt \right| \leq \sqrt{\int_{-\infty}^{\infty} |x(t)|^2\, dt} \sqrt{\int_{-\infty}^{\infty} |y(t)|^2\, dt}$$

$$\left| \sum_{n=-\infty}^{\infty} x[n]\, y^*[n] \right| \leq \sqrt{\sum_{n=-\infty}^{\infty} |x[n]|^2} \sqrt{\sum_{n=-\infty}^{\infty} |y[n]|^2}$$

- Equality holds if the two vectors are linearly dependent (proportional)

$$\mathbf{y} = \alpha \times \mathbf{x}, \text{ for any } \alpha \in \mathbb{C}$$

  - Particulation for energy signal spaces L_2 and ℓ_2

$$y(t) = \alpha \times x(t) \text{ or } y[n] = \alpha \times x[n]$$

Representation in a basis of the vector space

- Basis for a Hilbert space $\mathbb{H}$ of dimension D: subset of D vectors

$$\{\phi_0, \phi_1, \cdots, \phi_{D-1}\} \in \mathbb{H}$$

  - Allow to represent any vector in the space as a linear combination of them

$$\mathbf{x} = \sum_{n=0}^{D-1} c_n(\mathbf{x})\ \phi_n$$

    - ★ D unique coefficients $c_n(\mathbf{x})$ ($n \in \{0, 1, \cdots, D-1\}$) for each $\mathbf{x} \in \mathbb{H}$ (coordinates)

- Orthogonal basis:

$$\langle \phi_n, \phi_m \rangle = 0,\ \forall\ n \neq m$$

- Orthonormal basis: orthogonal base with normalized elements

$$\langle \phi_n, \phi_m \rangle = 0,\ \forall\ n \neq m \text{ and also } \langle \phi_n, \phi_n \rangle = 1 \rightarrow ||\phi_n|| = 1$$

  - Coefficients in an orthonormal basis:

$$c_n(\mathbf{x}) = \langle \mathbf{x}, \phi_n \rangle$$

Gram-Schmidt orthogonalization procedure

- Objective (general)
  - To find an orthonormal basis that allows representing a set of vectors
- Objective (particular)
  - To find an orthonormal basis that allows representing a set of M signals
    - ★ Signals (M)

$$\{s_i(t)\}_{i=0}^{M-1} = \{s_0(t), s_1(t), \cdots, s_{M-1}(t)\}$$

    - ★ Orthonormal basis - N signals (dimension N) - $N \leq M$

$$\{\phi_j(t)\}_{j=0}^{N-1} = \{\phi_0(t), \phi_1(t), \cdots, \phi_{N-1}(t)\}$$

Properties of the elements of the basis

$$\langle \phi_j(t), \phi_k(t) \rangle = \int_{-\infty}^{\infty} \phi_j(t)\ \phi_k^*(t)\ dt = \begin{cases} 0, & \text{if } k \neq j \\ 1, & \text{if } k = j \end{cases} = \delta[j-k]$$

$$||\phi_j(t)|| = \sqrt{\langle \phi_j(t), \phi_j(t) \rangle} = \sqrt{\int_{-\infty}^{\infty} |\phi_j(t)|^2\ dt} = \sqrt{\mathcal{E}\{\phi_j(t)\}} = 1$$

Vector representation of $s_i(t)$

$$s_i(t) = \sum_{j=0}^{N-1} a_{i,j}\ \phi_j(t)$$

$$a_{i,j} = \langle s_i(t), \phi_j(t) \rangle$$

$$\mathbf{a}_i = \begin{bmatrix} a_{i,0} \\ a_{i,0} \\ \vdots \\ a_{i,N-1} \end{bmatrix}$$

Obtaining the basis

- Step 0: Choose $s_0(t)$ with non-zero energy

$$\phi_0(t) = \frac{s_0(t)}{\sqrt{\mathcal{E}_0}}, \qquad \mathcal{E}_0 = \mathcal{E}\{s_0(t)\} : \text{ Energy of } s_0(t)$$

- Step 1
  - Projection of $s_1(t)$ onto $\phi_0(t)$

$$a_{1,0} = \langle s_1(t), \phi_0(t) \rangle = \int_{-\infty}^{\infty} s_1(t)\ \phi_0^*(t)\ dt$$

  - Orthogonalization - This projection is subtracted

$$d_1(t) = s_1(t) - a_{1,0}\ \phi_0(t)$$

  - Normalization

$$\phi_1(t) = \frac{d_1(t)}{\sqrt{\mathcal{E}_1}}, \qquad \mathcal{E}_1 = \mathcal{E}\{d_1(t)\} = \int_{-\infty}^{\infty} |d_1(t)|^2\ dt$$

Obtaining the basis (II)

- Step k
  - Projection of $s_k(t)$ over $\{\phi_0(t), \phi_1(t), \cdots, \phi_{k-1}(t)\}$, the already available elements of the basis

$$a_{k,j} = \langle s_k(t), \phi_j(t) \rangle = \int_{-\infty}^{\infty} s_k(t)\ \phi_j^*(t)\ dt,\ j = 0, 1, \cdots, k-1$$

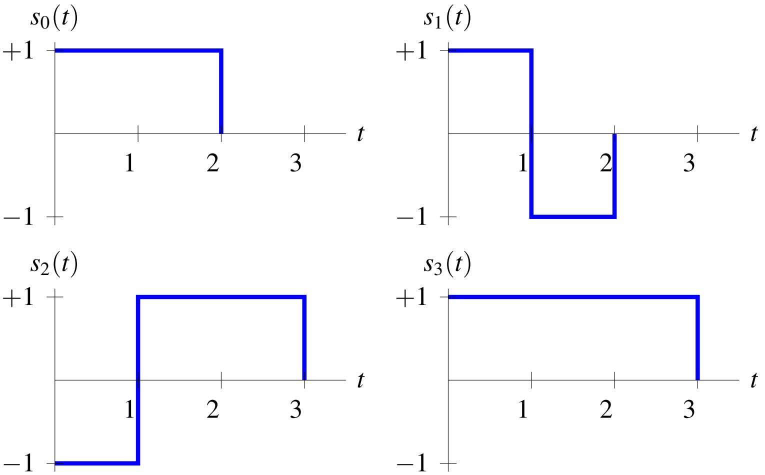
  - Orthogonalization - Subtraction of projections

$$d_k(t) = s_k(t) - \sum_{j=0}^{k-1} a_{k,j}\ \phi_j(t)$$

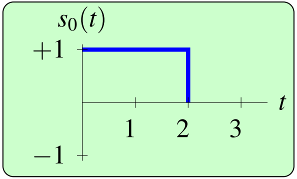
    - ★ If $d_k(t) = 0$: A new element in the basis is not necessary to represent $s_k(t)$
  - Normalization

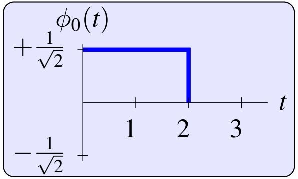
$$\phi_k(t) = \frac{d_k(t)}{\sqrt{\mathcal{E}_k}}, \qquad \mathcal{E}_k = \mathcal{E}\{d_k(t)\} = \int_{-\infty}^{\infty} |d_k(t)|^2\ dt$$

Example: Gram-Schmidt - Signals

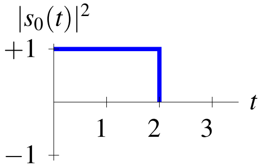


Example: Gram-Schmidt - Step 0



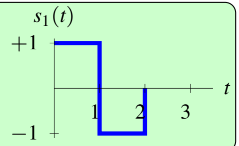


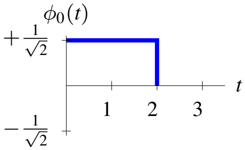
$$\mathcal{E}_0 = \mathcal{E}\{s_0(t)\} = \int_{-\infty}^{\infty} |s_0(t)|^2 \, dt = 2$$

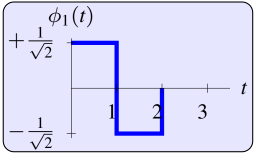


$$\phi_0(t) = \frac{s_0(t)}{\sqrt{\mathcal{E}_0}} = \frac{s_0(t)}{\sqrt{2}}$$

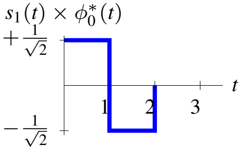
Example: Gram-Schmidt - Step 1



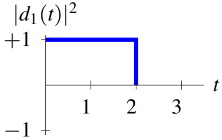




$$a_{1,0} = \langle s_1(t), \phi_0(t) \rangle = \int_{-\infty}^{\infty} s_1(t) \times \phi_0^*(t)\, dt = 0$$

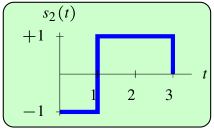


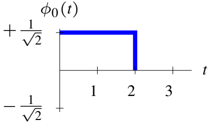
$$d_1(t) = s_1(t) - a_{1,0}\, \phi_0(t) = s_1(t) \qquad \mathcal{E}_1 = \mathcal{E}\{d_1(t)\} = \int_{-\infty}^{\infty} |d_1(t)|^2\, dt = 2$$

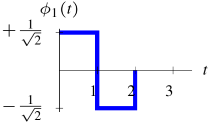


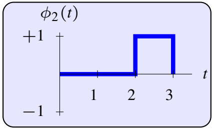
$$\phi_1(t) = \frac{d_1(t)}{\sqrt{\mathcal{E}_1}} = \frac{s_1(t)}{\sqrt{2}}$$

Example: Gram-Schmidt - Step 2

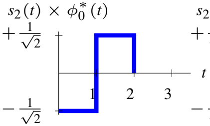


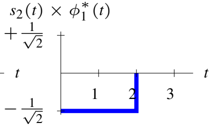




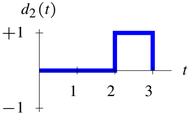


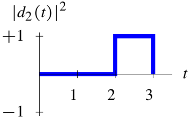
$$a_{2,0} = \langle s_2(t), \phi_0(t) \rangle = \int_{-\infty}^{\infty} s_2(t) \times \phi_0^*(t)\, dt = 0$$





$$a_{2,1} = \langle s_2(t), \phi_1(t) \rangle = \int_{-\infty}^{\infty} s_2(t) \times \phi_1^*(t)\, dt = -\sqrt{2}$$



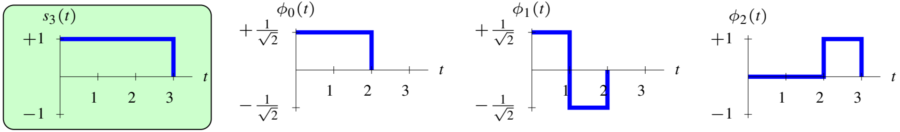


$$d_2(t) = s_2(t) - a_{2,0}\, \phi_0(t) - a_{2,1}\, \phi_1(t)$$

$$\mathcal{E}_2 = \mathcal{E}\{d_2(t)\} = \int_{-\infty}^{\infty} |d_2(t)|^2\, dt = 1$$

$$\phi_2(t) = \frac{d_2(t)}{\sqrt{\mathcal{E}_2}}$$

Example: Gram-Schmidt - Step 3

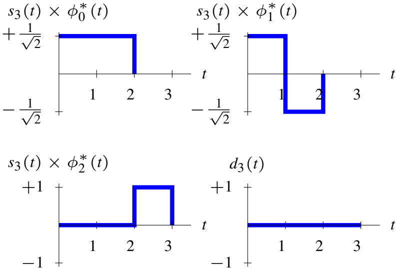


$$a_{3,0} = \langle s_3(t), \phi_0(t) \rangle = \int_{-\infty}^{\infty} s_3(t) \times \phi_0^*(t)\, dt = \sqrt{2}$$

$$a_{3,1} = \langle s_3(t), \phi_1(t) \rangle = \int_{-\infty}^{\infty} s_3(t) \times \phi_1^*(t)\, dt = 0$$

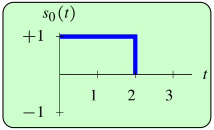
$$a_{3,2} = \langle s_3(t), \phi_2(t) \rangle = \int_{-\infty}^{\infty} s_3(t) \times \phi_2^*(t)\, dt = 1$$

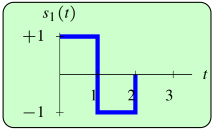
$$d_3(t) = s_3(t) - a_{3,0}\, \phi_0(t) - a_{3,1}\, \phi_1(t) - a_{3,2}\, \phi_2(t) = 0$$

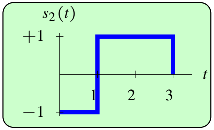


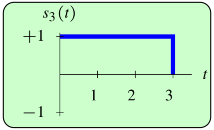
$s_3(t)$ does NOT generate a new element of the basis $\phi_3(t)$

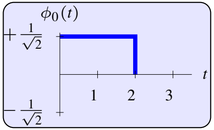
Example: Gram-Schmidt - Basis

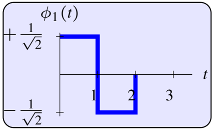


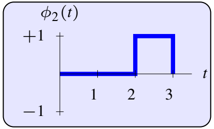












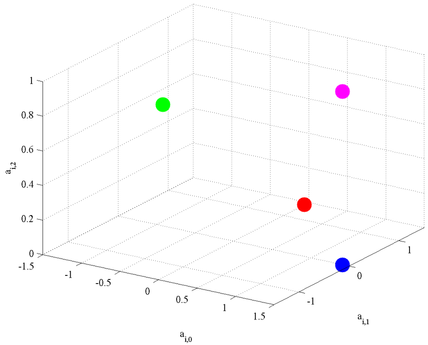
$$\mathbf{a}_i = \begin{bmatrix} a_{i,0} \\ a_{i,1} \\ a_{i,2} \end{bmatrix} \Rightarrow s_i(t) = a_{i,0}\ \phi_0(t) + a_{i,1}\ \phi_1(t) + a_{i,2}\ \phi_2(t)$$

$$\mathbf{a}_0 = \begin{bmatrix} \sqrt{2} \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{a}_1 = \begin{bmatrix} 0 \\ \sqrt{2} \\ 0 \end{bmatrix} \quad \mathbf{a}_2 = \begin{bmatrix} 0 \\ -\sqrt{2} \\ 1 \end{bmatrix} \quad \mathbf{a}_3 = \begin{bmatrix} \sqrt{2} \\ 0 \\ 1 \end{bmatrix}$$

Example: Gram-Schmidt - Coordinates

- Vector representation of the signals

$$\mathbf{a}_0 = \begin{bmatrix} \sqrt{2} \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{a}_1 = \begin{bmatrix} 0 \\ \sqrt{2} \\ 0 \end{bmatrix} \quad \mathbf{a}_2 = \begin{bmatrix} 0 \\ -\sqrt{2} \\ 1 \end{bmatrix} \quad \mathbf{a}_3 = \begin{bmatrix} \sqrt{2} \\ 0 \\ 1 \end{bmatrix}$$



Example: Gram-Schmidt - Alternative basis

- Basis

$$\phi_0'(t) = \begin{cases} 1, & \text{if } 0 \leq t < 1 \\ 0, & \text{other case} \end{cases}$$

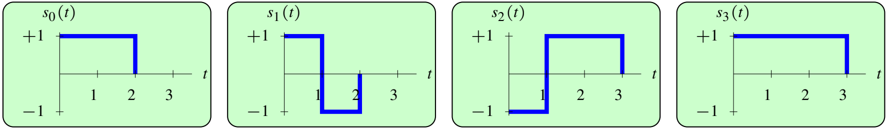
$$\phi_1'(t) = \begin{cases} 1, & \text{if } 1 \leq t < 2 \\ 0, & \text{other case} \end{cases}$$

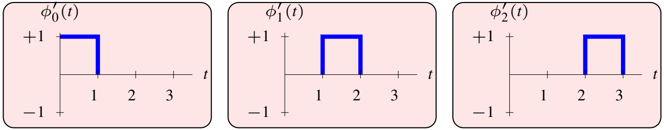
$$\phi_2'(t) = \begin{cases} 1, & \text{if } 2 \leq t < 3 \\ 0, & \text{other case} \end{cases}$$

- Coordinates in the new basis

$$\mathbf{a}_0' = \begin{bmatrix} +1 \\ +1 \\ 0 \end{bmatrix} \quad \mathbf{a}_1' = \begin{bmatrix} +1 \\ -1 \\ 0 \end{bmatrix} \quad \mathbf{a}_2' = \begin{bmatrix} -1 \\ +1 \\ +1 \end{bmatrix} \quad \mathbf{a}_3' = \begin{bmatrix} +1 \\ +1 \\ +1 \end{bmatrix}$$

Example: Gram-Schmidt - Alternative basis





$$\langle \phi'_j(t), \phi'_k(t) \rangle = \int_{-\infty}^{\infty} \phi'_j(t)\ \phi'^{*}_k(t)\ dt = \begin{cases} 0, & \text{si } k \neq j \\ 1, & \text{si } k = j \end{cases} = \delta[j-k]$$

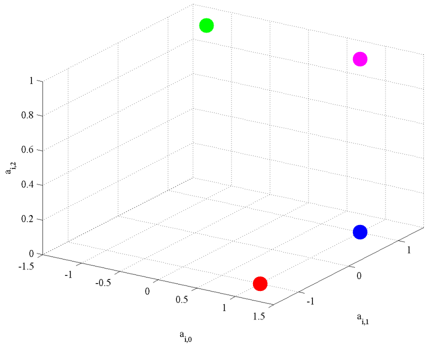
$$\mathbf{a}'_i = \begin{bmatrix} a'_{i,0} \\ a'_{i,1} \\ a'_{i,2} \end{bmatrix} \Rightarrow s_i(t) = a'_{i,0}\ \phi'_0(t) + a'_{i,1}\ \phi'_1(t) + a'_{i,2}\ \phi'_2(t)$$

$$\mathbf{a}'_0 = \begin{bmatrix} +1 \\ +1 \\ 0 \end{bmatrix} \quad \mathbf{a}'_1 = \begin{bmatrix} +1 \\ -1 \\ 0 \end{bmatrix} \quad \mathbf{a}'_2 = \begin{bmatrix} -1 \\ +1 \\ +1 \end{bmatrix} \quad \mathbf{a}'_3 = \begin{bmatrix} +1 \\ +1 \\ +1 \end{bmatrix}$$

Example: Gram-Schmidt - Coordinates (Alt. basis)

- New vector representation of the signals

$$\mathbf{a}_0' = \begin{bmatrix} +1 \\ +1 \\ 0 \end{bmatrix} \quad \mathbf{a}_1' = \begin{bmatrix} +1 \\ -1 \\ 0 \end{bmatrix} \quad \mathbf{a}_2' = \begin{bmatrix} -1 \\ +1 \\ +1 \end{bmatrix} \quad \mathbf{a}_3' = \begin{bmatrix} +1 \\ +1 \\ +1 \end{bmatrix}$$



Example: Gram-Schmidt - Energies and distances

- Energy of a signal and distances between signals are efficiently calculated from the vector representations of the signals
  - Energy of a signal

$$\mathcal{E}_i = \mathcal{E}\left\{s_i(t)\right\} = \int_{-\infty}^{\infty} |s_i(t)|^2\, dt = ||\mathbf{a}_i||^2 = \sum_{j=0}^{N-1} |a_{i,j}|^2$$

  - Distance among two signals

$$d\left(s_i(t), s_k(t)\right) = \sqrt{\int_{-\infty}^{\infty} |s_i(t) - s_k(t)|^2\, dt} = ||\mathbf{a}_i - \mathbf{a}_k|| = \sqrt{\sum_{j=0}^{N-1} |a_{i,j} - a_{k,j}|^2}$$

Example: Gram-Schmidt - Energies and distances (II)

- Coordinates in the first basis

$$\mathbf{a}_0 = \begin{bmatrix} \sqrt{2} \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{a}_1 = \begin{bmatrix} 0 \\ \sqrt{2} \\ 0 \end{bmatrix} \quad \mathbf{a}_2 = \begin{bmatrix} 0 \\ -\sqrt{2} \\ 1 \end{bmatrix} \quad \mathbf{a}_3 = \begin{bmatrix} \sqrt{2} \\ 0 \\ 1 \end{bmatrix}$$

- Coordinates in the second basis

$$\mathbf{a}'_0 = \begin{bmatrix} +1 \\ +1 \\ 0 \end{bmatrix} \quad \mathbf{a}'_1 = \begin{bmatrix} +1 \\ -1 \\ 0 \end{bmatrix} \quad \mathbf{a}'_2 = \begin{bmatrix} -1 \\ +1 \\ +1 \end{bmatrix} \quad \mathbf{a}'_3 = \begin{bmatrix} +1 \\ +1 \\ +1 \end{bmatrix}$$

- Energies and distances (independent of the chosen basis)

$$\mathcal{E}_0 = 2,\ \mathcal{E}_1 = 2,\ \mathcal{E}_2 = 3,\ \mathcal{E}_3 = 3$$

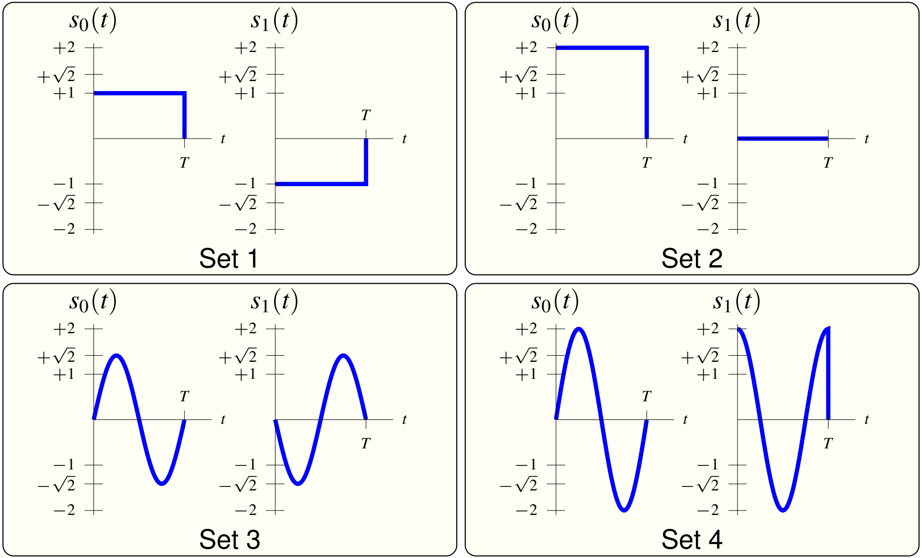
$$d(\mathbf{s}_0, \mathbf{s}_1) = 2,\ d(\mathbf{s}_0, \mathbf{s}_2) = \sqrt{5},\ d(\mathbf{s}_0, \mathbf{s}_3) = 1$$

$$d(\mathbf{s}_1, \mathbf{s}_2) = \sqrt{9},\ d(\mathbf{s}_1, \mathbf{s}_3) = \sqrt{5},\ d(\mathbf{s}_2, \mathbf{s}_3) = 2$$

Gram-Schmidt Example - Conclusions

- The orthonormal basis that allows representing the M signals is not unique
  - Any set of N orthonormal signals that allow each of the M signals to be represented exactly is valid.
- The energy of each of the signals and the distance between them will be the same for any orthonormal basis
  - The choice of one base or another will only mean a rotation of the reference system

Candidates $\{s_i(t)\}_{i=0}^{M-1}$ - Example for $M = 2$: $\{s_0(t), s_1(t)\}$



Distances between signals

$$d(s_i(t), s_j(t)) = \sqrt{\int_{-\infty}^{\infty} |s_i(t) - s_j(t)|^2 \, dt}$$

- First set

$$d(s_0(t), s_1(t)) = \sqrt{\int_0^T |1 - (-1)|^2 \, dt} = 2\sqrt{T}$$

- Second set

$$d(s_0(t), s_1(t)) = \sqrt{\int_0^T |2 - 0|^2 \, dt} = 2\sqrt{T}$$

Distances between signals

- Third set

$$d(s_0(t), s_1(t)) = \sqrt{\int_0^T \left|\sqrt{2}\sin\left(\frac{2\pi t}{T}\right) - \left(-\sqrt{2}\sin\left(\frac{2\pi t}{T}\right)\right)\right|^2 dt}$$
$$= \sqrt{\int_0^T 8\sin^2\left(\frac{2\pi t}{T}\right) dt} = \sqrt{4\left[t - \frac{T}{2\pi}\sin\left(\frac{2\pi t}{T}\right)\cos\left(\frac{2\pi t}{T}\right)\right]_0^T} = 2\sqrt{T}$$

- Fourth set

$$d(s_0(t), s_1(t)) = \sqrt{\int_0^T \left|2\sin\left(\frac{2\pi t}{T}\right) - \left(2\cos\left(\frac{2\pi t}{T}\right)\right)\right|^2 dt}$$
$$= \sqrt{\int_0^T 4 - 8\sin\left(\frac{2\pi t}{T}\right)\cos\left(\frac{2\pi t}{T}\right) dt} = 2\sqrt{T}$$

given that

$$\int_0^T 8\sin\left(\frac{2\pi t}{T}\right)\cos\left(\frac{2\pi t}{T}\right) dt = \left[\frac{2T}{\pi}\sin^2\left(\frac{2\pi t}{T}\right)\right]_0^T = 0$$

Average energy per symbol

$$E_s = E\left[\mathcal{E}\{s(t)\}\right] = E\left[\int_{-\infty}^{\infty} |s(t)|^2\, dt\right]$$
$$= \sum_{i=0}^{M-1} P\left(s(t) = s_i(t)\right) \int_{-\infty}^{\infty} |s_i(t)|^2\, dt$$

- Set 1

$$E_s = \frac{1}{2}\int_0^T |1|^2\, dt + \frac{1}{2}\int_0^T |-1|^2\, dt = \frac{1}{2}T + \frac{1}{2}T = T$$

- Set 2

$$E_s = \frac{1}{2}\int_0^T |2|^2\, dt + \frac{1}{2}\int_0^T |0|^2\, dt = \frac{1}{2}4T + \frac{1}{2}0 = 2T$$

Average energy per symbol

$$\int_0^T \cos^2\left(\frac{2\pi t}{T}\right)\, dt = \frac{T}{2\pi}\left[\frac{\pi t}{T} + \frac{1}{2}\cos\left(\frac{2\pi t}{T}\right)\sin\left(\frac{2\pi t}{T}\right)\right]_0^T = \frac{T}{2}$$

$$\int_0^T \sin^2\left(\frac{2\pi t}{T}\right)\, dt = \frac{T}{2\pi}\left[\frac{\pi t}{T} - \frac{1}{2}\cos\left(\frac{2\pi t}{T}\right)\sin\left(\frac{2\pi t}{T}\right)\right]_0^T = \frac{T}{2}$$

- Set 3

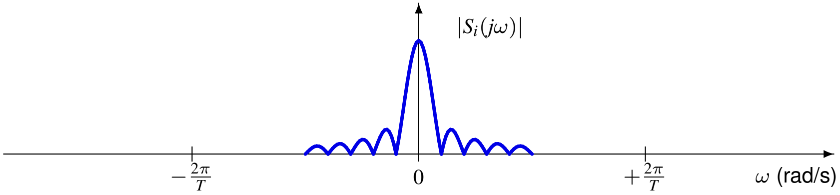
$$E_s = \frac{1}{2}\,(\sqrt{2})^2\frac{T}{2} + \frac{1}{2}\,(\sqrt{2})^2\frac{T}{2} = T$$

- Set 4

$$E_s = \frac{1}{2}\,(2)^2\frac{T}{2} + \frac{1}{2}\,(2)^2\frac{T}{2} = 2T$$

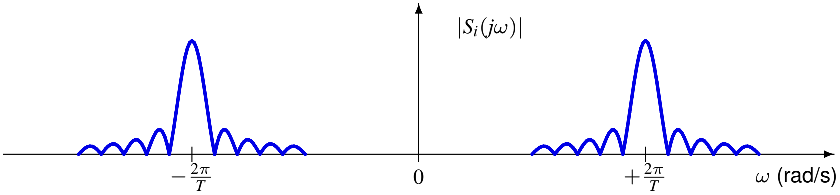
Frequency response of signals

- Set 1 and Set 2



  - Signals suitable for transmission on channels with "*good response*" at low frequencies

- Set 3 and Set 4



  - Signals suitable for transmission on channels with "*good response*" around the frequency $\frac{2\pi}{T}$ radians/s

Basis and constellation

- An orthonormal basis for the space of M signals
  - Vector representation of each signal

$$s_i(t) \leftrightarrow \mathbf{a}_i = \begin{bmatrix} a_{i,0} \\ a_{i,1} \\ \vdots \\ a_{i,N-1} \end{bmatrix}$$

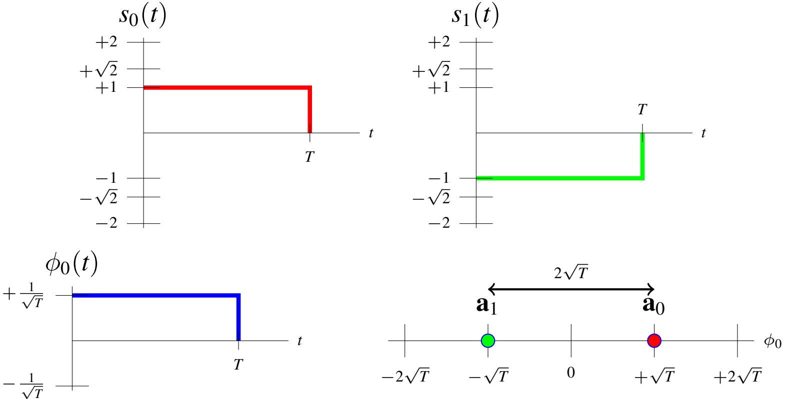
$$s_i(t) = \sum_{j=0}^{N-1} a_{i,j}\, \phi_j(t)$$

- The norm facilitates the measurement of energy and distances

$$\mathcal{E}\{s_i(t)\} = ||\mathbf{a}_i||^2 = \sum_{j=0}^{N-1} |a_{i,j}|^2$$

$$d(s_i(t), s_k(t)) = ||\mathbf{a}_i - \mathbf{a}_k|| = \sqrt{\sum_{j=1}^{N-1} |a_{i,j} - a_{k,j}|^2}$$

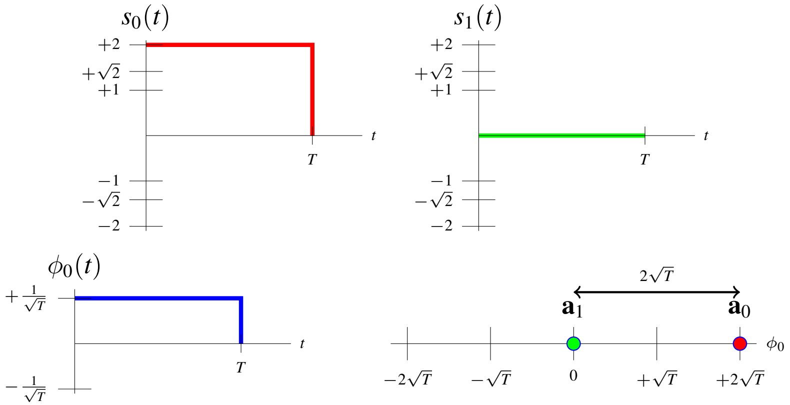
Basis and constellation - Set 1



$$\mathbf{a}_0 = [a_{0,0}] = +\sqrt{T}, \qquad \mathbf{a}_1 = [a_{1,0}] = -\sqrt{T}$$

$$s_0(t) = \mathbf{a}_0\ \phi_0(t) \qquad s_1(t) = \mathbf{a}_1\ \phi_0(t)$$

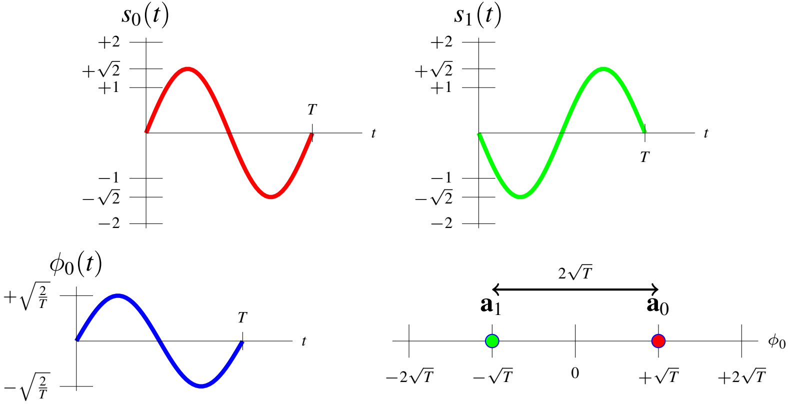
Basis and constellation - Set 2



$$\mathbf{a}_0 = [a_{0,0}] = +2\sqrt{T}, \qquad \mathbf{a}_1 = [a_{1,0}] = 0$$

$$s_0(t) = \mathbf{a}_0 \; \phi_0(t) \qquad s_1(t) = \mathbf{a}_1 \; \phi_0(t)$$

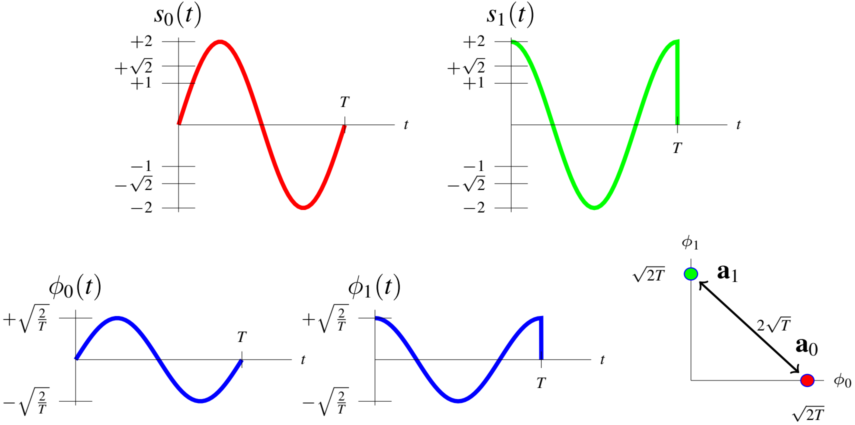
Basis and constellation - Set 3



$$\mathbf{a}_0 = [a_{0,0}] = +\sqrt{T}, \qquad \mathbf{a}_1 = [a_{1,0}] = -\sqrt{T}$$

$$s_0(t) = \mathbf{a}_0\ \phi_0(t) \qquad s_1(t) = \mathbf{a}_1\ \phi_0(t)$$

Basis and constellation - Set 4



$$\mathbf{a}_0 = \begin{bmatrix} a_{0,0} \\ a_{0,1} \end{bmatrix} = \begin{bmatrix} \sqrt{2T} \\ 0 \end{bmatrix}, \quad \mathbf{a}_1 = \begin{bmatrix} a_{1,0} \\ a_{1,1} \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{2T} \end{bmatrix}$$

$$s_0(t) = a_{0,0}\ \phi_0(t) + a_{0,1}\ \phi_1(t) \qquad s_1(t) = a_{1,0}\ \phi_0(t) + a_{1,1}\ \phi_1(t)$$

Distances between signals - Constellation

$$d(\mathbf{a}_i, \mathbf{a}_k) = ||\mathbf{a}_i - \mathbf{a}_j|| = \sqrt{\sum_{j=0}^{N-1} |a_{i,j} - a_{k,j}|^2}$$

- Set 1: $\mathbf{a}_0 = +\sqrt{T},\ \mathbf{a}_1 = -\sqrt{T}$ $\quad d(\mathbf{a}_0, \mathbf{a}_1) = \sqrt{\left|(+\sqrt{T}) - (-\sqrt{T})\right|^2} = 2\sqrt{T}$
- Set 2: $\mathbf{a}_0 = 2\sqrt{T},\ \mathbf{a}_1 = 0$ $\quad d(\mathbf{a}_0, \mathbf{a}_1) = \sqrt{\left|(+2\sqrt{T}) - (0)\right|^2} = 2\sqrt{T}$
- Set 3: $\mathbf{a}_0 = +\sqrt{T},\ \mathbf{a}_1 = -\sqrt{T}$ $\quad d(\mathbf{a}_0, \mathbf{a}_1) = \sqrt{\left|(+\sqrt{T}) - (-\sqrt{T})\right|^2} = 2\sqrt{T}$
- Set 4: $\mathbf{a}_0 = \begin{bmatrix} \sqrt{2T} \\ 0 \end{bmatrix}, \mathbf{a}_1 = \begin{bmatrix} 0 \\ \sqrt{2T} \end{bmatrix}$ $\quad d(\mathbf{a}_0, \mathbf{a}_1) = \sqrt{\left|(\sqrt{2T}) - (0)\right|^2 + \left|(0) - (\sqrt{2T})\right|^2} = 2\sqrt{T}$

Average energy per symbol - Constellation

$$E_s = E\left[\mathcal{E}\{s(t)\}\right] = \sum_{i=0}^{M-1} p_A(\mathbf{a}_i)\ \mathcal{E}\{\mathbf{a}_i\}$$
$$= \sum_{i=0}^{M-1} p_A(\mathbf{a}_i) \sum_{j=0}^{N-1} |a_{i,j}|^2$$

- Set 1 (equiprobable symbols)

$$E_s = \frac{1}{2}\left(+\sqrt{T}\right)^2 + \frac{1}{2}\left(-\sqrt{T}\right)^2 = \frac{1}{2}T + \frac{1}{2}T = T$$

- Set 2 (equiprobable symbols)

$$E_s = \frac{1}{2}\left(+2\sqrt{T}\right)^2 + \frac{1}{2}(0)^2 = \frac{1}{2}4T + \frac{1}{2}0 = 2T$$

Average energy per symbol - Constellation (II)

- Set 3 (equiprobable symbols)

$$E_s = \frac{1}{2}\left(+\sqrt{T}\right)^2 + \frac{1}{2}\left(-\sqrt{T}\right)^2 = \frac{1}{2}T + \frac{1}{2}T = T$$

- Set 4 (equiprobable symbols)

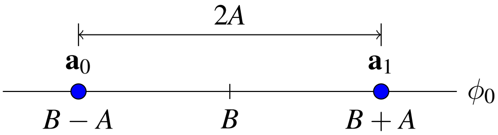
$$E_s = \frac{1}{2}\left[\left(\sqrt{2T}\right)^2 + (0)^2\right] + \frac{1}{2}\left[(0)^2 + \left(\sqrt{2T}\right)^2\right] = \frac{1}{2}2T + \frac{1}{2}2T = 2T$$

- Minimum energy for given distances between symbols
  - In this case Set 1 and Set 3 require less energy for the same distance
    - ⋆ For some fixed distances between symbols, the energy is minimized if the mean of the constellation is zero

$$E\left[\mathbf{a}_i\right] = \begin{bmatrix} E[a_{i,0}] \\ E[a_{i,1}] \\ \vdots \\ E[a_{i,N-1}] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{0}$$

Minimum E_s for some distances between symbols - Zero mean

- Example in 1D space: symbols $\mathbf{a}_0 = B - A, \quad \mathbf{a}_1 = B + A$



Mean B

Distance $2A$

- Average energy per symbol (equiprobable symbols)

$$\begin{aligned} E_s &= \frac{1}{2}\,\mathcal{E}\{\mathbf{a}_0\} + \frac{1}{2}\,\mathcal{E}\{\mathbf{a}_1\} = \frac{1}{2}\,(B-A)^2 + \frac{1}{2}\,(B+A)^2 \\ &= \frac{1}{2}\,\left(B^2 + A^2 - 2AB\right) + \frac{1}{2}\,\left(B^2 + A^2 + 2AB\right) = B^2 + A^2 \end{aligned}$$

  - Contribution of the mean: B^2
  - Contribution of the distance: A^2
- Minimum energy per symbol for a distance $2A$

Zero Mean $(B = 0) \rightarrow E_s = A^2$

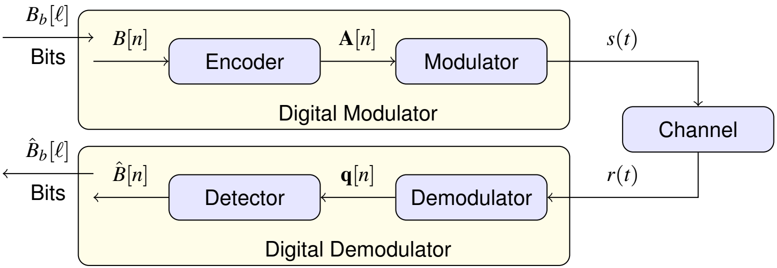
Design of the digital modulator - Selection of the M signals

- Constraints to take into account
  - Signal energy E_s
  - Distance (measure of similarity) between signals: $d(s_i(t), s_j(t))$
  - Adaptation to the characteristics of the channel: $s_i(t) * h(t) = s_i(t)$
- Discrete representation (vector) of the signals
  - Constellation of M points
    - M vectors $\mathbf{a}_i$ of dimension N (representing the M signals)
    - It allows to evaluate energies and distances (independently of the basis)
  - N-dimensional orthonormal basis
    - N orthonormal functions, $\phi_j(t)$
    - It allows to evaluate the adaptation to the channel characteristics (regardless of the constellation)

$$\text{If } \phi_j(t) * h(t) = \phi_j(t),\ \forall j,\ \text{then } s_i(t) * h(t) = s_i(t),\ \forall i$$

  - Constellation vs basis: common design contraint
    - N: Dimension of the signal space

Digital Communication Model



- The digital modulator is split into two modules
  - Encoder + Modulator
- The digital demodulator is split into two modules
  - Demodulator + Detector
- Intermediate vector representations: $\mathbf{A}[n]$ and $\mathbf{q}[n]$
  - Representation of the signals in a vector space of dimension N
  - Significantly simplifies the design of transmitter and receiver

Description of each module - Transmitter

- Encoder
  - Defines the constellation : $\{\mathbf{a}_0, \mathbf{a}_1, \cdots, \mathbf{a}_{M-1}\}$ M vectors
    - Vector representation of the signal associated with each symbol
    - Interval for index n: vector $\mathbf{A}[n]$ representing $s(t)$ in $nT \leq t < (n+1)T$
  - Design criteria (to select the constellation)
    - Energy
    - Distance ("*similarity*") between signals (performance)
- Modulator
  - Defines the orthonormal basis of the signal space:

$$\{\phi_0(t), \phi_1(t), \cdots, \phi_{N-1}(t)\} \text{ } N \text{ signals}$$

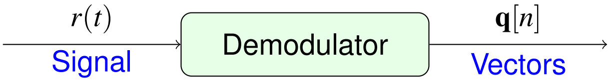
  - Design criteria (to select the N functions $\phi_k(t)$)
    - Adaptation to the characteristics of the channel

$$\phi_k(t) * h(t) = \phi_k(t) \overset{\mathcal{FT}}{\leftrightarrow} \Phi_k(j\omega)\, H(j\omega) = \Phi_k(j\omega)$$

Description of each module - Receiver

- Demodulator
  - Converts the received signal, by symbol intervals, into vectors in the signal space defined by the basis $\{\phi_k(t)\}_{k=0}^{N-1}$
    - ★ Interval for index n: vector $\mathbf{q}[n]$ representing $r(t)$ in $nT \leq t < (n+1)T$
- Detector
  - Compares the "*similarity*" between the received signal and the M possible signals $\{s_0(t), s_1(t), \cdots, s_{M-1}(t)\}$ to decide symbols
    - ★ Measure of distance on vectorial representations
    - ★ Compare the distances of:
      - Vector of the received signal in the symbol interval: $\mathbf{q}[n]$
      - Vectors of the M possible symbols: $\mathbf{a}_i$, for $i \in \{0, 1, \cdots, M-1\}$

Demodulator



- Gets the discrete-time representation of the received signal $r(t)$
  - Projection in the N-dimendional signal space of the modulator

$$\mathbf{q}[n] = \begin{bmatrix} q_0[n] \\ q_1[n] \\ \vdots \\ q_{N-1}[n] \end{bmatrix} \equiv r(t) \text{ in orthonormal basis } \{\phi_0(t), \phi_1(t), \cdots, \phi_{N-1}(t)\}$$

- Piecewise signal processing: by symbol intervals
  - Vector $\mathbf{q}[n]$: vector representation of $r(t)$ in $nT \leq t < (n+1)T$

Demodulator by correlation

- $\mathbf{q}[n]$: inner product with $\{\phi_0(t-nT), \phi_1(t-nT), \cdots, \phi_{N-1}(t-nT)\}$
  - Suport of $\phi_k(t)$:

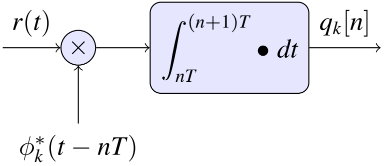
$$0 \leq t < T$$

  - Suport of $\phi_k(t-nT)$:

$$nT \leq t < (n+1)T$$

$$q_k[n] = \langle r(t), \phi_k(t-nT)\rangle = \int_{nT}^{(n+1)T} r(t)\ \phi_k^*(t-nT)\ dt$$

- Correlation: Direct implementation of the inner product



Demodulator using matched filters

$$q_k[n] = \langle r(t), \phi_k(t-nT)\rangle = \int_{nT}^{(n+1)T} r(t)\ \phi_k^*(t-nT)\ dt = \int_{nT}^{(n+1)T} r(\tau)\ \phi_k^*(\tau-nT)\ d\tau$$

- Filtering the signal with a filter with impulse response $h_k(t)$

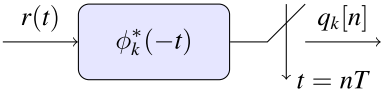
$$y_k(t) = r(t) * h_k(t) = \int_{-\infty}^{\infty} r(\tau)\ h_k(t-\tau)\ d\tau$$

  - Response of the matched filter $h_k(t) = \phi_k^*(-t)$

$$y_k(t) = \int_{-\infty}^{\infty} r(\tau)\ \phi_k^*(-(t-\tau))\ d\tau = \int_{-\infty}^{\infty} r(\tau)\ \phi_k^*(\tau-t)\ d\tau$$

  - Value of the signal $y_k(t)$ at the instant $t = nT$

$$y_k(nT) = \int_{-\infty}^{\infty} r(\tau)\ \phi_k^*(\tau-nT)\ d\tau = q_k[n]$$



Demodulator with causal matched filters

- Support of the time response: elements of the basis
  - Functions $\phi_k(t)$ are defined in $0 \leq t < T$
  - Support for matched filters $h_k(t) = \phi_k^*(-t)$
    - ★ Functions $\phi_k^*(-t)$ are defined in $-T < t \leq 0$
    - ★ NON-causal (anticausal) impulse responses
    - ★ Real implementation is NOT possible
- Real implementation of the matched filters
  - Conversion to causal response: delay of T seconds

$$h_k^T(t) = h_k(t-T) = \phi_k^*(-(t-T)) = \phi_k^*(T-t)$$

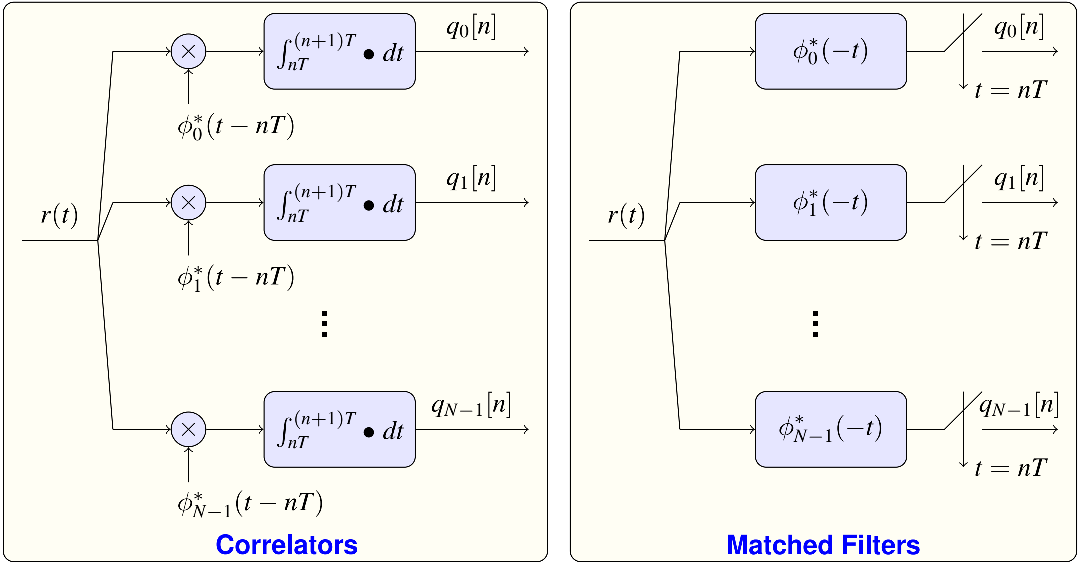
  - Inner product (to obtain coordinate $q_k[n]$)
    - ★ The causal filter $h_k^T(t)$ delays the output signal by T seconds

$$y_k^T(t) = r(t) * h_k^T(t) = y_k(t-T)$$

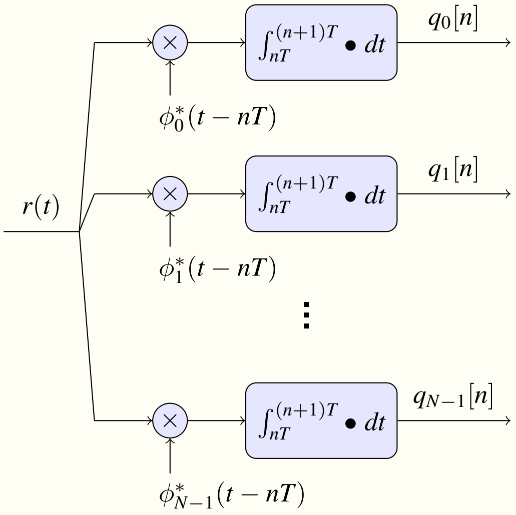
    - ★ The sampling instant must be delayed T seconds

$$q_k[n] = \langle r(t), \phi_k(t-nT)\rangle = y_k(nT) = y_k^T((n+1)T)$$

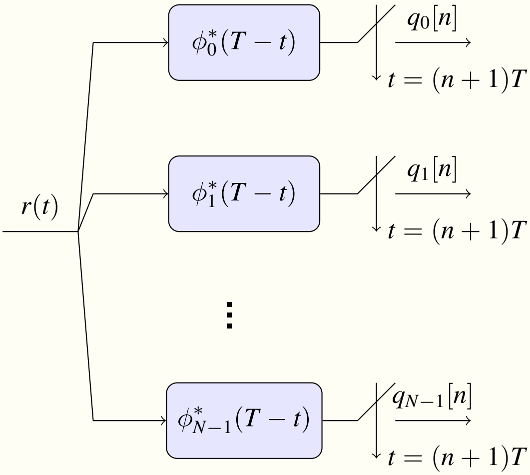
Implementation of the N-dimensional demodulator



Implementation of the N-dimensional demodulator

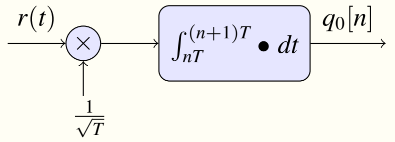


Correlators

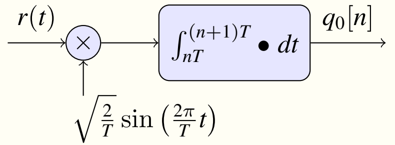


Matched Filters (Causal)

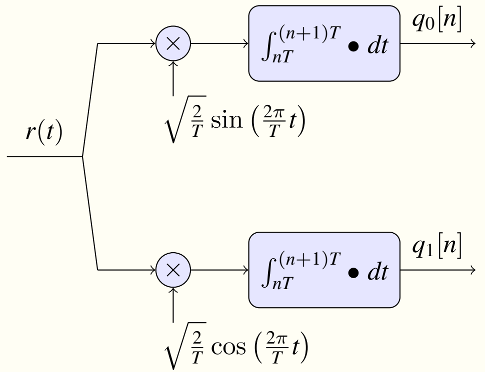
Implementation examples (Correlators)



(a) Sets 1, 2



(b) Set 3

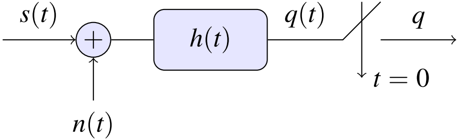


(c) Set 4

REMARK: $\sin\left(\frac{2\pi}{T} t\right) = \sin\left(\frac{2\pi}{T}(t - nT)\right)$

Property of the matched filter - Maximum S/N ratio

- Filtering a signal $s(t)$ plus noise $n(t)$ with filter $h(t)$



  - S : Energy in q due to signal $s(t)$
    - ★ $s(t)$ is a deterministic real signal
  - N : Energy in q due to noise $n(t)$
    - ★ Noise model: stationary random process, white and Gaussian, with statistics

$$S_n(j\omega) = \frac{N_0}{2},\ R_n(\tau) = \frac{N_0}{2}\delta(\tau)$$

  - Calculation of the signal to noise ratio (S/N)

$$\frac{S}{N} \equiv \frac{\text{Energy in } q \text{ due to } s(t)}{\text{Energy in } q \text{ due to } n(t)}$$

    - ★ Search for the real filter $h(t)$ that maximizes the S/N ratio

Property of the matched filter - Maximum S/N ratio (II)

- Filter output

$$q(t) = (s(t)+n(t)) * h(t) = s(t) * h(t) + n(t) * h(t) = \int_{-\infty}^{\infty} s(\tau)\, h(t-\tau)\, d\tau + \int_{-\infty}^{\infty} n(\tau)\, h(t-\tau)\, d\tau$$

- Value at instant $t=0$

$$q = q(0) = \underbrace{\int_{-\infty}^{\infty} s(\tau)\, h(-\tau)\, d\tau}_{\text{signal term}\equiv s} + \underbrace{\int_{-\infty}^{\infty} n(\tau)\, h(-\tau)\, d\tau}_{\text{noise term}\equiv z} = s + z$$

- Signal to noise ratio in q

$$\left(\frac{S}{N}\right)_q = \frac{E[|s|^2]}{E[|z|^2]}$$

  - Calculation of $E[|s|^2]$
    - Processing of $s(t)$, deterministic signal

$$E\left[|s|^2\right] = |s|^2 = \left|\int_{-\infty}^{\infty} s(\tau)\, h(-\tau)\, d\tau\right|^2 \text{ (deterministic value)}$$

  - Calculation of $E[|z|^2]$
    - Processing of $n(t)$, random signal
      Calculation of the expected value of $|z|^2$ taking into account the statistics of the noise signal $n(t)$

Property of the matched filter - Maximum S/N ratio (III)

- Statistics of $n(t)$

$$\text{Power Spectral Density } S_n(j\omega) = \frac{N_0}{2};\ \text{Autocorrelation } R_n(\tau) = \frac{N_0}{2}\delta(\tau)$$

- Calculation of $E\left[|z|^2\right]$

$$\begin{aligned}
E\left[|z|^2\right] =& E\left[\left(\int_{-\infty}^{+\infty} n(\tau)\ h(-\tau)\ d\tau\right)\left(\int_{-\infty}^{+\infty} n(\theta)\ h(-\theta)\ d\theta\right)\right] \\
=& \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \underbrace{E[n(\tau)\ n(\theta)]}_{R_n(\tau-\theta)}\ h(-\tau)\ h(-\theta)\ d\tau\ d\theta \\
=& \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \underbrace{\frac{N_0}{2}\delta(\tau-\theta)}_{R_n(\tau-\theta)}\ h(-\tau)\ h(-\theta)\ d\tau\ d\theta \\
=& \frac{N_0}{2}\int_{-\infty}^{+\infty} |h(-\tau)|^2\ d\tau = \frac{N_0}{2}\int_{-\infty}^{+\infty} |h(\tau)|^2\ d\tau = \frac{N_0}{2}\ \mathcal{E}\{h(t)\}
\end{aligned}$$

NOTE: the integral of the product of a function with a delta is

$$\int_{-\infty}^{+\infty} f(x)\ \delta(x-x_0)\ dx = f(x_0)$$

Property of the matched filter - Maximum S/N ratio (IV)

- Signal to noise ratio
$$\left(\frac{S}{N}\right)_q = \frac{|s|^2}{E[z^2]} = \frac{\left|\int_{-\infty}^{\infty} s(\tau)\ h(-\tau)\ d\tau\right|^2}{\frac{N_0}{2}\ \mathcal{E}\{h(t)\}}$$
- Cauchy-Schwarz ineq. $(s(t), h(t) \in \boldsymbol{R})$
$$\left|\int_{-\infty}^{\infty} s(\tau)\ h(-\tau)\ d\tau\right|^2 \leq \left(\int_{-\infty}^{\infty} |s(\tau)|^2\ d\tau\right)\left(\int_{-\infty}^{\infty} |h(-\tau)|^2\ d\tau\right)$$
  - Equality (maximum value) is obtained for $h(-t) = \alpha \times s(t)$, $\alpha \in \boldsymbol{R}$
- Maximum signal to noise ratio:

$$\max_{h(t)} \left(\frac{S}{N}\right)_q = \left.\frac{\left(\int_{-\infty}^{\infty} s(\tau)\ h(-\tau)\ d\tau\right)^2}{\frac{N_0}{2}\ \mathcal{E}\{h(t)\}}\right|_{h(-t)=\alpha\ s(t)}$$
$$= \frac{\left(\int_{-\infty}^{\infty} |s(\tau)|^2\ d\tau\right)\ \left(\alpha^2 \int_{-\infty}^{\infty} |s(\tau)|^2\ d\tau\right)}{\frac{N_0}{2}\ \alpha^2\ \mathcal{E}\{s(t)\}} = \frac{\mathcal{E}\{s(t)\}}{\frac{N_0}{2}}$$

Property of the matched filter - Maximum S/N ratio (IV)

Two conclusions can be drawn from this result:

1. For real signals, the signal-to-noise ratio at the output is makes maximum when $h(t) = \alpha\, s(-t)$ for any value of α (except $\alpha = 0$) and, particularly, for the matched filter

$$h(t) = s(-t)$$

For complex signals the same conclusion is reached but with the complex matched filter

$$h(t) = s^*(-t)$$

2. The signal-to-noise ratio at the output of the matched filter does not depends on the specific shape of $s(t)$
   - S/N depends on $\mathcal{E}\{s(t)\}$
   - S/N depends on N_0

Demodulator - Statistical model of $\mathbf{q}[n]$ for $\mathbf{A}[n] = \mathbf{a}_i$

- Model of the output of the demodulator $\mathbf{q}[n]$ assuming:
  - Optimal choice of modulator for the channel
    - ★ Orthonormal Basis:

$$\langle \phi_j(t), \phi_k(t) \rangle = \int_0^T \phi_j(t)\ \phi_k^*(t)\ dt = \begin{cases} 0, & \text{if } k \neq j \\ 1, & \text{if } k = j \end{cases} \equiv \delta[j-k]$$

    - ★ Adaptation to the channel $\rightarrow$ Gaussian channel:

$$\phi_j(t) * h(t) = \phi_j(t) \rightarrow r(t) = s(t) + n(t)$$

  - Transmitted symbols is

$$\mathbf{A}[n] = \mathbf{a}_i = \begin{bmatrix} a_{i,0} \\ a_{i,1} \\ \vdots \\ a_{i,N-1} \end{bmatrix} \rightarrow \underset{nT \leq t < (n+1)T}{s(t)} = \sum_{j=0}^{N-1} a_{i,j}\ \phi_j(t - nT)$$

    - ★ For ease of notation, $n = 0$. First symbol interval $0 \leq t < T$:

$$\mathbf{A} \equiv \mathbf{A}[0] = \mathbf{a}_i \rightarrow \underset{0 \leq t < T}{s(t)} = \sum_{j=0}^{N-1} a_{i,j}\ \phi_j(t)$$

Demodulator - Statistical model of $\mathbf{q}[n]$ for $\mathbf{A}[n] = \mathbf{a}_i$

- Coordinate of index k of $\mathbf{q}[n]$ (for ease of notation, $n = 0$)

$$q_k = \langle r(t), \phi_k(t)\rangle = \int_0^T r(t)\ \phi_k^*(t)\ dt = \int_0^T (s(t) + n(t))\ \phi_k^*(t)\ dt$$

$$= \int_0^T \left(\sum_{j=0}^{N-1} a_{i,j}\ \phi_j(t)\right) \phi_k^*(t)\ dt + \underbrace{\int_0^T n(t)\ \phi_k^*(t)\ dt}_{z_k}$$

$$= \sum_{j=0}^{N-1} a_{i,j} \int_0^T \phi_j(t)\ \phi_k^*(t)\ dt + z_k = \sum_{j=0}^{N-1} a_{i,j}\ \delta[j-k] + z_k = a_{i,k} + z_k$$

$$q_k = a_{i,k} + z_k$$

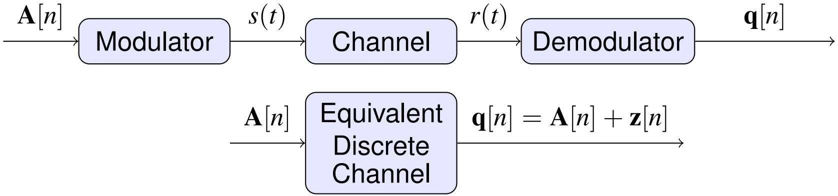
$$z_k = \int_0^T n(t)\ \phi_k^*(t)\ dt$$

Equivalent discrete channel

- Complete model of the observation $\mathbf{q}[n]$ given that $\mathbf{A}[n]=\mathbf{a}_i$

$$\mathbf{q}[n]=\begin{bmatrix} a_{i,0}\\ a_{i,1}\\ \vdots\\ a_{i,N-1}\end{bmatrix}+\begin{bmatrix} z_0\\ z_1\\ \vdots\\ z_{N-1}\end{bmatrix}=\mathbf{a}_i+\mathbf{z}$$

- Equivalent discrete channel



Equivalent discrete channel - Statistical model for $\mathbf{z}$

- Model for $n(t)$ is a Gaussian random process
- Coordinates $\{z_0, z_1, \cdots, z_{N-1}\}$ are jointly-Gaussian random variables
  - Characterization: Jointly Gaussian probability density function
  - Parameters:
    - ★ Vector of means
    - ★ Covariance matrix
- Mean of each coordinate

$$E[z_k] = E\left[\int_0^T n(t)\ \phi_k^*(t)\ dt\right] = \int_0^T \underbrace{E[n(t)]}_{m_n(t)=0}\ \phi_k^*(t)\ dt = 0$$

Equivalent discrete channel - Statistical model for $\mathbf{z}$ (II)

- Covariance between two coordinates

$$\text{Cov}(z_j, z_k) = E[z_j\ z_k^*] = E\left[\left(\int_0^T n(t)\ \phi_j^*(t)\ dt\right)\left(\int_0^T n^*(\tau)\ \phi_k(\tau)\ d\tau\right)\right]$$

$$= \int_0^T \int_0^T \underbrace{E[n(t)\ n^*(\tau)]}_{R_n(t-\tau)=\frac{N_0}{2}\ \delta(t-\tau)}\ \phi_j^*(t)\ \phi_k(\tau)\ dt\ d\tau$$

$$= \int_0^T \int_0^T \frac{N_0}{2}\ \delta(t-\tau)\ \phi_j^*(t)\ \phi_k(\tau)\ dt\ d\tau$$

$$= \frac{N_0}{2} \int_0^T \phi_j^*(t)\ \phi_k(t)\ dt = \frac{N_0}{2}\ \delta[j-k]$$

  - Random variables z_j and z_k ($k \neq j$): Uncorrelated $\rightarrow$ Independent
    - ★ They are jointly Gaussian random variables with null covariance
  - Variance of each noise component z_k: $\sigma_{z_k}^2 = \frac{N_0}{2}$

Marginal and joint distributions for $\mathbf{z}$

- Marginal distribution

$$f_{z_k}(z_k) = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{z_k^2}{N_0}}$$

  - Gaussian: zero mean and variance $\frac{N_0}{2}$ $f_{z_k}(z_k) = \mathcal{N}\left(0, \frac{N_0}{2}\right)$

- Joint distribution for $\mathbf{z} = [z_0, z_1, \cdots, z_{N-1}]^T$

$$f_{\mathbf{z}}(\mathbf{z}) = \prod_{k=0}^{N-1} f_{z_k}(z_k) = \frac{1}{(\pi N_0)^{N/2}} e^{-\sum_{k=0}^{N-1} \frac{z_k^2}{N_0}} = \frac{1}{(\pi N_0)^{N/2}} e^{-\frac{||\mathbf{z}||^2}{N_0}}$$

  - Independent jointly Gaussian (N-dimensional): zero mean and variances $\frac{N_0}{2}$

$$f_{\mathbf{z}}(\mathbf{z}) = \mathcal{N}^N\left(\mathbf{0}, \frac{N_0}{2}\right)$$

Conditional distributions for $\mathbf{q}$ given $\mathbf{A} = \mathbf{a}_i$

- Equivalent discrete channel $\boxed{\mathbf{q}[n] = \mathbf{A}[n] + \mathbf{z}[n]}$
- Distribution for each coordinate if $\mathbf{A}[n] = \mathbf{a}_i$

$$q_k = a_{i,k} + z_k, \quad \text{with } f_{z_k}(z_k) = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{z_k^2}{N_0}} \equiv \mathcal{N}\left(0, \frac{N_0}{2}\right)$$

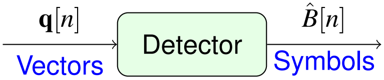
$$f_{q_k|\mathbf{A}}(q_k|\mathbf{a}_i) = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(q_k-a_{i,k})^2}{N_0}} \equiv \mathcal{N}\left(a_{i,k}, \frac{N_0}{2}\right)$$

  - Gaussian: mean $a_{i,k}$ and variance $\frac{N_0}{2}$
- Distribution of the whote observation $\mathbf{q}[n]$ given $\mathbf{A}[n] = \mathbf{a}_i$

$$f_{\mathbf{q}|\mathbf{A}}(\mathbf{q}|\mathbf{a}_i) = \prod_{k=0}^{N-1} f_{q_k|A_k}(q_k|a_{i,k}) = \frac{1}{(\pi N_0)^{N/2}} e^{-\sum_{k=0}^{N-1} \frac{(q_k-a_{i,k})^2}{N_0}}$$
$$= \frac{1}{(\pi N_0)^{N/2}} e^{-\frac{||\mathbf{q}-\mathbf{a}_i||^2}{N_0}} \equiv \mathcal{N}^N\left(\mathbf{a}_i, \frac{N_0}{2}\right)$$

  - Gaussian (N-dimensional): mean $\mathbf{a}_i$ and variances $\frac{N_0}{2}$

Detector



- Estimation of the sequence of transmitted symbols $B[n]$
  - Symbol-by-symbol decisions
  - At the discrete instant n:
    - The observation at n, i.e. $\mathbf{q}[n]$, is processed
    - The transmitted symbol at n is estimated, i.e., $\hat{B}[n]$
- Design goal
  - Minimization of the symbol error rate P_e

Decision to minimize $P_e = P(\hat{B}[n] \neq B[n])$

Detector - Decision Regions

- Alphabet of M possible values

$$B[n] \in \{b_0, b_1, \cdots, b_{M-1}\}$$

- Decision for observation $\mathbf{q}[n]$
  - Domain of $\mathbf{q}[n]$ is splitted into M disjoint regions
    - ★ Each region (I_k) is associated with a symbol (b_k)

$$\hat{B}[n] = b_k \text{ if } \mathbf{q}[n] \in I_k$$

    - ★ That is why they are called decision regions
    - ★ REMARK: there is a one-to-one association $B[n] = b_i \leftrightarrow \mathbf{A}[n] = \mathbf{a}_i$

$$\hat{B}[n] = b_k \leftrightarrow \hat{\mathbf{A}}[n] = \mathbf{a}_k$$

- Design of the decision regions
  - Split of the domain of $\mathbf{q}[n]$ to satisfy the design criterion of the detector
    - ★ Minimization of the symbol error probability P_e

Detector design

- Ease of notation: $\mathbf{q} \equiv \mathbf{q}[n] \rightarrow \hat{B} \equiv \hat{B}[n]$
- Probability of error for a specific case
  - Case: $\mathbf{q} = \mathbf{q}_0 \rightarrow \hat{B} = b_i$

$$\begin{aligned} P_e(\mathbf{q} = \mathbf{q}_0 \rightarrow \hat{B} = b_i) =& P(B \neq b_i|\mathbf{q} = \mathbf{q}_0) = 1 - P(B = b_i|\mathbf{q} = \mathbf{q}_0) \\ =& 1 - p_{B|\mathbf{q}}(b_i|\mathbf{q}_0) \end{aligned}$$

  - Conditional probability $p_{B|\mathbf{q}}(b_i|\mathbf{q}_0)$: posterior probability (*a posteriori*)
- Probability of error for a "*dumb*" detector (constant decision)
  - Decision is always $\hat{B} = b_i$, for any value of $\mathbf{q}$
    - ⋆ Average probability of error when deciding $\hat{B} = b_i$, for all possible values of $\mathbf{q}$

$$\begin{aligned} P_e(\hat{B} = b_i, \forall \mathbf{q}) =& E_{f_\mathbf{q}(\mathbf{q}_0)}\left[P_e(\mathbf{q} = \mathbf{q}_0 \rightarrow \hat{B} = b_i)\right] = \int_{-\infty}^{\infty} \left[1 - p_{B|\mathbf{q}}(b_i|\mathbf{q}_0)\right] f_\mathbf{q}(\mathbf{q}_0)\, d\mathbf{q}_0 \\ =& \int_{-\infty}^{\infty} f_\mathbf{q}(\mathbf{q}_0)\, d\mathbf{q}_0 - \int_{-\infty}^{\infty} p_{B|\mathbf{q}}(b_i|\mathbf{q}_0)\, f_\mathbf{q}(\mathbf{q}_0)\, d\mathbf{q}_0 \\ =& 1 - \int_{-\infty}^{\infty} p_{B|\mathbf{q}}(b_i|\mathbf{q}_0)\, f_\mathbf{q}(\mathbf{q}_0)\, d\mathbf{q}_0 \end{aligned}$$

Detector design (II)

- Extension to general decision: To decide $\hat{B} = b_i$ if $\mathbf{q} \in I_i$

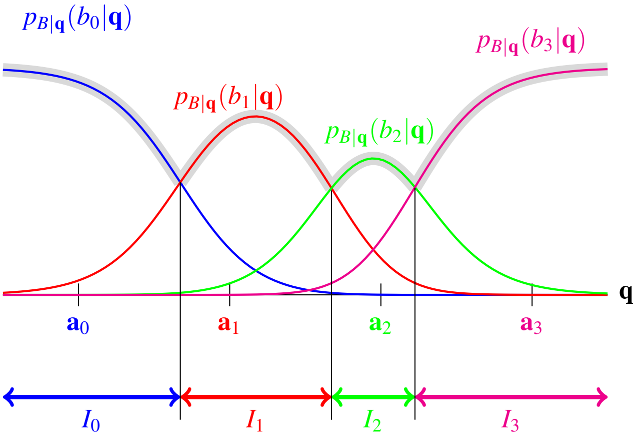
$$P_e = 1 - \sum_{i=0}^{M-1} \int_{I_i} p_{B|\mathbf{q}}(b_i|\mathbf{q}_0)\, f_{\mathbf{q}}(\mathbf{q}_0)\, d\mathbf{q}_0$$

- Minimization of the probability of error
  - Minimized by maximizing the second term
  - $f_{\mathbf{q}}(\mathbf{q}_0)$ is independent of the symbol that is transmitted or decided
  - Given $\mathbf{q}_0$ the term $p_{B|\mathbf{q}}(b_i|\mathbf{q}_0)$ can be chosen
    - ★ This is equivalent to associate $\mathbf{q}_0$ to a decision region I_k ($k \in \{0, 1, ..., M-1\}$)
- Layout of the decision regions : MAXIMUM A POSTERIORI CRITERION
  - Assignment of $\mathbf{q} = \mathbf{q}_0$ to the decision region I_i that maximizes $p_{B|\mathbf{q}}(b_i|\mathbf{q}_0)$

$$\mathbf{q}_0 \in I_i \text{ if } p_{B|\mathbf{q}}(b_i|\mathbf{q}_0) > p_{B|\mathbf{q}}(b_j|\mathbf{q}_0), \quad \forall\, j \neq i$$

  - In the case: $p_{B|\mathbf{q}}(b_i|\mathbf{q}_0) = p_{B|\mathbf{q}}(b_k|\mathbf{q}_0) > p_{B|\mathbf{q}}(b_j|\mathbf{q}_0), \forall\, j \neq \{i, k\}$
    - ★ Arbitrary assignment of $\mathbf{q}_0$ to I_i or I_k

Decision regions for minimal P_e - An example



Maximum A Posteriori (MAP) Criterion - Development

- Posterior probabilities $p_{B|\mathbf{q}}(b_i|\mathbf{q}_0)$ - Bayes' Rule

$$p_{B|\mathbf{q}}(b_i|\mathbf{q}_0) = \frac{p_B(b_i)\, f_{\mathbf{q}|B}(\mathbf{q}_0|b_i)}{f_{\mathbf{q}}(\mathbf{q}_0)}$$

  - Taking into account that $B = b_j$ implies that $\mathbf{A} = \mathbf{a}_j$ and vice versa

$$f_{\mathbf{q}|B}(\mathbf{q}_0|b_i) \equiv f_{\mathbf{q}|\mathbf{A}}(\mathbf{q}_0|\mathbf{a}_i)$$

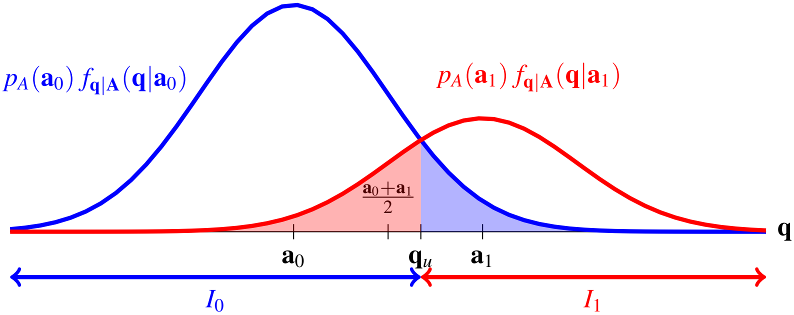
- MAP criterion: $\mathbf{q}_0$ is assigned to I_i if

$$\frac{p_B(b_i)\, f_{\mathbf{q}|\mathbf{A}}(\mathbf{q}_0|\mathbf{a}_i)}{f_{\mathbf{q}}(\mathbf{q}_0)} > \frac{p_B(b_j)\, f_{\mathbf{q}|\mathbf{A}}(\mathbf{q}_0|\mathbf{a}_j)}{f_{\mathbf{q}}(\mathbf{q}_0)} \quad j = 0, \cdots, M-1,\ j \neq i$$

Since $f_{\mathbf{q}}(\mathbf{q}_0)$ is a non-negative quantity

$$\mathbf{q}_0 \in I_i \text{ if } \begin{cases} p_B(b_i)\, f_{\mathbf{q}|\mathbf{A}}(\mathbf{q}_0|\mathbf{a}_i) > p_B(b_j)\, f_{\mathbf{q}|\mathbf{A}}(\mathbf{q}_0|\mathbf{a}_j) \\ p_{\mathbf{A}}(\mathbf{a}_i)\, f_{\mathbf{q}|\mathbf{A}}(\mathbf{q}_0|\mathbf{a}_i) > p_{\mathbf{A}}(\mathbf{a}_j)\, f_{\mathbf{q}|\mathbf{A}}(\mathbf{q}_0|\mathbf{a}_j) \end{cases} \quad \forall j \neq i$$

MAP Criterion: Example for binary case and Gaussian $f_{\mathbf{q}|\mathbf{A}}(\mathbf{q}_0|\mathbf{a}_i)$

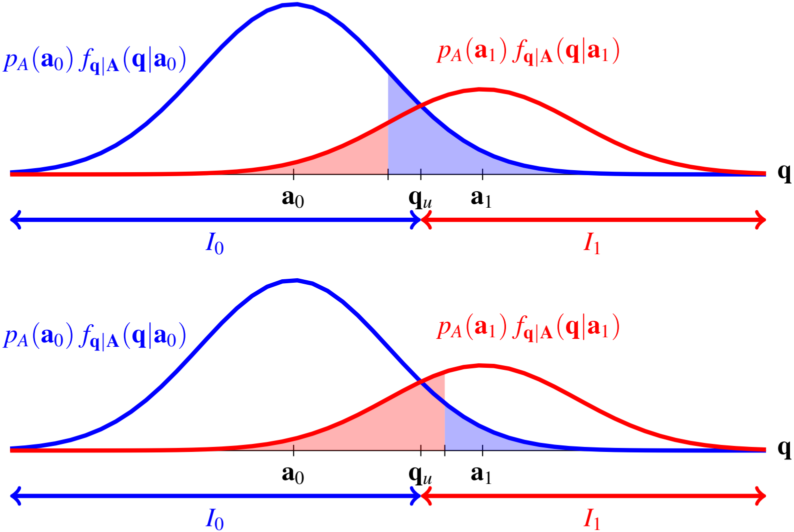


$$\mathbf{q}_u = \frac{\mathbf{a}_0 + \mathbf{a}_1}{2} + \frac{N_0}{2} \frac{\ln \frac{p_A(\mathbf{a}_0)}{p_A(\mathbf{a}_1)}}{(\mathbf{a}_1 - \mathbf{a}_0)}, \;\; I_0 = (-\infty, \mathbf{q}_u), \;\; I_1 = [\mathbf{q}_u, \infty)$$

$$P_e = p_A(\mathbf{a}_0) \int_{I_1} f_{\mathbf{q}|\mathbf{A}}(\mathbf{q}|\mathbf{a}_0)\, dq + p_A(\mathbf{a}_1) \int_{I_0} f_{\mathbf{q}|\mathbf{A}}(\mathbf{q}|\mathbf{a}_1)\, dq$$

MAP Criterion: Example for binary case and Gaussian $f_{\mathbf{q}|\mathbf{A}}(\mathbf{q}_0|\mathbf{a}_i)$ (II)

- Any other decision threshold will increase the probability of error



Maximum Likelihood (ML) criterion

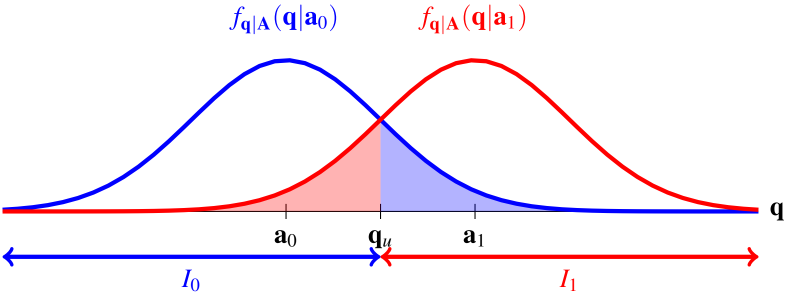
- ML is valid when the symbols are equiprobable

$$p_B(b_i) = p_{\mathbf{A}}(\mathbf{a}_i) = \frac{1}{M}, \ \forall i$$

- In that case, $\mathbf{q}_0$ is assigned to region I_i if

$$f_{\mathbf{q}|\mathbf{A}}(\mathbf{q}_0|\mathbf{a}_i) > f_{\mathbf{q}|\mathbf{A}}(\mathbf{q}_0|\mathbf{a}_j) \quad \forall j \neq i$$

ML Criterion: Example for binary case and Gaussian $f_{\mathbf{q}|\mathbf{A}}(\mathbf{q}_0|\mathbf{a}_i)$



$$\mathbf{q}_u = \frac{\mathbf{a}_0 + \mathbf{a}_1}{2},\ I_0 = (-\infty, \mathbf{q}_u),\ I_1 = [\mathbf{q}_u, \infty)$$

$$P_e = \frac{1}{2}\int_{I_1} f_{\mathbf{q}|\mathbf{A}}(\mathbf{q}|\mathbf{a}_0)\ dq + \frac{1}{2}\int_{I_0} f_{\mathbf{q}|\mathbf{A}}(\mathbf{q}|\mathbf{a}_1)\ dq$$

Minimum Euclidean Distance criterion: Gaussian $f_{\mathbf{q}|\mathbf{A}}(\mathbf{q}|\mathbf{a}_i)$ and $p_B(b_i) = 1/M$

- ML Criterion with Gaussian $f_{\mathbf{q}|\mathbf{A}}(\mathbf{q}|\mathbf{a}_i)$: $\mathbf{q}_0$ belongs to I_i if

$$\frac{1}{(\pi N_0)^{N/2}} e^{-\frac{||\mathbf{q}_0 - \mathbf{a}_i||^2}{N_0}} > \frac{1}{(\pi N_0)^{N/2}} e^{-\frac{||\mathbf{q}_0 - \mathbf{a}_j||^2}{N_0}} \quad \forall j \neq i$$

$$e^{-\frac{||\mathbf{q}_0 - \mathbf{a}_i||^2}{N_0}} > e^{-\frac{||\mathbf{q}_0 - \mathbf{a}_j||^2}{N_0}} \quad \forall j \neq i$$

  - Exponential is a monotonic increasing function $e^a > e^b \Leftrightarrow a > b$

$$||\mathbf{q}_0 - \mathbf{a}_i||^2 < ||\mathbf{q}_0 - \mathbf{a}_j||^2 \quad \forall j \neq i$$

  - Applying the definition of the norm of a vector

$$||\mathbf{q}_0 - \mathbf{a}_i||^2 = \sum_{k=0}^{N-1} |q_{0,k} - a_{i,k}|^2 = |d(\mathbf{q}_0, \mathbf{a}_i)|^2$$

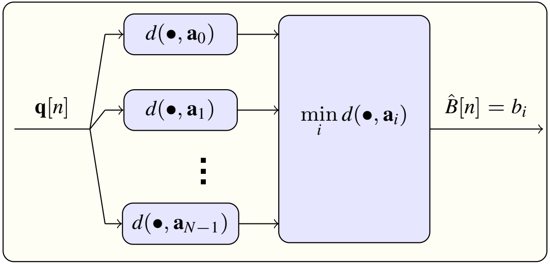
  - Minimum Euclidean Distance (MED) criterion

$$\mathbf{q} \in I_i \text{ if } d(\mathbf{q}_0, \mathbf{a}_i) < d(\mathbf{q}_0, \mathbf{a}_j), \ \forall j \neq i$$

Minimum Euclidean Distance detector

- Equiprobable symbols
- $f_{\mathbf{q}|\mathbf{A}}(\mathbf{q}|\mathbf{a}_i)$ has a Gaussian distribution (Gaussian noise)

  NOTE: Actually, it will also hold for any distribution that is symmetric with respect to the origin (even function) and decreasing in the 1-D case, since in that case two distributions with different mean " *will intersect*" at the midpoint between both means, or for decreasing radial basis functions in the case N-D



Calculation of the symbol error rate

- When the symbol $B[n] = b_i$ (or $\mathbf{A}[n] = \mathbf{a}_i$) is transmitted
  - Distribution of the observation $f_{\mathbf{q}|\mathbf{A}}(\mathbf{q}|\mathbf{a}_i)$
  - Conditional error probability: $P_{e|B=b_i} = P_{e|\mathbf{A}=\mathbf{a}_i} \equiv P_{e|\mathbf{a}_i}$
    - ⋆ Error occurs if $\hat{B}[n] = b_k \neq b_i$
    - ⋆ This occurs when $\mathbf{q} \notin I_i$

$$P_{e|\mathbf{a}_i} = \int_{\mathbf{q}\notin I_i} f_{\mathbf{q}|\mathbf{A}}(\mathbf{q}|\mathbf{a}_i)\, d\mathbf{q}$$

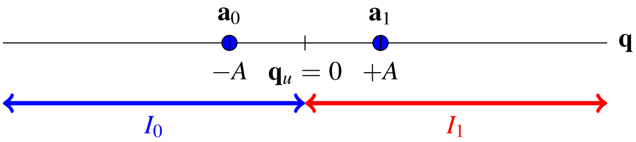
- Total error probability
  - Conditional error probabilities are averaged

$$P_e = \sum_{i=0}^{M-1} p_{\mathbf{A}}(\mathbf{a}_i)\, P_{e|\mathbf{a}_i}$$

    - ⋆ For equiprobable symbols $p_A(\mathbf{a}_i) = \frac{1}{M} \rightarrow P_e = \frac{1}{M}\sum_{i=0}^{M-1} P_{e|\mathbf{a}_i}$

Example

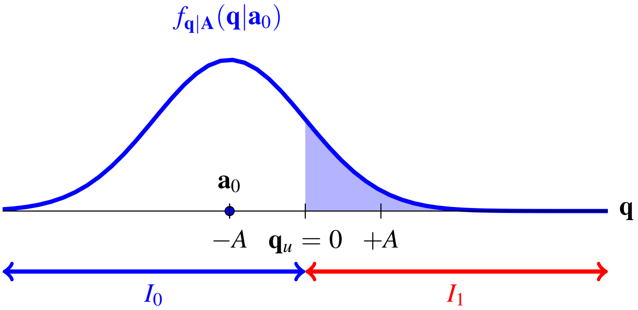
- One-dimensional ($N = 1$) and binary ($M = 2$) case
  - Constellation $\mathbf{a}_0 = -A, \ \mathbf{a}_1 = +A$
  - Equiprobable symbols $p_{\mathbf{A}}(\mathbf{a}_0) = p_{\mathbf{A}}(\mathbf{a}_1) = \frac{1}{2}$
  - Decision regions Threshold $\mathbf{q}_u = 0 \rightarrow I_0 = (-\infty, 0), \ I_1 = [0, \infty)$



- Error probability

$$\begin{aligned} P_e =& p_{\mathbf{A}}(\mathbf{a}_0)\ P_{e|\mathbf{a}_0} + p_{\mathbf{A}}(\mathbf{a}_1)\ P_{e|\mathbf{a}_1} \\ =& \frac{1}{2}\ P_{e|\mathbf{a}_0} + \frac{1}{2}\ P_{e|\mathbf{a}_1} \end{aligned}$$

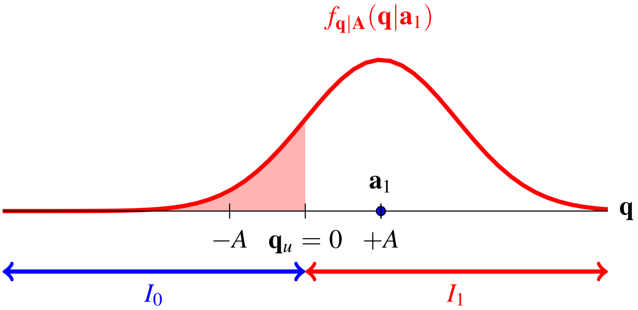
Probability of error $B = b_0$



- Gaussian distribution $f_{\mathbf{q}|\mathbf{A}}(\mathbf{q}|\mathbf{a}_0)$ with mean $+A$ and variance $N_0/2$

$$P_{e|\mathbf{a}_0} = \int_{\mathbf{q}\notin I_0} f_{\mathbf{q}|\mathbf{A}}(\mathbf{q}|\mathbf{a}_0)\, d\mathbf{q} = Q\left(\frac{A}{\sqrt{N_0/2}}\right)$$

Probability of error $B = b_1$

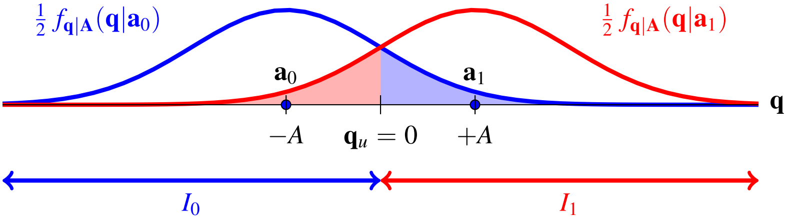


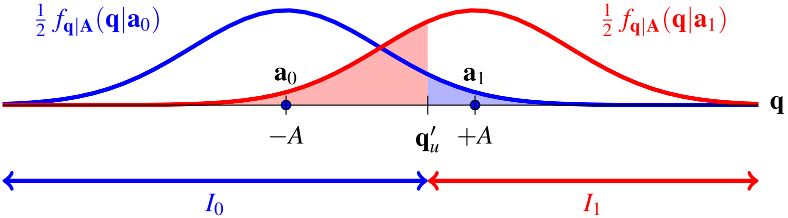
- Gaussian distribution $f_{\mathbf{q}|\mathbf{A}}(\mathbf{q}|\mathbf{a}_1)$ with mean $-A$ and variance $N_0/2$

$$P_{e|\mathbf{a}_1} = \int_{\mathbf{q}\notin I_1} f_{\mathbf{q}|\mathbf{A}}(\mathbf{q}|\mathbf{a}_1)\, d\mathbf{q} = Q\left(\frac{A}{\sqrt{N_0/2}}\right)$$

Error Probability - Graphical interpretation

$$P_e = \frac{1}{2}P_{e|a_0} + \frac{1}{2}P_{e|a_1} = \frac{1}{2}\int_{\mathbf{q}\notin I_0} f_{\mathbf{q}|\mathbf{A}}(q|a_0)\ d\mathbf{q} + \frac{1}{2}\int_{\mathbf{q}\notin I_1} f_{\mathbf{q}|\mathbf{A}}(q|a_1)\ d\mathbf{q}$$





General result for 1-D binary equiprobable case

- In this case, we have the following conditions
  - Decision regions
    - Threshold at the midpoint of the two symbols

$$\mathbf{q}_u = \frac{\mathbf{a}_0 + \mathbf{a}_1}{2}$$

    - Distance from each symbol to the threshold

$$d(\mathbf{a}_0, \mathbf{q}_u) = d(\mathbf{a}_1, \mathbf{q}_u) = \frac{d(\mathbf{a}_0, \mathbf{a}_1)}{2}$$

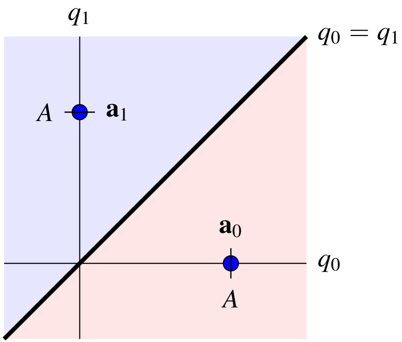
- Probability of error

$$P_e = Q\left(\frac{d(\mathbf{a}_0, \mathbf{a}_1)}{2\sqrt{N_0/2}}\right)$$

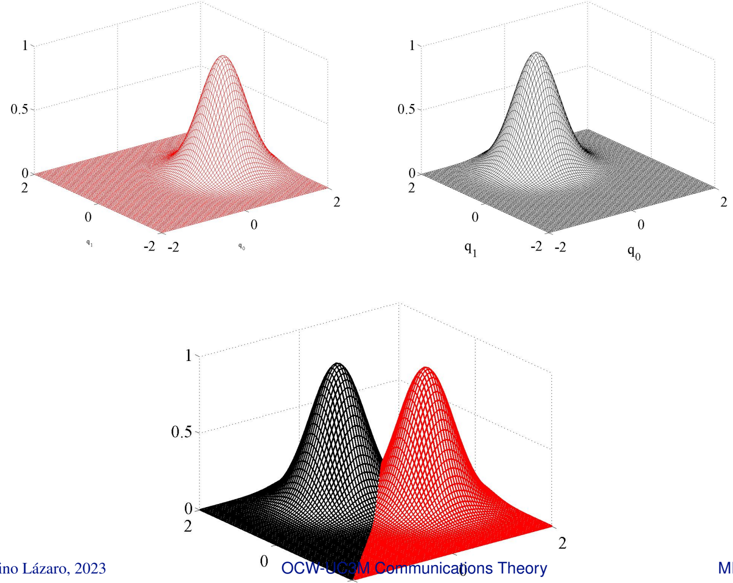
2D binary example

- Equiprobable symbols $p_{\mathbf{A}}(\mathbf{a}_0) = p_{\mathbf{A}}(\mathbf{a}_1) = \frac{1}{2}$
- Constellation $\mathbf{a}_0 = \begin{bmatrix} A \\ 0 \end{bmatrix}, \ \mathbf{a}_1 = \begin{bmatrix} 0 \\ A \end{bmatrix}$
- Decision regions: boundary $q_0 = q_1$

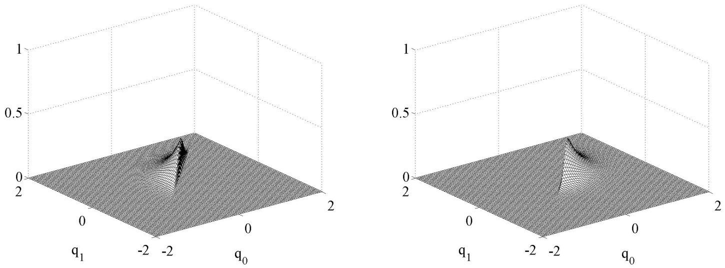
$$I_0 = \left\{ \mathbf{q} = \begin{bmatrix} q_0 \\ q_1 \end{bmatrix} : q_0 \geq q_1 \right\} I_1 = \left\{ \mathbf{q} = \begin{bmatrix} q_0 \\ q_1 \end{bmatrix} : q_0 < q_1 \right\}$$



Distributions $f_{\mathbf{q}|\mathbf{A}}(\mathbf{q}|\mathbf{a}_0), f_{\mathbf{q}|\mathbf{A}}(\mathbf{q}|\mathbf{a}_1)$ ($A = 1$)



Calculation of error probabilities

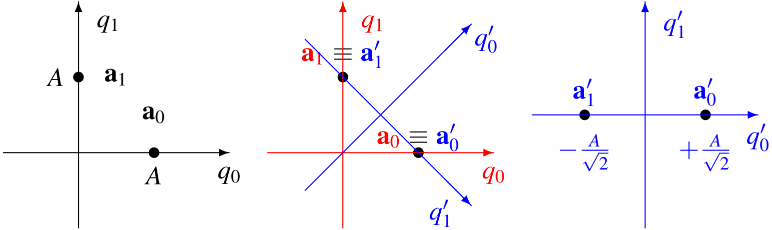


- Integrals of the conditional distributions given $\mathbf{a}_0$ and $\mathbf{a}_1$ out of their decision regions
- 2D Gaussian integrals in a half plane
  - There are no analytical expressions or numerical tables

Change of coordinate system

- A change of coordinates can be made
  - One of the axes passes through the two points of the constellation
  - The constellation is rotated 45º (without changing scale)

$$q'_0 = \frac{1}{\sqrt{2}}(q_0 - q_1),\; q'_1 = \frac{1}{\sqrt{2}}(q_0 + q_1 - A)$$



  - The value of the second coordinate is null for both symbols

Change of coordinate system

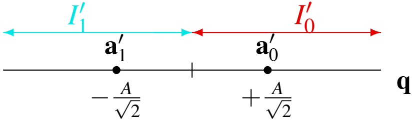
- A change of coordinates can be made
  - One of the axes passes through the two points of the constellation
  - The constellation is rotated 45º (without changing scale)

$$q_0' = \frac{1}{\sqrt{2}}(q_0 - q_1),\ q_1' = \frac{1}{\sqrt{2}}(q_0 + q_1 - A)$$

  - New constellation (distance is maintained)

$$\mathbf{a}_0' = \begin{bmatrix} +\frac{A}{\sqrt{2}} \\ 0 \end{bmatrix},\ \mathbf{a}_1' = \begin{bmatrix} -\frac{A}{\sqrt{2}} \\ 0 \end{bmatrix}$$

    - ⋆ The second coordinate can be eliminated: 1D space!!!



Formal expansion of the change of coordinates

- Transformation: $q_0' = \frac{1}{\sqrt{2}}(q_0 - q_1),\ q_1' = \frac{1}{\sqrt{2}}(q_0 + q_1 - A)$
- If $A[n] = \mathbf{a}_0$, then $q_0 = a_{0,0} + z_0 = A + z_0$ $q_1 = a_{0,1} + z_1 = 0 + z_1$

$$q_0'\Big|_{\mathbf{A}=\mathbf{a}_0} = \frac{1}{\sqrt{2}}\left((a_{0,0}+z_0) - (a_{0,1}+z_1)\right) = \frac{1}{\sqrt{2}}\left((A+z_0) - (0+z_1)\right) = \underbrace{\frac{A}{\sqrt{2}}}_{a_{0,0}'} + \underbrace{\frac{1}{\sqrt{2}}(z_0 - z_1)}_{z_0'}$$

$$q_1'\Big|_{\mathbf{A}=\mathbf{a}_0} = \frac{1}{\sqrt{2}}\left((a_{0,0}+z_0) + (a_{0,1}+z_1) - A\right) = \frac{1}{\sqrt{2}}\left((A+z_0) + (0+z_1) - A\right) = \underbrace{\frac{1}{\sqrt{2}}(z_0 + z_1)}_{z_1'}$$

- If $A[n] = \mathbf{a}_1$ then $q_0 = a_{1,0} + z_0 = 0 + z_0$ $q_1 = a_{1,1} + z_1 = A + z_1$

$$q_0'\Big|_{\mathbf{A}=\mathbf{a}_1} = \frac{1}{\sqrt{2}}\left((a_{1,0}+z_0) - (a_{1,1}+z_1)\right) = \frac{1}{\sqrt{2}}\left((0+z_0) - (A+z_1)\right) = \underbrace{-\frac{A}{\sqrt{2}}}_{a_{1,0}'} + \underbrace{\frac{1}{\sqrt{2}}(z_0 - z_1)}_{z_0'}$$

$$q_1'\Big|_{\mathbf{A}=\mathbf{a}_1} = \frac{1}{\sqrt{2}}\left((a_{1,0}+z_0) + (a_{1,1}+z_1) - A\right) = \frac{1}{\sqrt{2}}\left((0+z_0) + (A+z_1) - A\right) = \underbrace{\frac{1}{\sqrt{2}}(z_0 + z_1)}_{z_1'}$$

Formal expansion of the change of coordinates (II)

- Coordinates of the new constellation

$$\mathbf{a}'_0 = \begin{bmatrix} +\frac{A}{\sqrt{2}} \\ 0 \end{bmatrix},\ \mathbf{a}'_1 = \begin{bmatrix} -\frac{A}{\sqrt{2}} \\ 0 \end{bmatrix}$$

  - There is no signal term in the second coordinate

- Noise components: $z'_0 = \frac{1}{\sqrt{2}}(z_0 - z_1)$ $z'_1 = \frac{1}{\sqrt{2}}(z_0 + z_1)$
  - Terms z'_0 and z'_1 proportional to $z_0 - z_1$ and $z_0 + z_1$, respectively
  - They are independent (Gaussian and uncorrelated)

$$\text{Cov}(z_0 - z_1, z_0 + z_1) = E[(z_0 - z_1)(z_0 + z_1)] = E[z_0^2 - z_1^2] = \frac{N_0}{2} - \frac{N_0}{2} = 0$$

    - ★ The second coordinate can be discarded (there is no signal and the noise term z'_1 is independent of z'_0)
  - Statistics of z'_0

$$E[z'_0] = E\left[\frac{1}{\sqrt{2}}(z_0 - z_1)\right] = \frac{1}{\sqrt{2}}E[z_0] - \frac{1}{\sqrt{2}}E[z_1] = 0$$

$$\text{Var}(z'_0) = E\left[\left(\frac{1}{\sqrt{2}}(z_0 - z_1)\right)^2\right] = \frac{1}{2}E[z_0^2] + \frac{1}{2}E[z_1^2] - E[z_0 z_1] = \frac{1}{2}\frac{N_0}{2} + \frac{1}{2}\frac{N_0}{2} - 0 = \frac{N_0}{2}$$

General result: binary equiprobable case

- It is always possible to find a change of coordinates with one of the axes passing through the two points of the constellation
  - What is relevant is the distance between the points
- In this case, we have the following conditions
  - Decision regions for equiprobable symbols
    - Threshold at the midpoint of the two symbols

$$\mathbf{q}_u = \frac{\mathbf{a}_0 + \mathbf{a}_1}{2}$$

    - Distance from each symbol to the threshold

$$d(\mathbf{a}_0, \mathbf{q}_u) = d(\mathbf{a}_1, \mathbf{q}_u) = \frac{d(\mathbf{a}_0, \mathbf{a}_1)}{2}$$

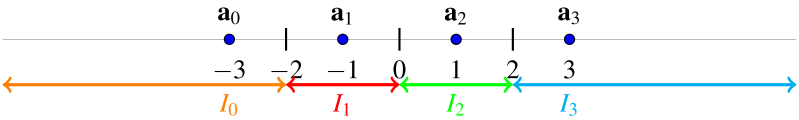
- Probability of error

$$P_e = Q\left(\frac{d(\mathbf{a}_0, \mathbf{a}_1)}{2\sqrt{N_0/2}}\right)$$

Example : 1-D space M-ary

- $M = 4$, equiprobable symbols $p_A(\mathbf{a}_i) = \frac{1}{4}$
- Constellation: $\mathbf{a}_0 = -3,\ \mathbf{a}_1 = -1,\ \mathbf{a}_2 = +1,\ \mathbf{a}_3 = +3$
- Decision regions: thresholds $\mathbf{q}_{u1} = -2,\ \mathbf{q}_{u2} = 0,\ \mathbf{q}_{u3} = +2$

$$I_0 = (-\infty, -2],\ I_1 = (-2.0],\ I_2 = (0, +2],\ I_3 = (+2, +\infty)$$



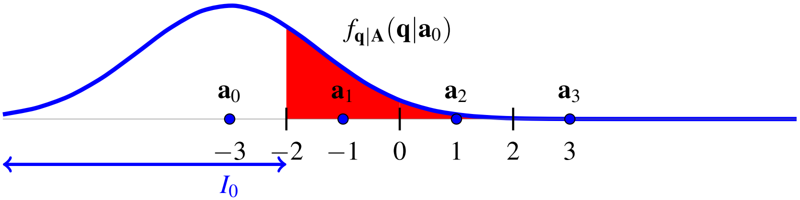
- Conditional error probabilities

$$P_{e|\mathbf{a}_0} = Q\left(\frac{1}{\sqrt{N_0/2}}\right),\ P_{e|\mathbf{a}_1} = 2Q\left(\frac{1}{\sqrt{N_0/2}}\right),\ P_{e|\mathbf{a}_2} = P_{e|\mathbf{a}_1},\ P_{e|\mathbf{a}_3} = P_{e|\mathbf{a}_0}$$

- Total error probability

$$P_e = \frac{1}{4}\sum_{i=0}^{M-1} P_{e|\mathbf{a}_i} = \frac{3}{2}Q\left(\frac{1}{\sqrt{N_0/2}}\right)$$

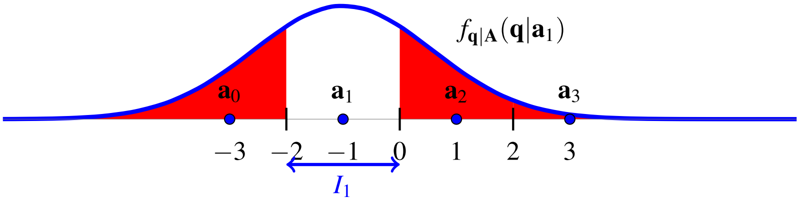
Calculation $P_{e|\mathbf{a}_0}$



- Distribution $f_{\mathbf{q}|\mathbf{A}}(\mathbf{q}|\mathbf{a}_0) = \mathcal{N}\left(\mathbf{a}_0, \frac{N_0}{2}\right)$
  - Gaussian: mean $\mathbf{a}_0 = -3$ and variance $N_0/2$
- Probability of error
  - Integrate $f_{\mathbf{q}|\mathbf{A}}(\mathbf{q}|\mathbf{a}_0)$ out of I_0

$$P_{e|\mathbf{a}_0} = \int_{\mathbf{q}\notin I_0} f_{\mathbf{q}|\mathbf{A}}(\mathbf{q}|\mathbf{a}_0)\, d\mathbf{q} = Q\left(\frac{1}{\sqrt{N_0/2}}\right)$$

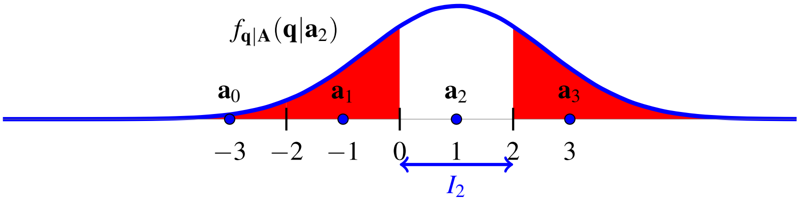
Calculation $P_{e|\mathbf{a}_1}$



- Distribution $$f_{\mathbf{q}|\mathbf{A}}(\mathbf{q}|\mathbf{a}_1) = \mathcal{N}\left(\mathbf{a}_1, \frac{N_0}{2}\right)$$
  - Gaussian: mean $\mathbf{a}_1 = -1$ and variance $N_0/2$
- Probability of error
  - Integrate $f_{\mathbf{q}|\mathbf{A}}(\mathbf{q}|\mathbf{a}_1)$ out of I_1

$$P_{e|\mathbf{a}_1} = \int_{\mathbf{q}\notin I_1} f_{\mathbf{q}|\mathbf{A}}(\mathbf{q}|\mathbf{a}_1)\, d\mathbf{q} = 2Q\left(\frac{1}{\sqrt{N_0/2}}\right)$$

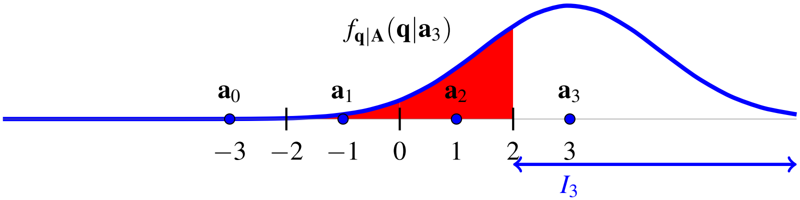
Calculation $P_{e|\mathbf{a}_2}$



- Distribution $f_{\mathbf{q}|\mathbf{A}}(\mathbf{q}|\mathbf{a}_2) = \mathcal{N}\left(\mathbf{a}_2, \frac{N_0}{2}\right)$
  - Gaussian: mean $\mathbf{a}_2 = +1$ and variance $N_0/2$
- Probability of error
  - Integrate $f_{\mathbf{q}|\mathbf{A}}(\mathbf{q}|\mathbf{a}_2)$ out of I_2

$$P_{e|\mathbf{a}_2} = \int_{\mathbf{q}\notin I_2} f_{\mathbf{q}|\mathbf{A}}(\mathbf{q}|\mathbf{a}_2)\, d\mathbf{q} = 2Q\left(\frac{1}{\sqrt{N_0/2}}\right)$$

Calculation $P_{e|\mathbf{a}_3}$

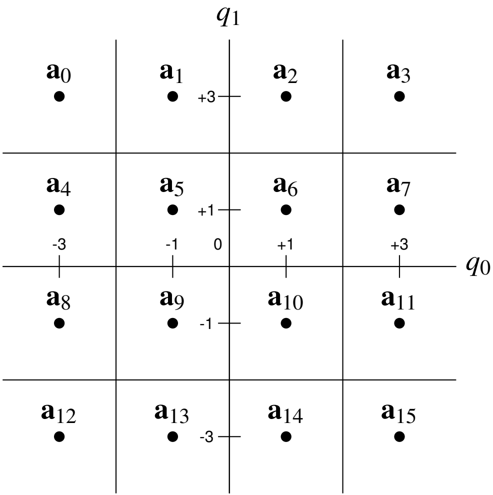


- Distribution $f_{\mathbf{q}|\mathbf{A}}(\mathbf{q}|\mathbf{a}_3) = \mathcal{N}\left(\mathbf{a}_3, \frac{N_0}{2}\right)$
  - Gaussian: mean $\mathbf{a}_3 = -3$ and variance $N_0/2$
- Probability of error
  - Integrate $f_{\mathbf{q}|\mathbf{A}}(\mathbf{q}|\mathbf{a}_3)$ out of I_3

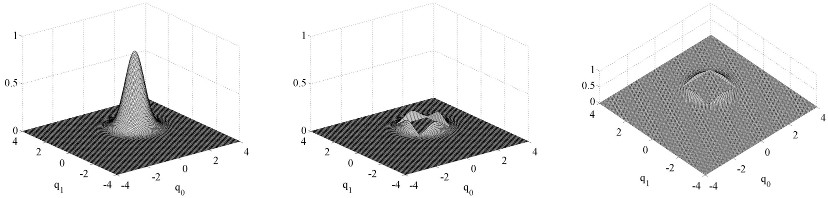
$$P_{e|\mathbf{a}_3} = \int_{\mathbf{q}\notin I_3} f_{\mathbf{q}|\mathbf{A}}(\mathbf{q}|\mathbf{a}_3)\, d\mathbf{q} = Q\left(\frac{1}{\sqrt{N_0/2}}\right)$$

M-ary detector in multidimensional space

- Example: 16 symbols in a lattice aligned with the q_0 and q_1 axes



Calculation of error probability for $\mathbf{a}_6$



- Symbol coordinates: $\mathbf{a}_6 = \begin{bmatrix} +1 \\ +1 \end{bmatrix}$
  - Distribution $f_{\mathbf{q}|\mathbf{A}}(\mathbf{q}|\mathbf{a}_6)$: 2D Gaussian in $\mathbf{a}_6$ ($\sigma^2 = N_0/2$)

$$f_{\mathbf{q}|\mathbf{A}}(\mathbf{q}|\mathbf{a}_6) = \mathcal{N}^2\left(\mathbf{a}_6, \frac{N_0}{2}\right) = \mathcal{N}^2\left(\begin{bmatrix} +1 \\ +1 \end{bmatrix}, \frac{N_0}{2}\right)$$

- Calculation of the probability of conditional error $P_{e|\mathbf{a}_6}$
  - Decision Region - Square: $0 \leq q_0 < 2$ and $0 \leq q_1 < 2$
    - ★ There are no analytic expressions to directly compute the integral of a Gaussian out of this square

2D space - Decision regions form a grid

- In this case the decision regions can be described by two independent conditions on q_0 and on q_1 that have to be fulfilled simultaneously
  - Examples

$$I_6 \equiv 0 \leq q_0 < 2 \text{ and } 0 \leq q_1 < 2$$

$$I_2 \equiv 0 \leq q_0 < 2 \text{ and } 2 \leq q_1 < \infty$$

  - The 2D problem can be decomposed into 2 coupled 1D problems

$$P_{e|\mathbf{a}_i} = 1 - P_{a|\mathbf{a}_i} = 1 - P_{a|a_{i,0}} \times P_{a|a_{i,1}} = 1 - \left[\left(1 - P_{e|a_{i,0}}\right)\left(1 - P_{e|a_{i,1}}\right)\right]$$

    - ★ The probability of success $P_{a|\mathbf{a}_i}$ can be written as the product of the probabilities in each of the two directions of the space $P_{a|a_{i,0}} \times P_{a|a_{i,1}}$
    - ★ The probability of success in one of the directions of space ($P_{a|a_{i,0}}$ or $P_{a|a_{i,1}}$) can be written as 1 minus the error probability in that direction ($P_{e|a_{i,0}}$ or $P_{e|a_{i,1}}$)

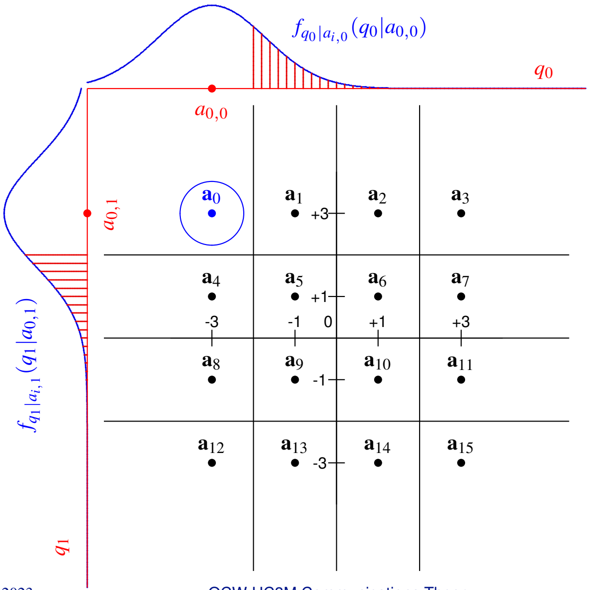
Three types of decision regions

- Type 1: $\{I_0, I_3, I_{12}, I_{15}\}$
  - A single decision boundary in each direction of space
- Type 2: $\{I_5, I_6, I_9, I_{10}\}$
  - Two decision boundaries in each direction of space
- Type 3: $\{I_1, I_2, I_4, I_7, I_8, I_{11}, I_{13}, I_{14}\}$
  - A boundary in one of the directions of space
  - Two decision boundaries in the other direction
- Calculation of the probability of error for equiprobable symbols
  - All symbols of the same type have the same probability of conditional error
    - ★ Examples of each type: $\mathbf{a}_0$ (Type 1), $\mathbf{a}_5$ (Type 2), $\mathbf{a}_7$ (Type 3)

$$P_e = \sum_{i=0}^{M-1} p_{\mathbf{A}}(\mathbf{a}_i)\, P_{e|\mathbf{a}_i} = 4 \times \frac{1}{16} P_{e|\mathbf{a}_0} + 4 \times \frac{1}{16} P_{e|\mathbf{a}_5} + 8 \times \frac{1}{16} P_{e|\mathbf{a}_7}$$

$$= 3Q\left(\frac{1}{\sqrt{N_0/2}}\right) - \frac{9}{4} Q^2\left(\frac{1}{\sqrt{N_0/2}}\right)$$

Type 1 Regions (example $\mathbf{a}_0$)



Type 1 Regions (example $\mathbf{a}_0$)

- First axis of space: q_0
  - Mean of the 1-D Gaussian distribution: $a_{0,0} = -3$
  - Decision region : $-\infty < q_0 < -2$

$$P_{a|a_{0,0}} = 1 - P_{e|a_{0,0}} = 1 - Q\left(\frac{1}{\sqrt{N_0/2}}\right)$$

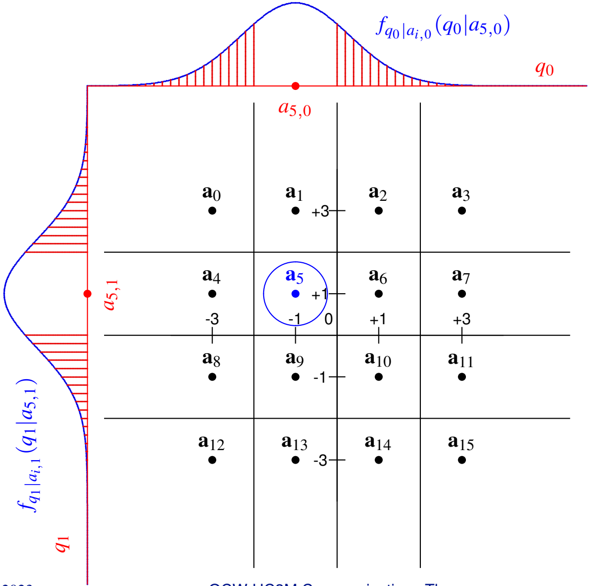
- Second axis of space: q_1
  - Mean of the 1-D Gaussian distribution: $a_{0,1} = +3$
  - Decision region : $+2 \leq q_1 < +\infty$

$$P_{a|a_{0,1}} = 1 - P_{e|a_{0,1}} = 1 - Q\left(\frac{1}{\sqrt{N_0/2}}\right)$$

- Conditional error probability

$$P_{e|\mathbf{a}_0} = 1 - \left[1 - Q\left(\frac{1}{\sqrt{N_0/2}}\right)\right]^2 = 2Q\left(\frac{1}{\sqrt{N_0/2}}\right) - Q^2\left(\frac{1}{\sqrt{N_0/2}}\right)$$

Type 2 Regions (example $\mathbf{a}_5$)



Type 2 Regions (example $\mathbf{a}_5$)

- First axis of space: q_0
  - Mean of the 1-D Gaussian distribution: $a_{5,0} = -1$
  - Decision region : $-2 \leq q_0 < 0$

$$P_{a|a_{5,0}} = 1 - P_{e|a_{5,0}} = 1 - 2Q\left(\frac{1}{\sqrt{N_0/2}}\right)$$

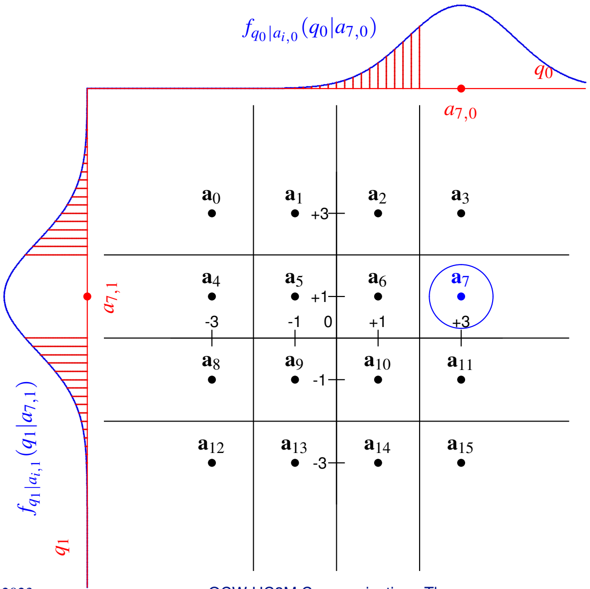
- Second axis of space: q_1
  - Mean of the 1-D Gaussian distribution: $a_{5,1} = +1$
  - Decision region : $0 \leq q_1 < +2$

$$P_{a|a_{5,1}} = 1 - P_{e|a_{5,1}} = 1 - 2Q\left(\frac{1}{\sqrt{N_0/2}}\right)$$

- Conditional error probability

$$P_{e|\mathbf{a}_5} = 1 - \left[1 - 2Q\left(\frac{1}{\sqrt{N_0/2}}\right)\right]^2 = 4Q\left(\frac{1}{\sqrt{N_0/2}}\right) - 4Q^2\left(\frac{1}{\sqrt{N_0/2}}\right)$$

Type 3 Regions (example $\mathbf{a}_7$)



Type 3 Regions (example $\mathbf{a}_7$)

- First axis of space: q_0
  - Mean of the 1-D Gaussian distribution: $a_{7,0} = +3$
  - Decision region : $+2 \leq q_0 < +\infty$

$$P_{a|a_{7,0}} = 1 - P_{e|a_{7,0}} = 1 - Q\left(\frac{1}{\sqrt{N_0/2}}\right)$$

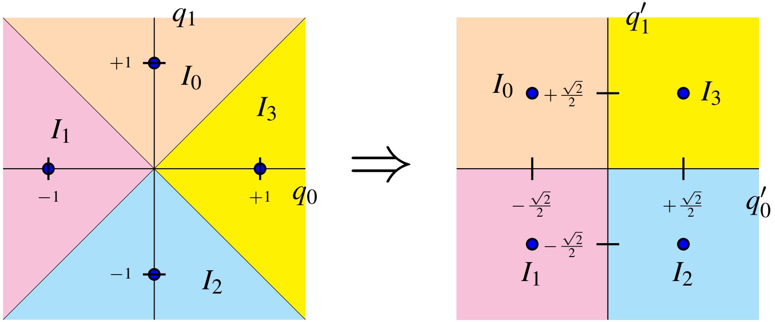
- Second axis of space: q_1
  - Mean of the 1-D Gaussian distribution: $a_{7,1} = +1$
  - Decision region : $0 \leq q_1 < +2$

$$P_{a|a_{7,1}} = 1 - P_{e|a_{7,1}} = 1 - 2Q\left(\frac{1}{\sqrt{N_0/2}}\right)$$

- Conditional error probability

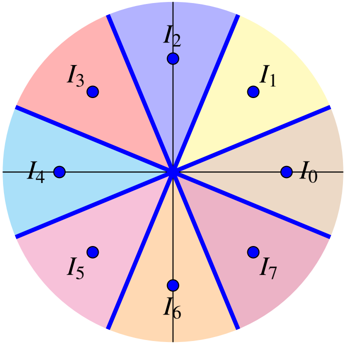
$$P_{e|\mathbf{a}_7} = 1 - \left[1 - Q\left(\frac{1}{\sqrt{N_0/2}}\right)\right]\left[1 - 2Q\left(\frac{1}{\sqrt{N_0/2}}\right)\right] = 3Q\left(\frac{1}{\sqrt{N_0/2}}\right) - 2Q^2\left(\frac{1}{\sqrt{N_0/2}}\right)$$

M-ary detector in multidimensional space (II)



- Decision regions form a grid, but not aligned with the axes (basis) of the space
  - Change of coordinates: 45º rotation
    - The decision regions are aligned with the new axes
    - Now it is possible to calculate the error probability

M-ary detector in multidimensional space (III)



- Decision regions do not form a grid
  - It is not possible to calculate analytically the probability of error, nor make simple transformations that allow it
    - Can be calculated exactly by numerical calculation (computers)
    - Analytically: approximations or bounds of the error probability

Approximations for the error probability

- They are useful when it is not possible or difficult to calculate the exact error probability
- Approximation of the error probability
  - It depends on the distances between symbols
    - Most likely errors: to decide a symbol at minimum distance from the transmitted symbol
  - Most common approximation
    - Assumption: errors only happen with symbols at minimum distance

$$P_e \approx k\, Q\left(\frac{d_{min}}{2\sqrt{N_0/2}}\right)$$

    - d_{min}: minimum distance between two constellation symbols
    - k: maximum number of symbols at minimum distance from a constellation symbol

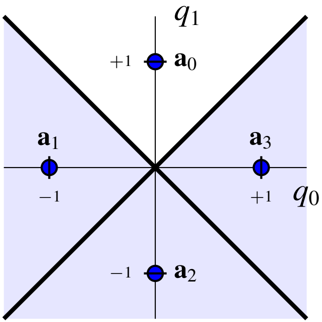
Bounds on error probability

- They are useful when it is not possible or difficult to calculate the exact error probability, and a minimum performance (maximum error probability) must be guaranteed
- An expression is obtained that bounds the probability of error

$$P_e \leq \text{Bound}$$

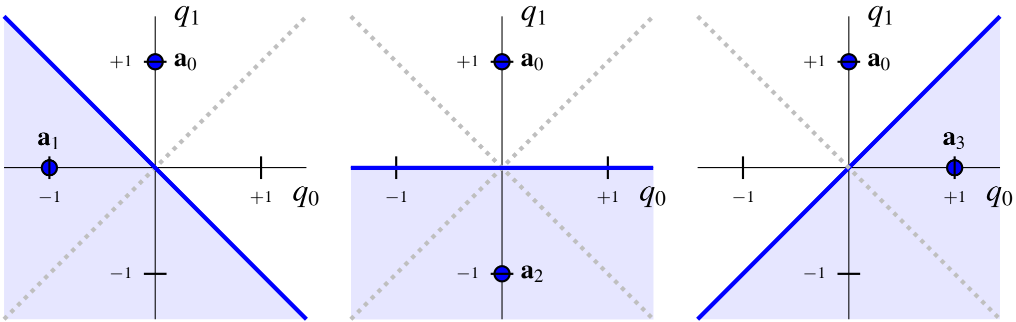
  - An upper bound guarantees a minimum performance
- Frequently used bounds
  - The Union Bound (tight bound)
    - ⋆ Bounds the error probability in a tight way
      - The error probability is not far from the value of the bound
    - ⋆ Its calculation becomes involved as the constellation size increases
  - Loose bound
    - ⋆ Less tight bound but easier to calculate

The Union Bound - Example



- Example: to find a bound of the error probability for this constellation
- Conditional probability of error $P_{e|\mathbf{a}_0}$
  - Integral of $f_{\mathbf{q}|\mathbf{A}}(\mathbf{q}|\mathbf{a}_0)$ out of I_0
    - ★ $f_{\mathbf{q}|\mathbf{A}}(\mathbf{q}|\mathbf{a}_0)$ is a 2D Gaussian centered at $\mathbf{a}_0$
    - ★ Integration of $f_{\mathbf{q}|\mathbf{A}}(\mathbf{q}|\mathbf{a}_0)$ in the highlighted area of the figure

The Union Bound - Graphical interpretation



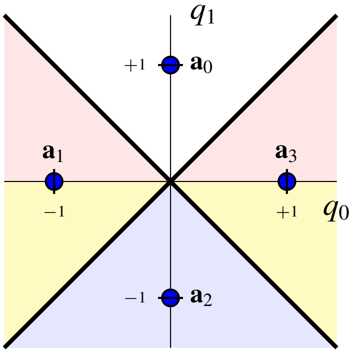
$$P_e(\mathbf{a}_0, \mathbf{a}_1) = Q\left(\frac{d(\mathbf{a}_0,\mathbf{a}_1)}{2\sqrt{N_0/2}}\right) = Q\left(\frac{\sqrt{2}}{2\sqrt{N_0/2}}\right) = Q\left(\frac{1}{\sqrt{N_0}}\right)$$

$$P_e(\mathbf{a}_0, \mathbf{a}_2) = Q\left(\frac{d(\mathbf{a}_0,\mathbf{a}_2)}{2\sqrt{N_0/2}}\right) = Q\left(\frac{2}{2\sqrt{N_0/2}}\right) = Q\left(\frac{\sqrt{2}}{\sqrt{N_0}}\right)$$

$$P_e(\mathbf{a}_0, \mathbf{a}_3) = Q\left(\frac{d(\mathbf{a}_0,\mathbf{a}_3)}{2\sqrt{N_0/2}}\right) = Q\left(\frac{\sqrt{2}}{2\sqrt{N_0/2}}\right) = Q\left(\frac{1}{\sqrt{N_0}}\right)$$

- Probabilities of error if there were only $\mathbf{a}_0$ and another symbol $\mathbf{a}_i$
  - Integrals of the Gaussian centered in $\mathbf{a}_0$ in the half planes of the figure
    - Added together, they form a bound on the probability of error

The Union Bound - Graphical interpretation (II)



- The Union Bound is an UPPER BOUND
  - The sum of the three terms integrates the Gaussian over the entire desired region
    - Only once in the areas in RED
    - Twice in the areas in YELLOW
    - Three times in the area in BLUE

The Union Bound - Upper Bound for P_e

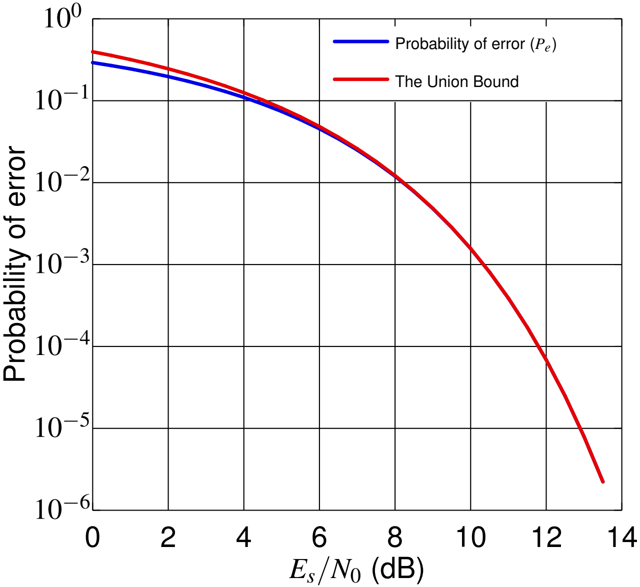
- Bound for the conditional error probabilities

$$P_{e|\mathbf{a}_0} \leq \sum_{j=1}^{M-1} P_e(\mathbf{a}_0, \mathbf{a}_j) \rightarrow \text{ In general } P_{e|\mathbf{a}_i} \leq \sum_{\substack{j=0 \\ j\neq i}}^{M-1} P_e(\mathbf{a}_i, \mathbf{a}_j)$$

- The Union Bound

$$P_e \leq \sum_{i=0}^{M-1} p_{\mathbf{A}}(\mathbf{a}_i) \sum_{\substack{j=0 \\ j\neq i}}^{M-1} P_e(\mathbf{a}_i, \mathbf{a}_j) = \sum_{i=0}^{M-1} p_{\mathbf{A}}(\mathbf{a}_i) \sum_{\substack{j=0 \\ j\neq i}}^{M-1} Q\left(\frac{d(\mathbf{a}_i, \mathbf{a}_j)}{2\sqrt{N_0/2}}\right)$$

The Union Bound - A tight bound of P_e



Complexity of the bound

- The Union Bound requires computing the distances between each pair of symbols and evaluate the function $Q(x)$ for each term
  - Minimum number of distances (considering the symmetry $d(\mathbf{a}_i, \mathbf{a}_j) = d(\mathbf{a}_j, \mathbf{a}_i)$)

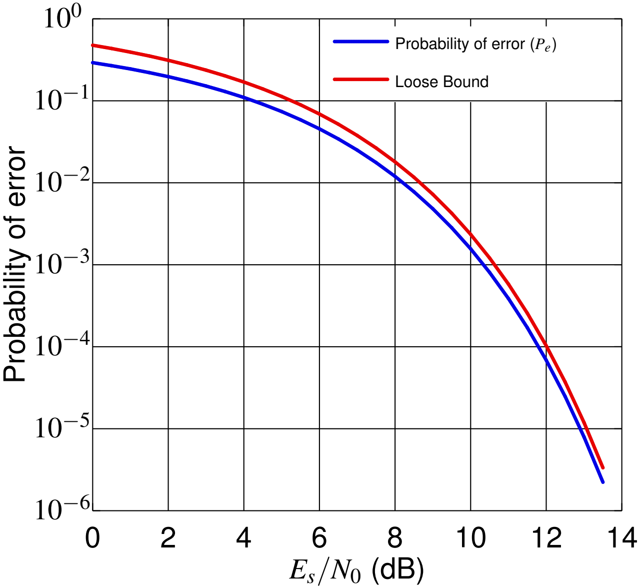
$$N_{distances} = \sum_{k=1}^{M-1} k$$

    - ★ $M = 4 \rightarrow N_{distances} = 6$
    - ★ $M = 8 \rightarrow N_{distances} = 28$
    - ★ $M = 16 \rightarrow N_{distances} = 120$
    - ★ $M = 64 \rightarrow N_{distances} = 2016$
- The Loose Bound
  - Assumption: all symbols are at d_{min} from the other $M-1$ symbols

$$P_e \leq (M-1)\, Q\left(\frac{d_{min}}{2\sqrt{N_0/2}}\right)$$

    - ★ As there will always be symbols at greater distances, this expression is a bound for the union bound, and therefore, is a bound for the probability of error

Loose Bound



Probability of error (P_e)
Loose Bound
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
Probability of error
0
2
4
6
8
10
12
14
E_s/N_0 (dB)

Expressions of P_e as a function of E_s/N_0

- Performance curves are often represented as a function of the E_s/N_0 ratio
- Common terms in error probabilities are usually written as

$$Q\left(\frac{d_{min}}{2\sqrt{N_0/2}}\right) \text{ or } Q\left(\frac{A}{\sqrt{N_0/2}}\right)$$

- In any case they can be rewritten as

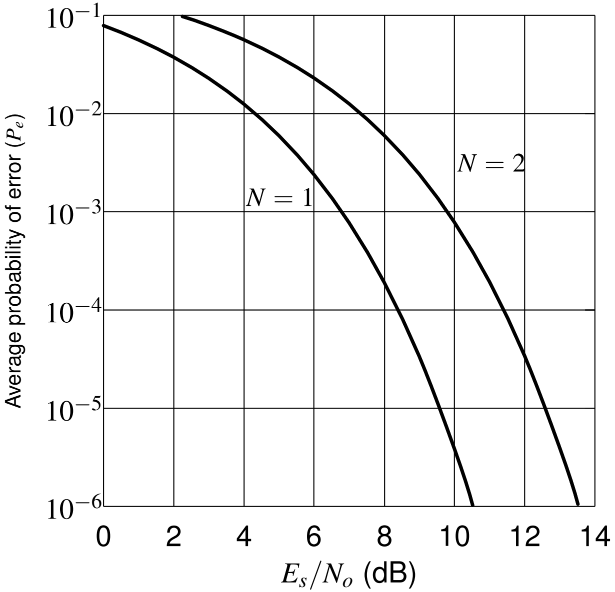
$$Q\left(v\sqrt{\frac{E_s}{N_0}}\right), \quad \text{where } Q\left(\underbrace{\frac{d_{min}}{\sqrt{2}\sqrt{E_s}}}_{v}\sqrt{\frac{E_s}{N_0}}\right) \text{ or } Q\left(\underbrace{\frac{A\sqrt{2}}{\sqrt{E_s}}}_{v}\sqrt{\frac{E_s}{N_0}}\right)$$

  - Factor v: constant value, which depends on the constellation
    - The higher v is, the more efficient the constellation is

Examples - Equiprobable Binary Constellations

- In this case, for any dimension N $$P_e = Q\left(\frac{d(\mathbf{a}_0,\mathbf{a}_1)}{2\sqrt{N_0/2}}\right)$$
- Two cases will be compared
  - Case (a): Binary symmetric constellation ($N=1$) $\mathbf{a}_0 = +A,\ \mathbf{a}_1 = -A$
  - Case (b): Orthogonal constellation ($N=2$) $\mathbf{a}_0 = \begin{bmatrix} A \\ 0 \end{bmatrix},\ \mathbf{a}_1 = \begin{bmatrix} 0 \\ A \end{bmatrix}$
- Distances and energies
  - Case (a): $E_s = A^2,\ d(\mathbf{a}_0,\mathbf{a}_1) = 2\,A$ Case (b): $E_s = A^2,\ d(\mathbf{a}_0,\mathbf{a}_1) = \sqrt{2}\,A$
- Calculation of the efficiency factor v
  - Case (a): $$v = \frac{d(\mathbf{a}_0,\mathbf{a}_1)}{\sqrt{2}\sqrt{E_s}} = \sqrt{2} \rightarrow P_e = Q\left(\sqrt{2\frac{E_s}{N_0}}\right)$$
  - Case (b): $$v = \frac{d(\mathbf{a}_0,\mathbf{a}_1)}{\sqrt{2}\sqrt{E_s}} = 1 \rightarrow P_e = Q\left(\sqrt{\frac{E_s}{N_0}}\right)$$

Examples - Plot of P_e as a function of E_s/N_0 (in dB)



Encoder

- Converts the sequence of symbols $B[n]$, (obtained from the sequence of bits $B_b[\ell]$), into a sequence of vectors with the discrete representation of the signals, $\mathbf{A}[n]$
  - Alphabet of $\mathbf{A}[n]$: Constellation: $\{\mathbf{a}_i\}_{i=0}^{M-1}, \ \mathbf{a}_i \in \boldsymbol{R}^N$
    - $M = 2^m$ symbols $\rightarrow m = \log_2 M$ bits per symbol
    - Binary rate: R_b - Symbol rate: R_s

$$R_b = \frac{1}{T_b} \text{ bits/s} \qquad R_s = \frac{1}{T} \text{ symbols/s (bauds)} \qquad R_b = m \times R_s$$

- Constraints on constellation design: Performance (P_e, BER) and Energy (E_s)
  - Distances between symbols (minimum distance)
  - Energy per symbol: squared norm of symbols

$$d(\mathbf{a}_i, \mathbf{a}_j) = \sqrt{\sum_{k=0}^{N-1} |a_{i,k} - a_{j,k}|^2}, \ d_{min} = \min_{i \neq j} d(\mathbf{a}_i, \mathbf{a}_j)$$

$$\mathcal{E}\{\mathbf{a}_i\} = ||\mathbf{a}_i||^2 = \sum_{k=0}^{N-1} |a_{i,k}|^2$$

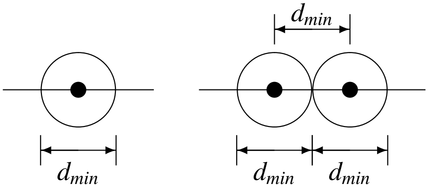
  - Trade-off: guaranteeing a minimum distance, with the symbols as close as possible to the origin of coordinates
    - Zero mean constellation: $E[\mathbf{a}_i] = [0, 0, \cdots, 0]^T = \mathbf{0}$

Encoder - Sphere Packing

- The tradeoff between distance and energy can be understood as a problem of packing spheres of a constant diameter
  - Guaranteeing minimum performance is equivalent to guaranteeing a minimum distance

$$P_e \approx k\, Q\left(\frac{d_{min}}{2\sqrt{N_0/2}}\right)$$

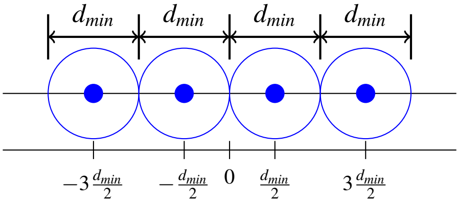
  - Symbol: sphere centered at $\mathbf{a}_i$ with diameter d_{min}
    - Two spheres in contact are at a distance d_{min}



  - The design problem can be stated as
    - Place M spheres of diameter d_{min} in the smallest possible space (N-D space) and centered at the origin (zero mean)

Sphere Packing

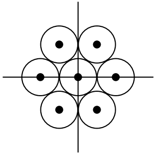
- Design in 1-D space



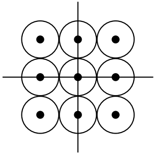
M equispaced points - Coordinates:

$$\left\{\pm\frac{d_{min}}{2}, 3\,\frac{d_{min}}{2}, \cdots, \pm(M-1)\,\frac{d_{min}}{2}\right\}$$

- Layout in 2-D space
  - Circles are formed from a central sphere
  - They are rearranged so that the mean is zero

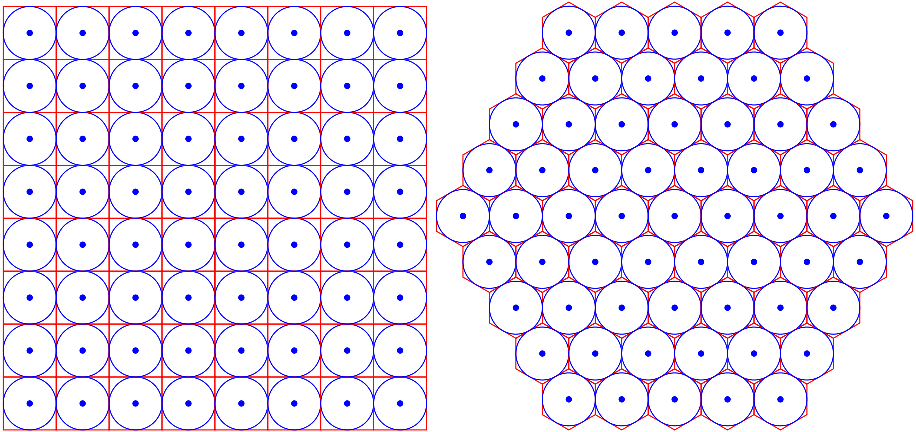


2D Space - YES



2D Space - NO

Rectangular vs Hexagonal arrangements



Equivalent constellation with zero mean

- The probability of error depends on the relative positions between symbols
  - These relative positions define the distances $d(\mathbf{a}_i, \mathbf{a}_j)$
- Fixed distances: the minimum E_s is obtained if the constellation has zero mean
- Transformation into equivalent zero mean constellation
  - Mean of the constellation $$E[\mathbf{a}_i] = \sum_{i=0}^{M-1} p_A(\mathbf{a}_i)\ \mathbf{a}_i$$
    - N dimensional vector (one value for each coordinate of $\mathbf{a}_i$)
  - This average is subtracted from each symbol of the initial constellation
$$\mathbf{a}_i' = \mathbf{a}_i - E[\mathbf{a}_i]$$
    - The mean of the new constellation is zero
$$E[\mathbf{a}_i'] = \sum_{i=0}^{M-1} \underbrace{p_{A'}(\mathbf{a}_i')}_{p_A(\mathbf{a}_i)}\ \mathbf{a}_i' = [0, 0, \cdots, 0]^T = \mathbf{0}$$

Equivalent constellation with zero mean - Example

- Constellation with $M = 4$ symbols, $p_A(\mathbf{a}_i) = \frac{1}{4}\ \forall i$

$$\mathbf{a}_0 = \begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix},\ \mathbf{a}_1 = \begin{bmatrix} +\frac{3}{2} \\ 0 \end{bmatrix},\ \mathbf{a}_2 = \begin{bmatrix} -\frac{1}{2} \\ +2 \end{bmatrix},\ \mathbf{a}_3 = \begin{bmatrix} +\frac{3}{2} \\ +2 \end{bmatrix}$$

  - Average energy per symbol

$$E_s = \frac{1}{4}\left[\left(-\frac{1}{2}\right)^2 + 0^2\right] + \frac{1}{4}\left[\left(\frac{3}{2}\right)^2 + 0^2\right] + \frac{1}{4}\left[\left(-\frac{1}{2}\right)^2 + 2^2\right] + \frac{1}{4}\left[\left(\frac{3}{2}\right)^2 + 2^2\right] = \frac{13}{4} = 3.25$$

- Mean of the constellation

$$E[\mathbf{a}_i] = \frac{1}{4}\begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} +\frac{3}{2} \\ 0 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} -\frac{1}{2} \\ +2 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} +\frac{3}{2} \\ +2 \end{bmatrix} = \begin{bmatrix} +\frac{1}{2} \\ +1 \end{bmatrix}$$

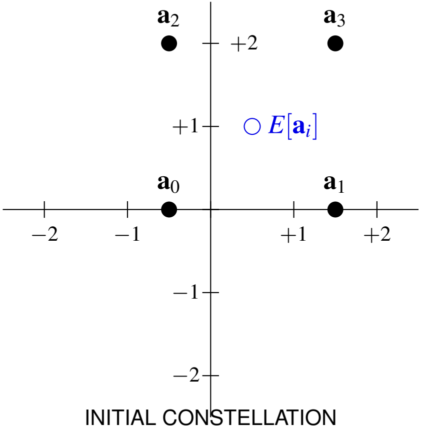
- Modified constellation: $\mathbf{a}_i = \mathbf{a}_i - E[\mathbf{a}_i]$

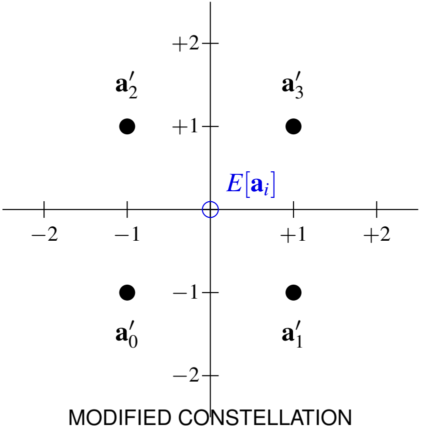
$$\mathbf{a}'_0 = \mathbf{a}_0 - E[\mathbf{a}_i] = \begin{bmatrix} -1 \\ -1 \end{bmatrix},\ \mathbf{a}'_1 = \mathbf{a}_1 - E[\mathbf{a}_i] = \begin{bmatrix} +1 \\ -1 \end{bmatrix},\ \mathbf{a}'_2 = \begin{bmatrix} -1 \\ +1 \end{bmatrix},\ \mathbf{a}'_3 = \begin{bmatrix} +1 \\ +1 \end{bmatrix}$$

  - Average energy per symbol

$$E'_s = \frac{1}{4}\left[(-1)^2 + (-1)^2\right] + \frac{1}{4}\left[(+1)^2 + (-1)^2\right] + \frac{1}{4}\left[(-1)^2 + (+1)^2\right] + \frac{1}{4}\left[(+1)^2 + (+1)^2\right] = 2$$

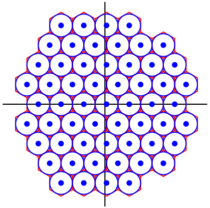
Equivalent constellation with zero mean - Example (II)





Encoder Layout (2D constellations)

- Sphere Packing
  - Optimal: minimum P_e for a given E_s
  - Hexagonal constellations
  - Hexagonal decision regions

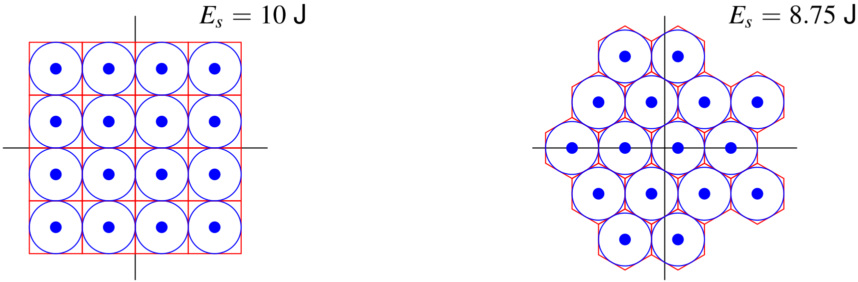


- Practical considerations
  - Simplicity of transmitter implementation
  - Simplicity of the receiver - Decision regions
  - Constant energy per symbol
  - Peak power / average power ratio

Other constellations: QAM, PSK, unipolar, orthogonal, $\cdots$

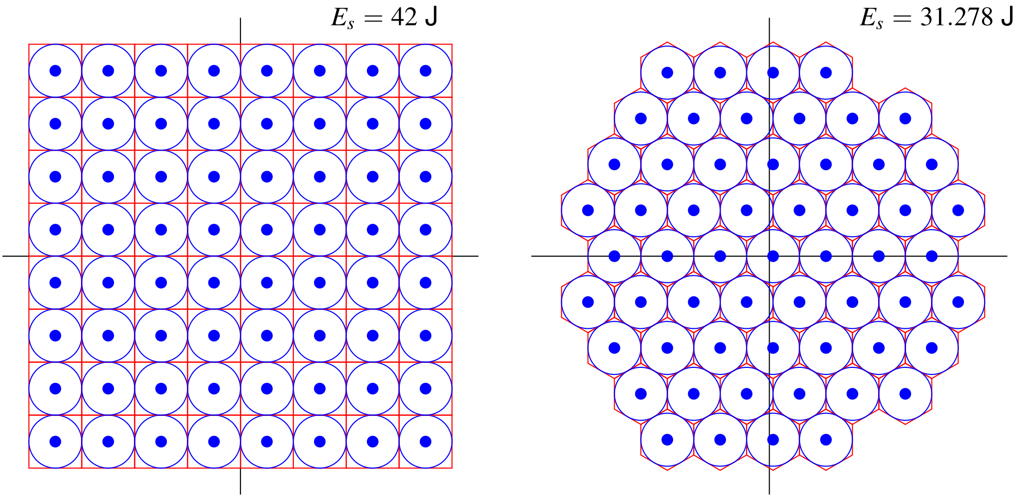
Hexagonal Constellations

QAM and Hexagonal constellations for $M = 4$ (unit ratio in E_s)



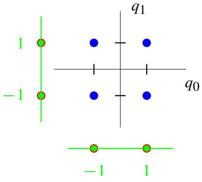
QAM and Hexagonal constellations for $M = 16$ (0.875 ratio in E_s)

Hexagonal Constellations (II)



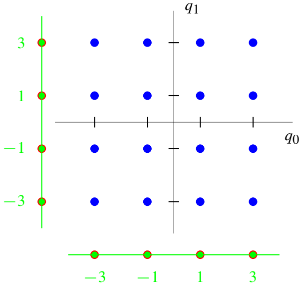
QAM and Hexagonal constellations for $M = 64$ (0.7447 ratio in E_s)

QAM Constellations



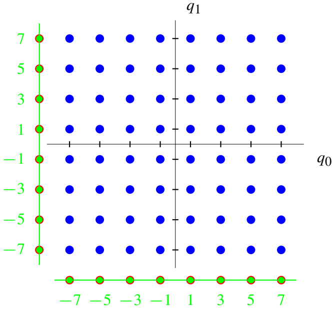
4-QAM

(2-PAM)×(2-PAM)



16-QAM

(4-PAM)×(4-PAM)

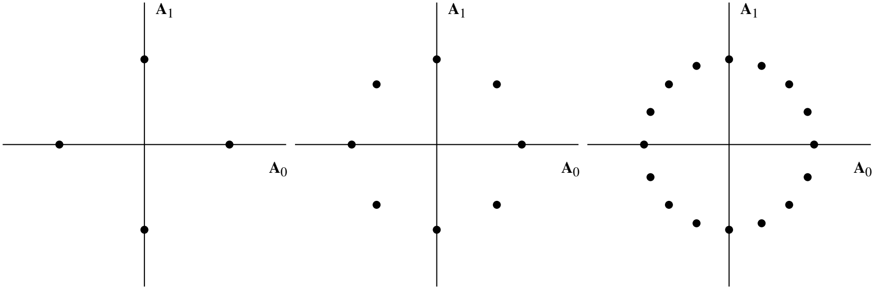


64-QAM

(8-PAM)×(8-PAM)

- Points on a grid
  - Ease of transmitter / receiver implementation
    - ★ Generation / Detection: independent for each coordinate

PSK Constellations



Constellations 4-PSK (QPSK), 8-PSK and 16-PSK

- Points on a circle of constant radius
  - Constant energy for all symbols

Binary Assignment

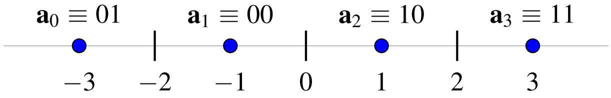
- Binary Assignment : Bits to Symbols conversion $B_b[\ell] \rightarrow B[n]$ $(\mathbf{A}[n])$
  - m bits are transmitted in each symbol b_i $(\mathbf{a}_i)$

$$M = 2^m \text{ symbols} \rightarrow m = \log_2 M \text{ bits/symbol}$$

- Objective of the binary assignment (values of the m bits for each symbol)
  - Bit Error Rate (BER) minimization
- Optimal binary assignment technique
  - Gray codes

Gray code

- The m bits assigned to symbols at minimum distance differ by only 1 bit

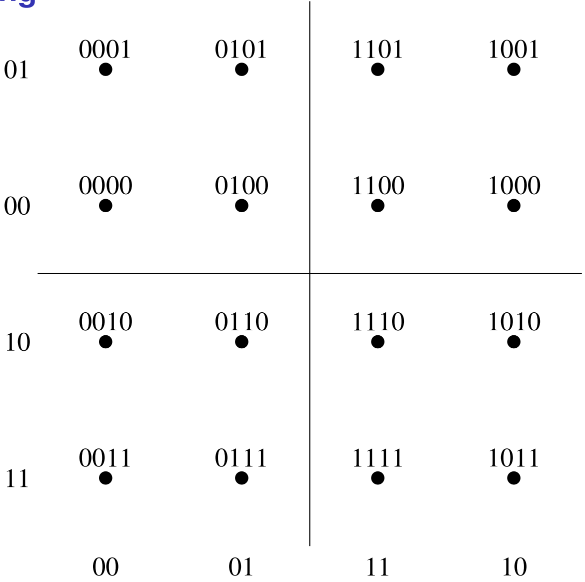


  - For a high signal-to-noise ratio
    - ★ Majority or symbol errors: decision of a symbol at minimum distance
    - ★ Such a symbol error: a single erroneous bit out of m bits

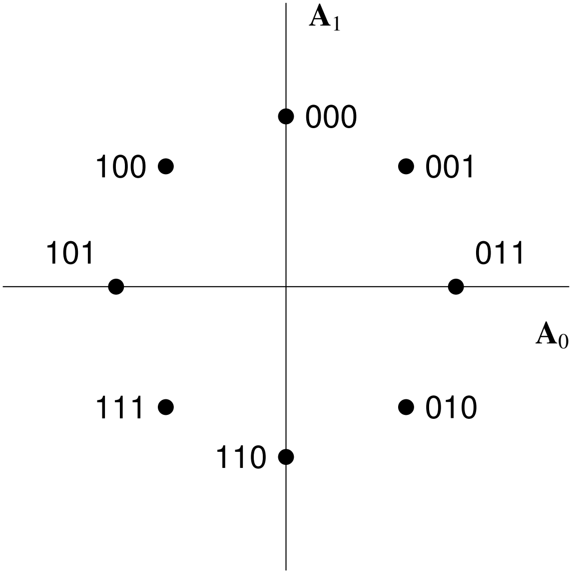
$$BER \approx \frac{1}{m} P_e$$

$m = \log_2 M$: number of bits per symbol

Gray QAM Encoding



Gray PSK Encoding



Calculation of the Bit Error Rate (BER)

- The conditional BER is averaged
$$BER = \sum_{i=0}^{M-1} p_A(a_i)\ BER_{a_i}$$

- Calculation of conditional BER
$$BER_{a_i} = \sum_{\substack{j=0 \\ j\neq i}}^{M-1} P_{e|a_i\rightarrow a_j} \frac{m_{e|a_i\rightarrow a_j}}{m}$$
  - $P_{e|a_i\rightarrow a_j}$: probability of transmitting $\mathbf{A}[n] = \mathbf{a}_i$, decide $\hat{\mathbf{A}}[n] = \mathbf{a}_j$
$$P_{e|a_i\rightarrow a_j} = \int_{\mathbf{q}_0\in I_j} f_{\mathbf{q}|\mathbf{A}}(\mathbf{q}_0|\mathbf{a}_i)\ d\mathbf{q}_0$$
  - $m_{e|a_i\rightarrow a_j}$: number of bit errors involved in that decision
  - m: number of bits per symbol of the constellation

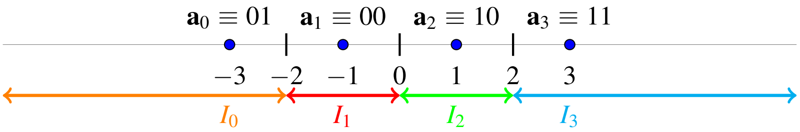
Example - 1-D space M-ary

- Example:
  - $M = 4$, equiprobable symbols $p_A(\mathbf{a}_i) = \frac{1}{4}$
  - Constellation: $\mathbf{a}_0 = -3,\ \mathbf{a}_1 = -1,\ \mathbf{a}_2 = +1,\ \mathbf{a}_3 = +3$
  - Decision regions: 3 thresholds

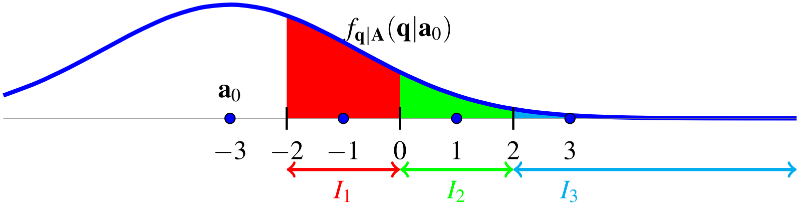
$$\mathbf{q}_{u1} = -2,\ \mathbf{q}_{u2} = 0,\ \mathbf{q}_{u3} = +2$$

$$I_0 = (-\infty, -2],\ I_1 = (-2, 0],\ I_2 = (0, +2],\ I_3 = (+2, +\infty)$$

  - Binary assignment



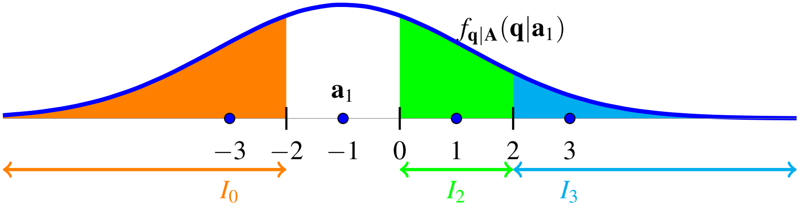
Calculation of $BER_{\mathbf{a}_0}$



- Binary assignment: $\mathbf{a}_0 \equiv 01,\ \mathbf{a}_1 \equiv 00,\ \mathbf{a}_2 \equiv 10,\ \mathbf{a}_3 \equiv 11$
- Distribution $f_{\mathbf{q}|\mathbf{A}}(\mathbf{q}|\mathbf{a}_0)$: Gaussian with mean $\mathbf{a}_0$ and variance $N_0/2$

$$BER_{\mathbf{a}_0} = \underbrace{\left[Q\left(\frac{1}{\sqrt{N_0/2}}\right) - Q\left(\frac{3}{\sqrt{N_0/2}}\right)\right]}_{P_{e|\mathbf{a}_0\rightarrow\mathbf{a}_1}} \times \underbrace{\frac{1}{2}}_{\frac{m_{e|\mathbf{a}_0\rightarrow\mathbf{a}_1}}{m}}$$
$$+ \underbrace{\left[Q\left(\frac{3}{\sqrt{N_0/2}}\right) - Q\left(\frac{5}{\sqrt{N_0/2}}\right)\right]}_{P_{e|\mathbf{a}_0\rightarrow\mathbf{a}_2}} \times \underbrace{\frac{2}{2}}_{\frac{m_{e|\mathbf{a}_0\rightarrow\mathbf{a}_2}}{m}} + \underbrace{\left[Q\left(\frac{5}{\sqrt{N_0/2}}\right)\right]}_{P_{e|\mathbf{a}_0\rightarrow\mathbf{a}_3}} \times \underbrace{\frac{1}{2}}_{\frac{m_{e|\mathbf{a}_0\rightarrow\mathbf{a}_3}}{m}}$$

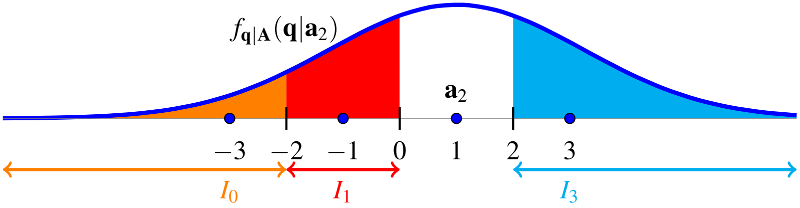
Calculation of $BER_{\mathbf{a}_1}$



- Binary assignment: $\mathbf{a}_0 \equiv 01,\ \mathbf{a}_1 \equiv 00,\ \mathbf{a}_2 \equiv 10,\ \mathbf{a}_3 \equiv 11$
- Distribution $f_{\mathbf{q}|\mathbf{A}}(\mathbf{q}|\mathbf{a}_1)$: Gaussian with mean $\mathbf{a}_1$ and variance $N_0/2$

$$BER_{\mathbf{a}_1} = \underbrace{\left[Q\left(\frac{1}{\sqrt{N_0/2}}\right)\right]}_{P_{e|\mathbf{a}_1\rightarrow\mathbf{a}_0}} \times \underbrace{\frac{1}{2}}_{\frac{m_{e|\mathbf{a}_1\rightarrow\mathbf{a}_0}}{m}} + \underbrace{\left[Q\left(\frac{1}{\sqrt{N_0/2}}\right) - Q\left(\frac{3}{\sqrt{N_0/2}}\right)\right]}_{P_{e|\mathbf{a}_1\rightarrow\mathbf{a}_2}} \times \underbrace{\frac{1}{2}}_{\frac{m_{e|\mathbf{a}_1\rightarrow\mathbf{a}_2}}{m}} + \underbrace{\left[Q\left(\frac{3}{\sqrt{N_0/2}}\right)\right]}_{P_{e|\mathbf{a}_1\rightarrow\mathbf{a}_3}} \times \underbrace{\frac{2}{2}}_{\frac{m_{e|\mathbf{a}_1\rightarrow\mathbf{a}_3}}{m}}$$

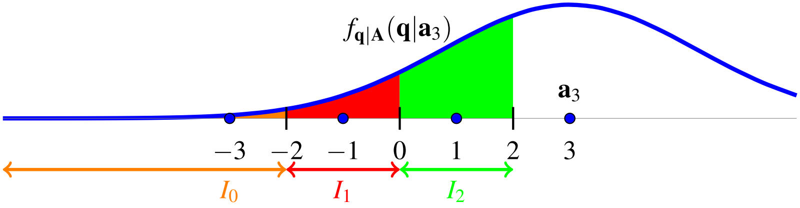
Calculation of $BER_{\mathbf{a}_2}$



- Binary assignment: $\mathbf{a}_0 \equiv 01,\ \mathbf{a}_1 \equiv 00,\ \mathbf{a}_2 \equiv 10,\ \mathbf{a}_3 \equiv 11$
- Distribution $f_{\mathbf{q}|\mathbf{A}}(\mathbf{q}|\mathbf{a}_2)$: Gaussian with mean $\mathbf{a}_2$ and variance $N_0/2$

$$BER_{\mathbf{a}_2} = \underbrace{\left[Q\left(\frac{3}{\sqrt{N_0/2}}\right)\right]}_{P_{e|\mathbf{a}_2\to\mathbf{a}_0}} \times \underbrace{\frac{2}{2}}_{\frac{m_{e|\mathbf{a}_2\to\mathbf{a}_0}}{m}} + \underbrace{\left[Q\left(\frac{1}{\sqrt{N_0/2}}\right) - Q\left(\frac{3}{\sqrt{N_0/2}}\right)\right]}_{P_{e|\mathbf{a}_2\to\mathbf{a}_1}} \times \underbrace{\frac{1}{2}}_{\frac{m_{e|\mathbf{a}_2\to\mathbf{a}_1}}{m}}$$
$$+ \underbrace{\left[Q\left(\frac{1}{\sqrt{N_0/2}}\right)\right]}_{P_{e|\mathbf{a}_2\to\mathbf{a}_3}} \times \underbrace{\frac{1}{2}}_{\frac{m_{e|\mathbf{a}_2\to\mathbf{a}_3}}{m}}$$

Calculation of $BER_{\mathbf{a}_3}$

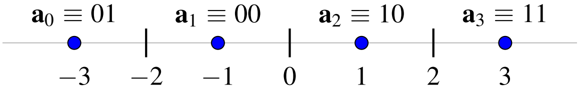


- Binary assignment: $\mathbf{a}_0 \equiv 01,\ \mathbf{a}_1 \equiv 00,\ \mathbf{a}_2 \equiv 10,\ \mathbf{a}_3 \equiv 11$
- Distribution $f_{\mathbf{q}|\mathbf{A}}(\mathbf{q}|\mathbf{a}_3)$: Gaussian with mean $\mathbf{a}_3$ and variance $N_0/2$

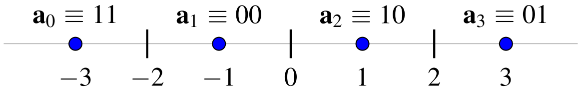
$$BER_{\mathbf{a}_3} = \underbrace{\left[Q\left(\frac{5}{\sqrt{N_0/2}}\right)\right]}_{P_{e|\mathbf{a}_3\to\mathbf{a}_0}} \times \underbrace{\frac{1}{2}}_{\frac{m_{e|\mathbf{a}_3\to\mathbf{a}_0}}{m}} + \underbrace{\left[Q\left(\frac{3}{\sqrt{N_0/2}}\right) - Q\left(\frac{5}{\sqrt{N_0/2}}\right)\right]}_{P_{e|\mathbf{a}_3\to\mathbf{a}_1}} \times \underbrace{\frac{2}{2}}_{\frac{m_{e|\mathbf{a}_3\to\mathbf{a}_1}}{m}}$$
$$+ \underbrace{\left[Q\left(\frac{1}{\sqrt{N_0/2}}\right) - Q\left(\frac{3}{\sqrt{N_0/2}}\right)\right]}_{P_{e|\mathbf{a}_3\to\mathbf{a}_2}} \times \underbrace{\frac{1}{2}}_{\frac{m_{e|\mathbf{a}_3\to\mathbf{a}_2}}{m}}$$

Influence of the binary assignment

- Previous assignment: $$BER = \frac{3}{4} Q\left(\frac{1}{\sqrt{N_o/2}}\right) + \frac{1}{2} Q\left(\frac{3}{\sqrt{N_o/2}}\right) - \frac{1}{4} Q\left(\frac{5}{\sqrt{N_o/2}}\right)$$



- If the binary assignment is changed



  - Terms that do not change: $P_{e|a_i \to a_j}$
  - Terms that do vary $m_{e|a_i \to a_j}$ $\Rightarrow$ The BER varies !!!

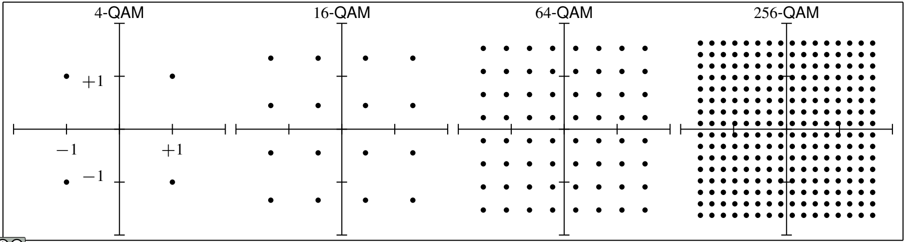
$$BER = \frac{5}{4} Q\left(\frac{1}{\sqrt{N_o/2}}\right) - \frac{1}{4} Q\left(\frac{3}{\sqrt{N_o/2}}\right)$$

Density of constellations

- Constellation size increase (M symbols):
  - Increase in binary rate
    - ★ Increase the number of bits per symbol $m = \log_2 M$
  - Performance reduction for a given E_s
    - ★ Reduction of the distance between constellation points

Example for constellations M-QAM

M (symbols)	m (bits/symbol)	E_s with normalized levels ($d_{min} = 2$)	d_{min} with $E_s = 2$
4	2	2	2
16	4	10	0.8944
64	8	42	0.4364
256	16	170	0.2169



BER as a function of E_s/N_0 or E_b/N_0

- Performance is often represented as a function of E_s/N_0 or E_b/N_0
  - E_b: Average energy per bit of information $E_b = \frac{E_s}{m}$
- Common terms in error probabilities are usually written as

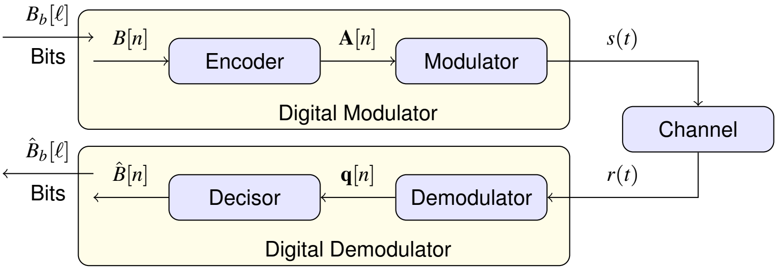
$$Q\left(\frac{A}{\sqrt{N_0/2}}\right)$$

- In any case they can be written as

$$Q\left(v\,\sqrt{\frac{E_s}{N_0}}\right) \text{ or } Q\left(v\sqrt{m}\,\sqrt{\frac{E_b}{N_0}}\right), \quad \text{where } v = \frac{A\sqrt{2}}{\sqrt{E_s}}$$

  - Factor v depends on the constellation

Digital Transmission - Transmission Rates



- Encoder: constellation of M symbols with $m = \log_2 M$ bits/symbol
- Two rates: transmission of 1 symbol (m bits) every T seconds
  - Symbol rate $R_s = \frac{1}{T}$ symbols/s (baud)
  - Bit rate $R_b = m \times R_s$ bits/s   Bit time: $T_b = \frac{T}{m}$ seconds

Signal Generation

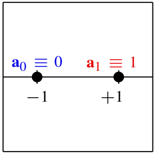
- Encoder: constellation of M symbols (dimension N)
  - Binary allocation of m bits per symbol
    - Gray coding
- Modulator: orthonormal basis of dimension N
  - Association of a signal (of T seconds) to each symbol $$s_i(t) = \sum_{j=0}^{N-1} a_{i,j} \times \phi_j(t)$$
- Generation of the signal by blocks of size T seconds (symbol intervals)
  - If $\mathbf{A}[n] = \mathbf{a}_k$ then $s(t) = s_k(t - nT)$ in $nT \leq (n+1)T$
    - In the symbol interval associated to $A[n]$ ($nT \leq (n+1)T$) the waveform associated to the corresponding symbol is placed
  - Analytical expression of the complete signal

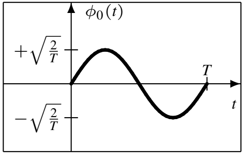
$$s(t) \sum_{n} \sum_{j=0}^{N-1} A_j[n] \times \phi_j(t - nT)$$

$A_j[n]$: coordinate of index j of symbol $\mathbf{A}[n]$

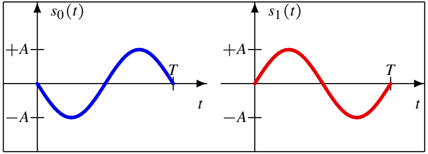
Signal Generation - Example A

Constellation (ENCODER) and orthonormal basis (MODULATOR)

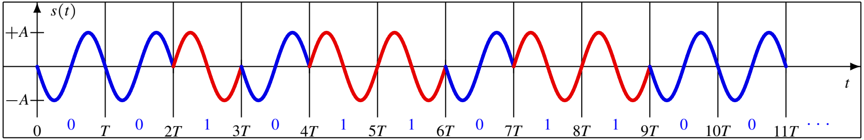




Signals associated with each of the symbols: $s_0(t) = -1 \times \phi_0(t),\, s_1(t) = +1 \times \phi_0(t)$

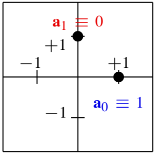


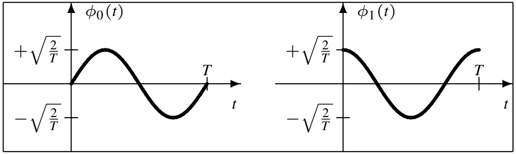
Modulated signal for the following binary sequence $B_b[\ell] = 0\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 0\ 0\ \cdots$



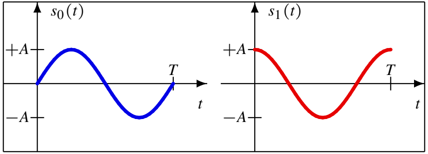
Signal Generation - Example B

Constellation (ENCODER) and orthonormal basis (MODULATOR)

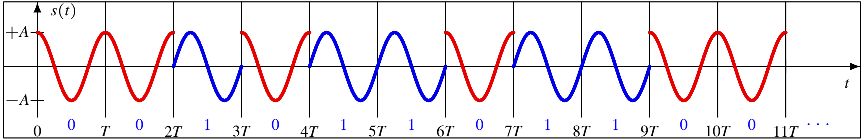




Signals associated with each of the symbols $s_0(t) = +1 \times \phi_0(t) + 0 \times \phi_1(t),\ s_1(t) = 0 \times \phi_0(t) + 1 \times \phi_1(t)$

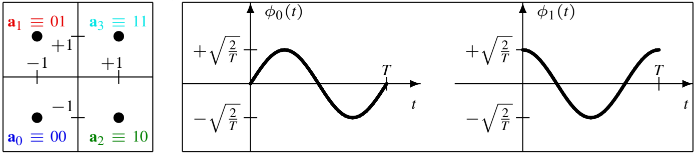


Modulated signal for the following binary sequence $B_b[\ell] = 0\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 0\ 0\ \cdots$

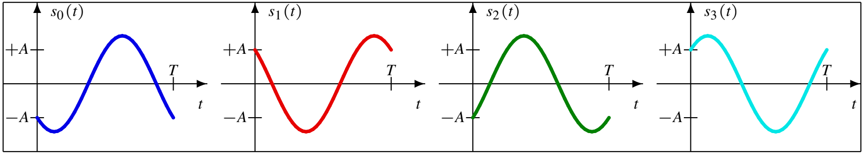


Signal Generation - Example C

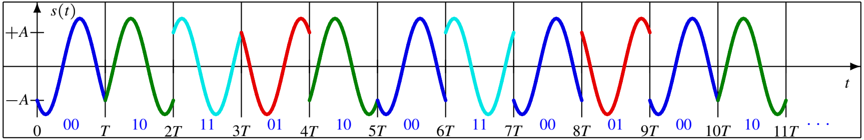
Constellation (ENCODER) and orthonormal basis (MODULATOR)



Signals: $s_0(t) = -1 \times \phi_0(t) - 1 \times \phi_1(t), s_1(t) = -1 \times \phi_0(t) + 1 \times \phi_1(t), s_2(t) = +1 \times \phi_0(t) - 1 \times \phi_1(t), s_3(t) = +1 \times \phi_0(t) + 1 \times \phi_1(t)$

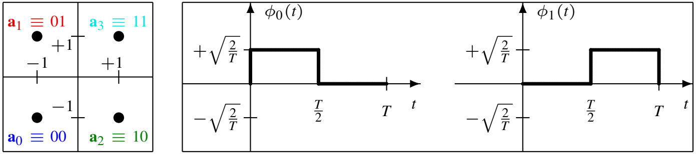


Modulated signal for the following binary sequence $B_b[\ell] = 00\ 10\ 11\ 01\ 10\ 00\ 11\ 00\ 01\ 00\ 10\ \cdots$

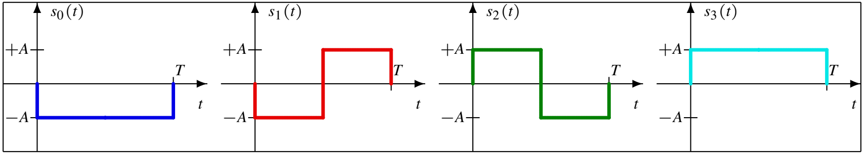


Signal Generation - Example D

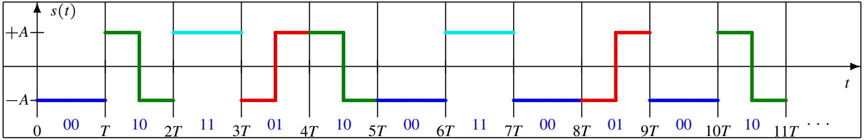
Constellation (ENCODER) and orthonormal basis (MODULATOR)



Signals: $s_0(t) = -1 \times \phi_0(t) - 1 \times \phi_1(t), s_1(t) = -1 \times \phi_0(t) + 1 \times \phi_1(t), s_2(t) = +1 \times \phi_0(t) - 1 \times \phi_1(t), s_3(t) = +1 \times \phi_0(t) + 1 \times \phi_1(t)$



Modulated signal for the following binary sequence $B_b[\ell] = 00\ 10\ 11\ 01\ 10\ 00\ 11\ 00\ 01\ 00\ 10\ \cdots$



Choice of modulator

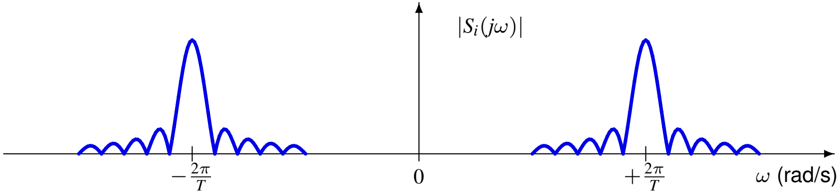
- It has been seen that the same encoder can be combined with multiple modulators (sets of N functions forming an orthonormal basis)
- An encoder (constellation+binary assignment) determines
  - Energy of the signals to be transmitted
  - Benefits
    - Probability (rate) of symbol error P_e
    - Probability (rate) of bit error BER
- Choice of modulator
  - Adaptation to the transmission channel
    - Perfect fit:

$$\phi_i(t) * h(t) = \phi_i(t) \leftrightarrow \Phi_i(j\omega) \times H(j\omega) = \Phi_i(j\omega)$$

    - Difficult to achieve exactly in practice
    - Search for signals whose frequency response is in the passband of the transmission channel
      - Distinction between baseband and bandpass channels

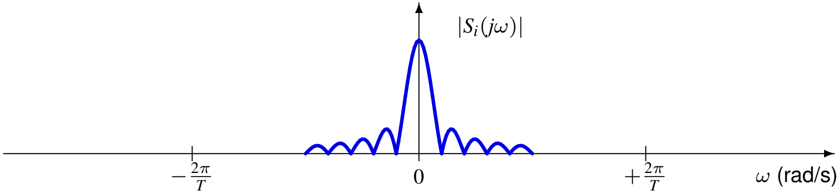
Frequency response of signals

- Depends on the frequency response of the base elements
- Example C



  - Signals suitable for transmission on channels with "*good response*" around the frequency $\frac{2\pi}{T}$ radians/s

- Example D



  - Signals suitable for transmission on channels with "*good response*" at low frequencies