Chapter 3 : Exercises

Exercise 3.1 For the signals given in Figure 3.1

a) Apply the Gram-Schmidt procedure to obtain an orthonormal basis that allows the vector representation of the signals

   i) Obtain the orthonormal basis, and say what the dimension of the signal space is.

   ii) Obtain the vector representation of the signals.

b) Calculate the energy of each signal from its vector representation (compare it with that obtained from the continuous time definition of the signal).

c) Calculate the energy of the difference between each of the six signals and the signal $s_0(t)$ from the vector representation of the signals (you can compare it with the one obtained in the time domain).

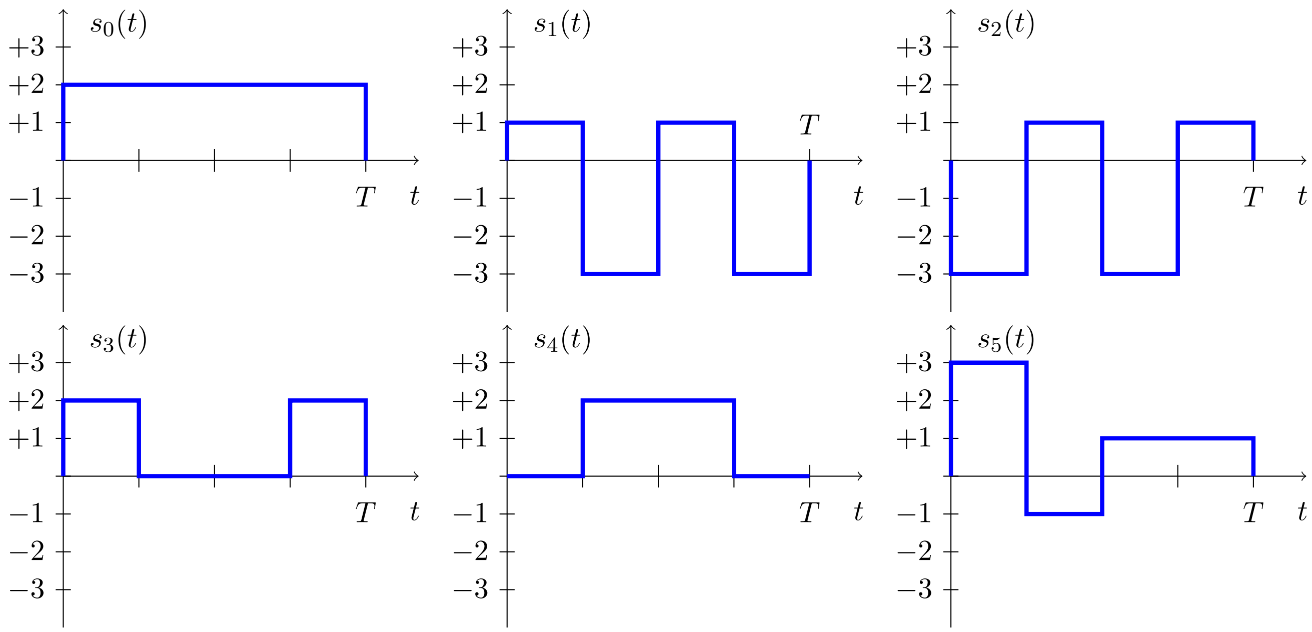


Figure 3.1: Signals for Exercise 3.1.

Exercise 3.2 A binary communications system employs causal rectangular pulses of duration T and amplitudes $\pm A$ to transmit information at a rate of 10 kbits/s. If the power spectral density of the Gaussian additive noise is $N_0/2$, with $N_0 = 10^{-4}$ W/Hz, and assuming that both symbols are transmitted with the same probability, determine the value of A needed to:

a) Get an approximate symbol error rate $P_e \approx 10^{-4}$.

b) Get an approximate bit error rate $BER \approx 10^{-6}$.

Exercise 3.3 A binary communications system uses a constellation with two symbols, $\mathbf{a}_0 = +A$ and $\mathbf{a}_1 = 0$. If it is transmitted over a Gaussian additive channel, with power spectral density $N_0/2$, and the probabilities of the symbols are $p_A(\mathbf{a}_0) = 1/3$, and $p_A(\mathbf{a}_1) = 2/3$:

a) Compute the optimal detector (i.e., compute the value of the decision threshold).

b) Calculate the probability of error.

Exercise 3.4 We have the constellation of Figure 3.2. The symbols are assumed to have the same probability. Taking $T = 1$

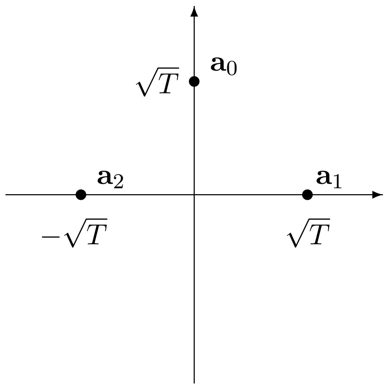


Figure 3.2: Constellation for Exercise 3.4.

a) Calculate the average energy per symbol, E_s.

b) Design an alternative constellation that, with the same probability of error, has the minimum mean energy per symbol. Calculate the new value of E_s.

c) For either constellation, obtain a limit on the probability of error using the union bound and the loose bound.

Exercise 3.5 For the implementation of a three-symbol equiprobable communications system, the modulator transmits the three signals shown in Figure 3.3.

a) Calculate the average energy per symbol.

b) Calculate some bound on the error probability.

Exercise 3.6 A communications system uses the one-dimensional constellation

$$\mathbf{a}_0 = 0,\ \mathbf{a}_1 = 1,\ \mathbf{a}_2 = 2,\ \mathbf{a}_3 = 3,$$

The observations provided by the demodulator are characterized by the equivalent discrete channel

$$q[n] = A[n] + z[n],$$

where each sample of the noise term, $z[n]$, has the probability density function represented in Figure 3.4.

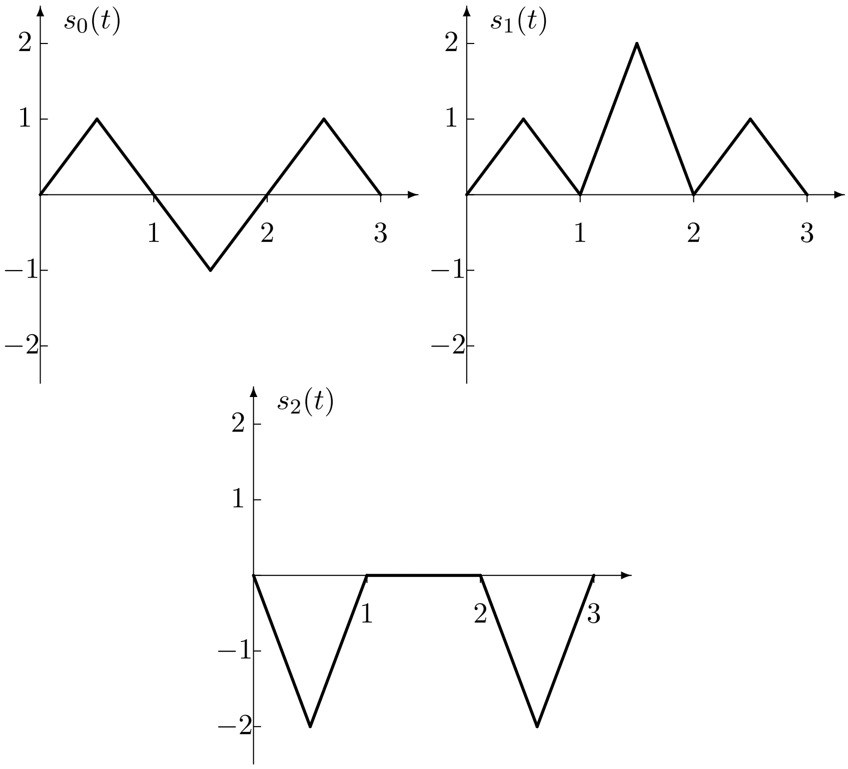


Figure 3.3: Signals for Exercise 3.5.

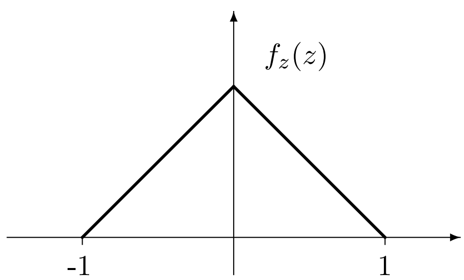


Figure 3.4: Constellation and probability density function of the noise term for Exercise 3.6.

a) Obtain the analytical expression of the probability density function of the conditional observation given the transmission of each symbol (i.e., $f_{\mathbf{q}|\mathbf{A}}(\mathbf{q}|\mathbf{a}_i)$, for all i).

b) Design the optimal detector for equiprobable symbols.

c) Calculate the probability of error for the previous detector.

d) Design the optimal detector for the following symbol probabilities: $p_A(\mathbf{a}_0) = p_A(\mathbf{a}_3) = 1/6$, $p_A(\mathbf{a}_1) = p_A(\mathbf{a}_2) = 1/3$.

e) Calculate the probability of error for the previous detector.

Exercise 3.7 A binary communications system uses the following signals to transmit the two symbols:

$$s_0(t) = -s_1(t) = \begin{cases} A, & 0 \leq t \leq T \\ 0, & \text{else} \end{cases}.$$

The receiver is implemented as shown in Figure 3.5 (representation for receiving the first symbol of the sequence).

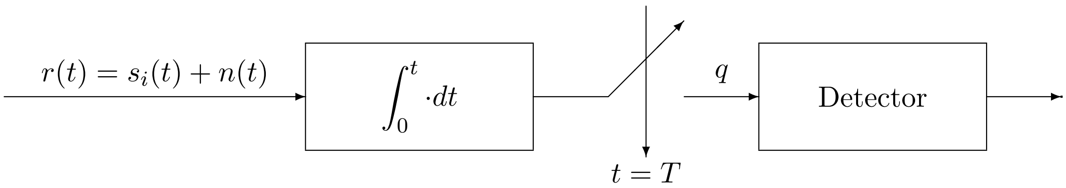


Figure 3.5: Receiver for Exercise 3.7.

Determine the signal-to-noise ratio at the output of the demodulator (at q), assuming that the additive noise is zero-mean, Gaussian, and with power spectral density $N_0/2$ W/Hz.

Exercise 3.8 The Manchester line code uses the signals in Figure 3.6 to transmit the two symbols of the code.

a) Determine the probability of error if both symbols are transmitted with equal probability.

b) Determine the probability of error if the probability of the symbols is $p_A(a_0) = p$ and $p_A(a_1) = 1 - p$.

Exercise 3.9 Consider a digital communications system that transmits information using a QAM constellation at a symbol rate of 2400 baud (symbols/s). Additive white and Gaussian noise is assumed.

a) Determine the ratio E_b/N_0 required to achieve an approximate symbol error probability of 10^{-5} for a bit rate of 4800 bits/s.

b) Repeat the calculation for a bit rate of 9600 bits/s.

c) Repeat the calculation for a bit rate of 19200 bits/s.

d) Discuss the conclusions obtained from these results.

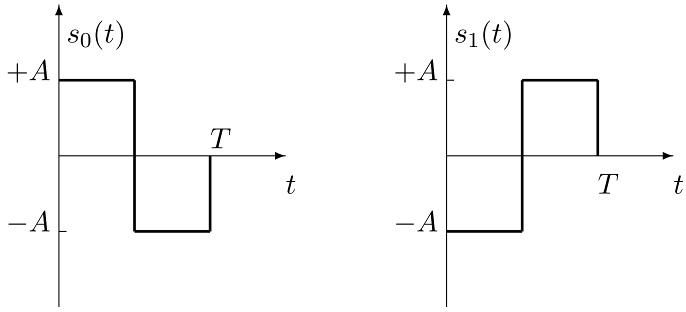


Figure 3.6: Signals for Exercise 3.8.

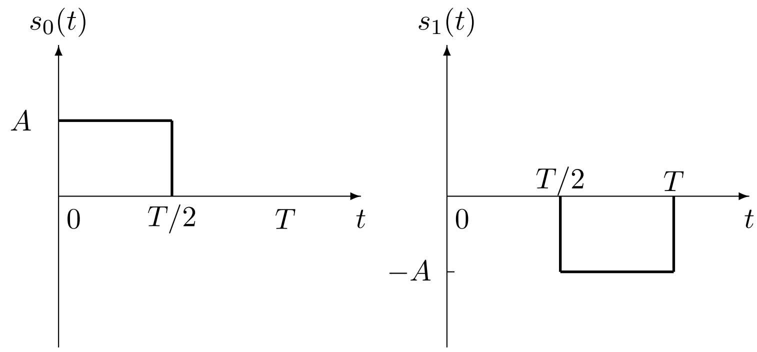


Figure 3.7: Signals for Exercise 3.10.

Exercise 3.10 A communications system transmits two symbols using the signals $s_0(t)$ and $s_1(t)$ shown below in Figure 3.7

a) We have an additive Gaussian channel (white noise with power spectral density $N_0/2$). Draw the constellation, design the optimal receiver (demodulator + detector), and calculate the error probability.

b) If the channel is such that its output to an input $s(t)$ is $\alpha s(t) + \alpha A$, again with the same type of additive noise, redesign the optimal detector and calculate the probability of error.

c) If in the situation of section a) the demodulator in Figure 3.8 is used, design the optimal detector and calculate the probability of error in this case. Comment on the results comparing them with those obtained in section a).

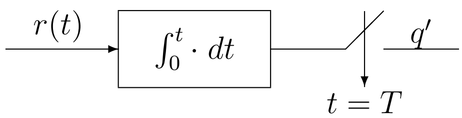


Figure 3.8: Demodulator for Exercise 3.10.

NOTE: Note that if a normalized demodulator is not used, the discrete noise variance is no longer $N_0/2$.

Exercise 3.11 The four signals in Figure 3.9 are used to transmit 4 equally probable symbols in a communications system. It is considered that these signals are transmitted through a Gaussian channel with power spectral density $N_0/2$.

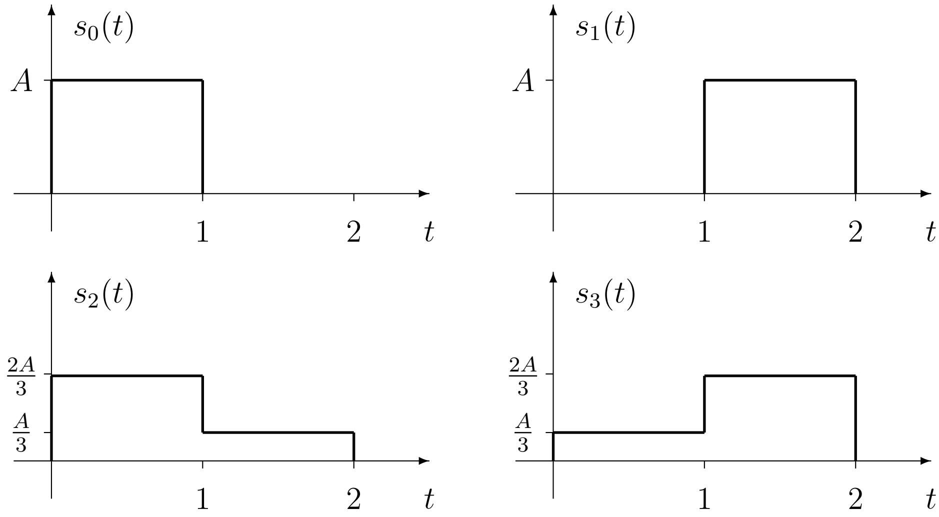


Figure 3.9: Signals for Exercise 3.11.

a) Design the transmitter: encoder (constellation) and modulator ($\{\phi_i(t)\}$, $i = 0.1, \cdots, N$).

b) Calculate the average energy per symbol, and make an optimal assignment of bits to each symbol justifying the assignment.

c) Design the optimal receiver (demodulator + detector) using causal matched filters (the analytical expression must be provided or drawn), obtain the expressions of the probability density function of the observation at the output of the demodulator conditioned on the transmission of each symbol ($f_{\mathbf{q}|\mathbf{A}}(\mathbf{q}|\mathbf{a}_i)$, $i = 0, 1, 2, 3$), and calculate the probability of error.

d) If the demodulator in Figure 3.10 is used, design the optimal detector, obtain the expressions of the conditional probability density function of the observation at the output of the demodulator given the transmission of each symbol ($f_{q|\mathbf{A}}(q|\mathbf{a}_i)$, $i = 0, 1, 2, 3$), and calculate the probability of error. Compare this value with the one obtained in the previous section and explain the results obtained.

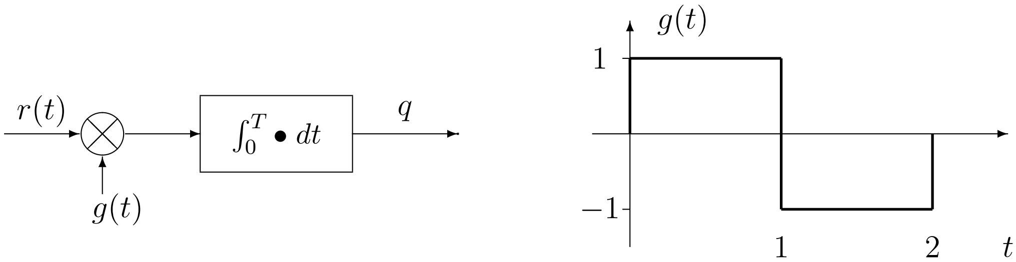


Figure 3.10: Demodulator for Exercise 3.11.

Exercise 3.12 The transmitter of a communications system has a symbol rate $R_s = 10^3$ bauds. Additive white noise is assumed, Gaussian, and with power spectral density $N_0/2$, with $N_0 = 2 \times 10^{-2}$.

a) An approximation commonly used in digital communications for the symbol error rate is

$$P_e \approx k \cdot Q\left(\frac{d_{min}}{2\sqrt{N_o/2}}\right), \tag{3.1}$$

where d_{min} is the minimum distance between two points on the constellation and k is the maximum number of symbols found within d_{min} of any constellation symbol.

i) Using the approximation (3.1), design the optimal one-dimensional encoder of the communications system, with the lowest average energy per symbol, to obtain an approximate symbol error rate $P_e \approx 2 \cdot 10^{-4}$ transmitting at a bit rate $R_b = 2 \times 10^3$ bits/s.

ii) Make an optimal assignment of bits to each symbol, explaining the reason for such assignment, and calculate the binary error rate, approximately.

b) If the communications system uses the encoder and the modulator defined in Figure 3.11:

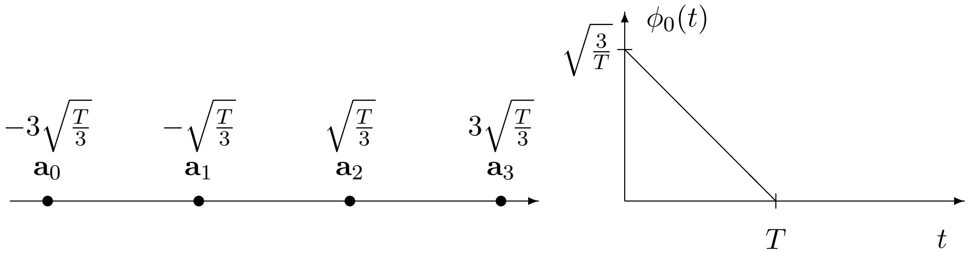


Figure 3.11: Encoder and modulator for Exercise 3.12.

i) Design the optimal demodulator using a correlator and using a causal matched filter (in the latter case give the analytical expression for the impulse response of the filter or draw it).

ii) If for simplicity, instead of the optimal demodulator, a demodulator that performs the following operation on the received signal ($r(t)$) is used

$$q = 2 \cdot \int_0^T r(t)\, dt,$$

design the optimal detector and calculate the symbol error rate assuming equiprobable symbols. Discuss if this probability of error will be greater or less than that obtained with the demodulator from the previous section.

Exercise 3.13 A communications system is going to be designed that will use the eight Figure 3.12 signals to transmit eight symbols with equal probability. The channel only introduces noise, which will be considered white, Gaussian, stationary and with density power spectral $N_0/2$. For simplicity in calculations, consider $T = 2$.

a) Assign the bits carried by each of the signals to obtain the minimum bit error rate, and justify the reason for choosing that assignment.

b) Obtain 8 other alternative signals (draw the eight signals or provide their analytical expressions) with which the same probability of error can be obtained as with the original set, but which require a minimum average energy per symbol, and calculate that energy.

c) Design the optimal receiver (demodulator + detector), and calculate the error probability (you can do this using both the original set of signals and the one obtained in section b), since in both cases the error probability must be the same).

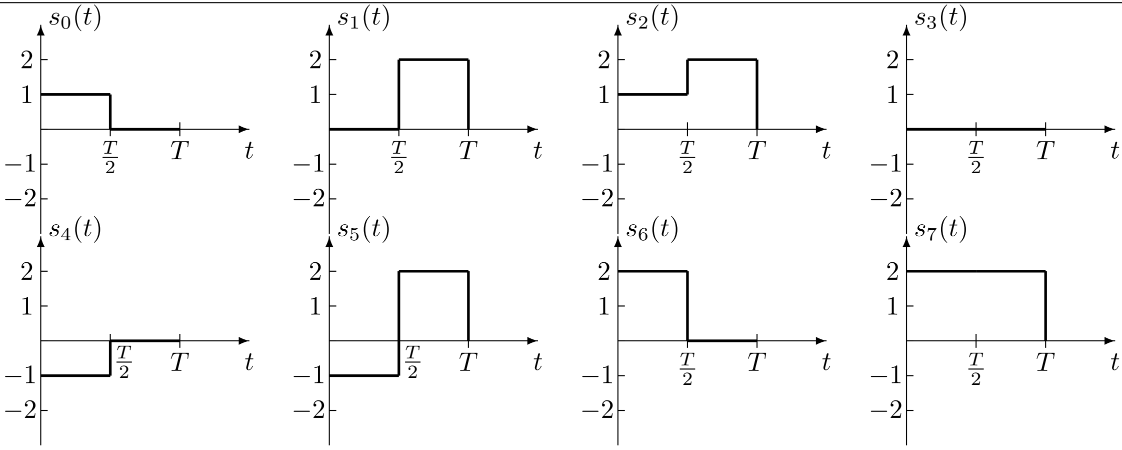


Figure 3.12: Signals for Exercise 3.13.

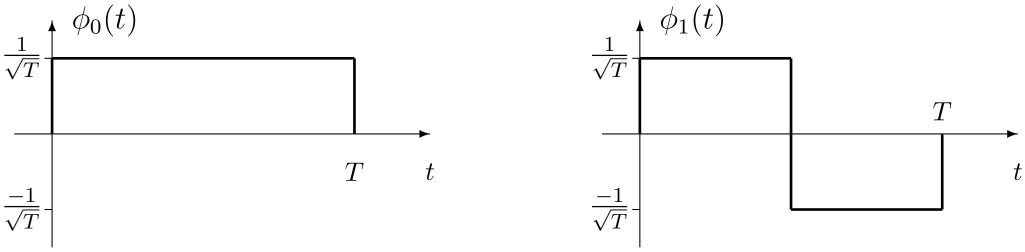


Figure 3.13: Basis for Exercise 3.14.

Exercise 3.14 A communications system uses a constellation made up of the following 4 symbols,

$$\mathbf{a}_0 = \begin{bmatrix} +1 \\ +1 \end{bmatrix}, \mathbf{a}_1 = \begin{bmatrix} -1 \\ +1 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \mathbf{a}_3 = \begin{bmatrix} +1 \\ -1 \end{bmatrix},$$

that are transmitted with equal probability, and a modulator given by the basis functions of Figure 3.13.

For simplicity in calculations, hereinafter consider $T = 1$. Consider also transmission over a Gaussian channel with white noise power spectral density $N_0/2$.

a) Perform the binary assignment that minimizes the probability of bit error, justifying said assignment (without the justification appropriate, the assignment will not be valued), and calculate the speed of symbol transmission, R_s, and the transmission rate binary, R_b.

b) Represent the four signals used for the transmission of each symbol, $s_i(t)$, $i = \{0, 1, 2, 3\}$, and the signal resulting from the transmission of the following sequence of symbols

$$\mathbf{A}[0] = \mathbf{a}_1,\ \mathbf{A}[1] = \mathbf{a}_0,\ \mathbf{A}[2] = \mathbf{a}_3,\ \mathbf{A}[3] = \mathbf{a}_0,\ \mathbf{A}[4] = \mathbf{a}_2.$$

c) Design the optimal receiver (demodulator + decider), and calculate the probability of error obtained with this receiver.

d) If instead of the optimal demodulator the optimal demodulator is used Figure 3.14, design the optimal decision maker for that demodulator and calculate the resulting error probability.

Exercise 3.15 A digital communications system uses a constellation of 8 equiprobable symbols that are shown in the figure.

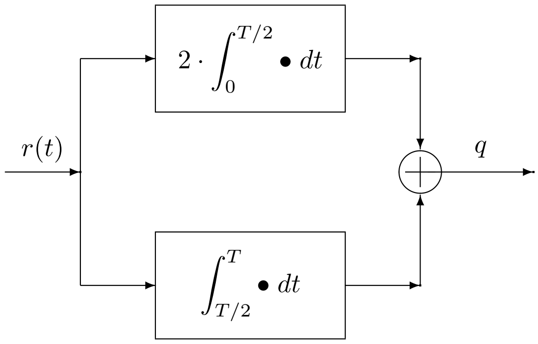
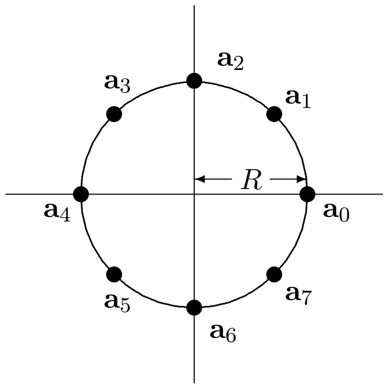


Figure 3.14: Demodulator for Exercise 3.14.



The noise associated with the transmission is modeled as usual: stationary, white, Gaussian thermal noise process, zero mean and with power spectral density $S_n(j\omega) = \frac{N_0}{2}$, where $N_0 = 0.03$ for this case.

a) If the symbol error rate must be approximately $P_e \approx 2 \times 10^{-6}$:

  i) Determine the value of the required radius R.

  ii) Calculate the average energy per symbol, E_s, of the resulting constellation.

  iii) Perform the optimal binary assignment justifying it, and estimate the approximated bir error rate (BER) using such assignment if it is assumed that the signal to noise ratio is high.

b) If you want to transmit at a binary rate $R_b = 1500$ bits per second through a baseband channel (initially without bandwidth limitation):

  i) Choose an appropriate orthonormal basis for the transmission justifying your choice (you must provide the analytic expressions for each element of the basis or a drawing of it.

  ii) Design an optimal demodulator for the communications system.

c) Regarding the detector:

  i) Design the optimal detector for the system.

  ii) Calculate the union bound (without neglecting any terms or making any approximation).

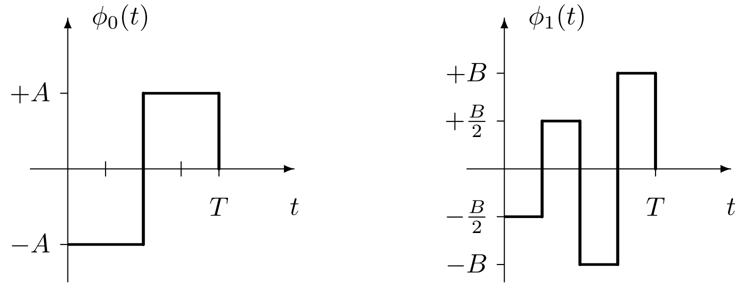
  iii) Obtain the loose bound and write the resul as a function of the $\frac{E_s}{N_0}$ ratio.

NOTE: the distance between two points on a circle of radius R is given by $d = 2R\sin\left(\frac{\theta}{2}\right)$, where θ is the angle formed by the radii that join these points with the center.

Exercise 3.16 A communications system uses a constellation with the following 4 symbols,

$$\mathbf{a}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathbf{a}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{a}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

that are transmitted with equal probability, and a modulator given by the basis functions of the figure.



For simplicity in calculations, hereinafter consider $T = 1$. Consider also transmission over a Gaussian channel with white noise with power spectral density $N_0/2$.

a) Determine the values of A and B, show that the functions form an orthonormal basis, and calculate the transmission rates, symbol rate and binary rate.

b) Represent the signals $s_i(t)$ associated with the transmission of each symbol $\mathbf{a}_i$, and the fragment of the signal resulting from the transmission of the next 5 initial symbols of the sequence

$$\mathbf{A}[0] = \mathbf{a}_1,\ \mathbf{A}[1] = \mathbf{a}_0,\ \mathbf{A}[2] = \mathbf{a}_3,\ \mathbf{A}[3] = \mathbf{a}_0,\ \mathbf{A}[4] = \mathbf{a}_2.$$

c) Perform the binary assignment for each symbol, justifying said assignment, and calculate the average energy per symbol of the system. Is the constellation used the most appropriate in terms of the trade-off between energy use and performance? (explain clearly why yes or why not).

d) Design the optimal demodulator, the optimal detector, and calculate the exact symbol error rate of the system.

e) Obtain the approximation of the symbol error rate and bound it probability using the union bound and the loose bound.

Exercise 3.17 A communications system uses the following constellation of 4 equiprobable symbols

$$\mathbf{a}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ \mathbf{a}_1 = \begin{bmatrix} 0 \\ 2 \end{bmatrix},\ \mathbf{a}_2 = \begin{bmatrix} -1 \\ +1 \end{bmatrix},\ \mathbf{a}_3 = \begin{bmatrix} +1 \\ +1 \end{bmatrix}$$

a) For this system

  i) Perform the binary assignment with the objective of having the minimum bit error rate, explaining why using that assignment and not another, implies minimizing the probability of error.

ii) Obtain the average energy per symbol and discuss whether it is a constellation with a good trade-off between performance and energy use.

b) Suppose for this section that you want to transmit through a low-pass channel (baseband transmission) with thermal additive noise (with the usual statistical model). Then

i) Design a modulator suitable for that type of channel, explaining the reason for the choice.

ii) Represent the signal $s_2(t)$ associated with the symbol $\mathbf{a}_2$ if that modulator is used.

iii) Design an optimal demodulator if that modulator is used.

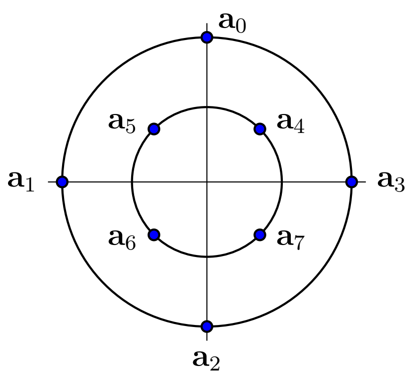
iv) Design the optimal detector (with minimum symbol error rate).

c) Assuming that with the modulator that was chosen in the previous section there is a perfect adaptation to the channel, or what is the same, that it is transmitted through a Gaussian channel with power spectral density $N_0/2$ with $N_0 = 0.02$.

i) Calculate the usual approximation of the symbol error rate, the approximation of the bit error rate if the binary assignment of the first section is used, and the loose bound expression, clearly indicating the numerical value of all the terms that appear in the expression.

ii) Calculate the exact symbol error rate, and write it as a function of the E_s/N_0 ratio.

Exercise 3.18 A communications system that transmits at a symbol rate $R_s = 4$ Mbauds uses the 8-symbol constellation shown in the figure, which is called circular 8-QAM. In this case $A = 1+\sqrt{3}$ and $B = 1$, and the 8 symbols are transmitted with equal probability.



$$\mathbf{a}_0 = \begin{bmatrix} 0 \\ +A \end{bmatrix} \quad \mathbf{a}_4 = \begin{bmatrix} +B \\ +B \end{bmatrix}$$

$$\mathbf{a}_1 = \begin{bmatrix} -A \\ 0 \end{bmatrix} \quad \mathbf{a}_5 = \begin{bmatrix} -B \\ +B \end{bmatrix}$$

$$\mathbf{a}_2 = \begin{bmatrix} 0 \\ -A \end{bmatrix} \quad \mathbf{a}_6 = \begin{bmatrix} -B \\ -B \end{bmatrix}$$

$$\mathbf{a}_3 = \begin{bmatrix} +A \\ 0 \end{bmatrix} \quad \mathbf{a}_7 = \begin{bmatrix} +B \\ -B \end{bmatrix}$$

The modulator, for a baseband transmission, is defined by

$$\phi_0(t) = \begin{cases} \frac{1}{\sqrt{T}}, & \text{if } 0 \leq t < T \\ 0, & \text{otherwise} \end{cases}, \quad \phi_1(t) = \begin{cases} +\frac{1}{\sqrt{T}}, & \text{if } 0 \leq t < \frac{T}{2} \\ -\frac{1}{\sqrt{T}}, & \text{if } \frac{T}{2} \leq t < T \\ 0, & \text{otherwise} \end{cases}.$$

a) For this system

i) Calculate the bit error rate and average energy per symbol.

ii) Design the optimal demodulator and detector. For the demodulator, use causal matched filters, and draw the block diagram of the demodulator and the response of the causal filters.

iii) Calculate the union bound and the usual approximation of the symbol error probability.

b) Temporarily an error occurs in the operation of the demodulator, so that it cannot obtain the second coordinate of the vector representation of the received signal, and only the first is available (that is, the vector $\mathbf{q}[n] = \begin{bmatrix} q_0[n] \\ q_1[n] \end{bmatrix}$ only $q_0[n]$ is available). While the problem is being solved, the user decides not send all 8 symbols, but only 4, at the cost of losing binary transmission rate, but with the objective of having a better probability of error.

i) Choose the subset of 4 symbols that you would transmit if you want to obtain the best performance taking into account the demodulator problem.

ii) Design the detector you would use in that case.

iii) Calculate the exact symbol error probability that the system would have in that case.