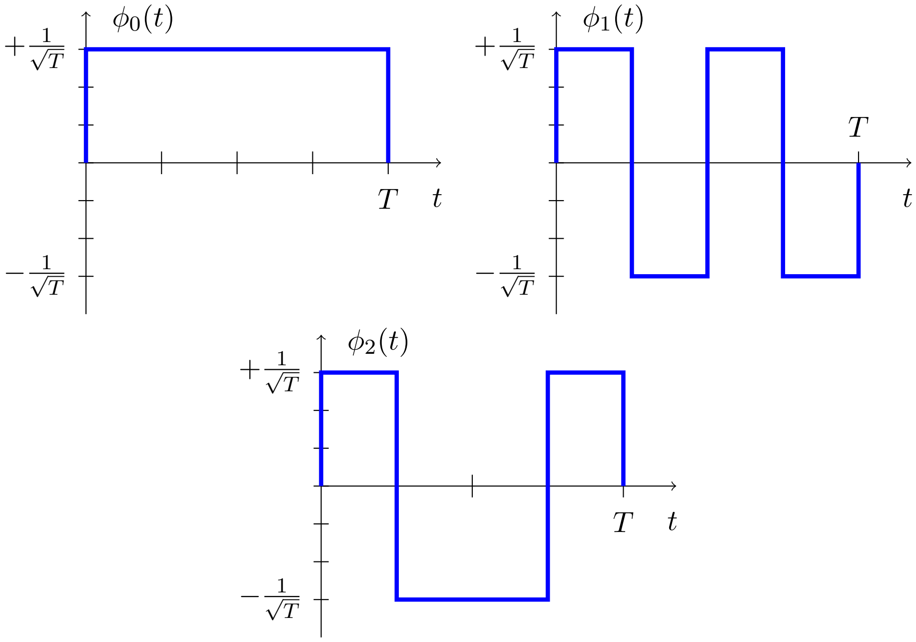
Chapter 3 : Solutions to the exercises

Exercise 3.1 Solution

a) Result of the application of the Gram-Schmidt procedure following the order given by the signal indices

i) The dimension of the signal space is $N = 3$, and the three elements that form the orthonormal basis are those in the figure



ii) The vector representation of the signals is

$$\mathbf{a}_0 = \begin{bmatrix} 2\sqrt{T} \\ 0 \\ 0 \end{bmatrix}, \ \mathbf{a}_1 = \begin{bmatrix} -\sqrt{T} \\ 2\sqrt{T} \\ 0 \end{bmatrix}, \ \mathbf{a}_2 = \begin{bmatrix} -\sqrt{T} \\ -2\sqrt{T} \\ 0 \end{bmatrix} \ \mathbf{a}_3 = \begin{bmatrix} \sqrt{T} \\ 0 \\ \sqrt{T} \end{bmatrix}, \ \mathbf{a}_4 = \begin{bmatrix} \sqrt{T} \\ 0 \\ -\sqrt{T} \end{bmatrix}, \ \mathbf{a}_5 = \begin{bmatrix} \sqrt{T} \\ \sqrt{T} \\ \sqrt{T} \end{bmatrix}$$

b)

$$\mathcal{E}\{\mathbf{a}_0\} = 4T, \ \mathcal{E}\{\mathbf{a}_1\} = 5T, \ \mathcal{E}\{\mathbf{a}_2\} = 5T, \ \mathcal{E}\{\mathbf{a}_3\} = 2T, \ \mathcal{E}\{\mathbf{a}_4\} = 2T, \ \mathcal{E}\{\mathbf{a}_5\} = 3T$$

c) The energy of the difference between each pair of signals

$$\mathcal{E}\left\{s_0(t) - s_0(t)\right\} = 0, \ \mathcal{E}\left\{s_0(t) - s_1(t)\right\} = 13T, \ \mathcal{E}\left\{s_0(t) - s_2(t)\right\} = 13T$$

$$\mathcal{E}\left\{s_0(t) - s_3(t)\right\} = 2T, \ \mathcal{E}\left\{s_0(t) - s_4(t)\right\} = 2T, \ \mathcal{E}\left\{s_0(t) - s_5(t)\right\} = 3T$$

Exercise 3.2 Solution

In this case, since the system is binary, the probability of symbol and bit error coincide

a) $A = 2.6163$

b) $A = 3.3588$

Exercise 3.3 Solution

a) The threshold that defines the optimal detector and the decision regions are

$$q_u = \frac{A^2 + N_0 \ln 2}{2A},\ I_0 = \{q : q \geq q_u\},\ I_1 = \{q : q < q_u\}$$

b)

$$P_e = \frac{1}{3} Q\left(\frac{A - q_u}{\sqrt{N_0/2}}\right) + \frac{2}{3} Q\left(\frac{q_u}{\sqrt{N_0/2}}\right)$$

Exercise 3.4 Solution

a) $E_s = 1$ J.

b) The alternative constellation, which has energy $E_s' = \frac{8}{9}$ J, is

$$\mathbf{a}_0' = \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix},\ \mathbf{a}_1' = \begin{bmatrix} 1 \\ -\frac{1}{3} \end{bmatrix},\ \mathbf{a}_2' = \begin{bmatrix} -1 \\ -\frac{1}{3} \end{bmatrix}$$

c) The union bound is

$$P_e \leq \frac{4}{3} Q\left(\frac{1}{\sqrt{N_0}}\right) + \frac{2}{3} Q\left(\frac{1}{sqrt N_0/2}\right)$$

The loose bound is

$$P_e \leq 2Q\left(\frac{1}{\sqrt{N_0}}\right)$$

Exercise 3.5 Solution

a) $E_s = \frac{17}{9}$ J.

b) The loose boud is

$$P_e \leq 2Q\left(\frac{\sqrt{3}}{2\sqrt{N_0/2}}\right)$$

Exercise 3.6 Solution

a) If $\Lambda(x)$ denotes the triangle function, with support between -1 and $+1$ and unit amplitude

$$\Lambda(x) = \begin{cases} 1 + x, & \text{si } -1 \leq x < 0 \\ 1 - x, & \text{si } 0 \leq x \leq 1 \end{cases}$$

The expressions of the conditional probability density functions are as follows

$$f_{\mathbf{q}|\mathbf{A}}(q|\mathbf{a}_0) = \Lambda\left(q\right),\ f_{\mathbf{q}|\mathbf{A}}(q|\mathbf{a}_1) = \Lambda\left(q-1\right),\ f_{\mathbf{q}|\mathbf{A}}(q|\mathbf{a}_2) = \Lambda\left(q-2\right),\ f_{\mathbf{q}|\mathbf{A}}(q|\mathbf{a}_3) = \Lambda\left(q-3\right)$$

b) Decision regions for equiprobable symbols

$$I_0 = \left\{q : q < \frac{1}{2}\right\},\ I_1 = \left\{q : \frac{1}{2} \leq q < \frac{3}{2}\right\},\ I_2 = \left\{q : \frac{3}{2} \leq q < \frac{8}{3}\right\},\ I_3 = \left\{q : q \geq \frac{8}{3}\right\}$$

c) Error probability for equiprobable symbols

$$P_e = \frac{3}{16}.$$

d) Decision regions for non-equiprobable symbols

$$I_0 = \left\{q : q < \frac{1}{3}\right\},\ I_1 = \left\{q : \frac{1}{3} \leq q < \frac{3}{2}\right\},\ I_2 = \left\{q : \frac{3}{2} \leq q < \frac{5}{2}\right\},\ I_3 = \left\{q : q \geq \frac{5}{2}\right\}$$

e) Error probability for non-equiprobable symbols

$$P_e = \frac{7}{36}.$$

Exercise 3.7 Solution

$$\left.\frac{S}{N}\right|_q = \frac{2A^2T}{N_0}$$

Exercise 3.8 Solution

a) $P_e = Q\left(\frac{A\sqrt{T}}{\sqrt{N_0/2v}}\right)$

b)

$$P_e = p\, Q\left(\frac{A\sqrt{T} - q_u}{\sqrt{N_0/2}}\right) + (1-p)\, Q\left(\frac{A\sqrt{T} + q_u}{\sqrt{N_0/2}}\right)$$

where

$$q_u = \frac{N_0 \ln \frac{1-p}{p}}{4A\sqrt{T}}$$

Exercise 3.9 Solution

a) $\frac{E_b}{N_0} \approx 9.68$ ($\frac{E_b}{N_0}$(dB) ≈ 9.85 dB)

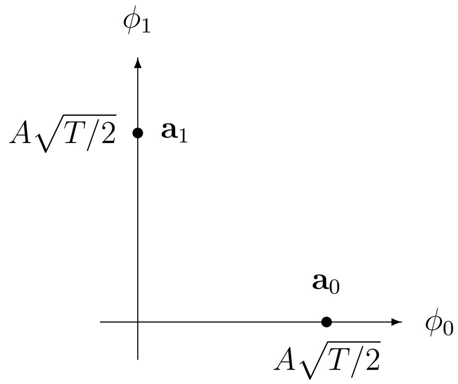
b) $\frac{E_b}{N_0} \approx 25.31$ ($\frac{E_b}{N_0}$(dB) ≈ 14.03 dB)

c) $\frac{E_b}{N_0} \approx 439.928$ ($\frac{E_b}{N_0}$(dB) ≈ 26.43 dB)

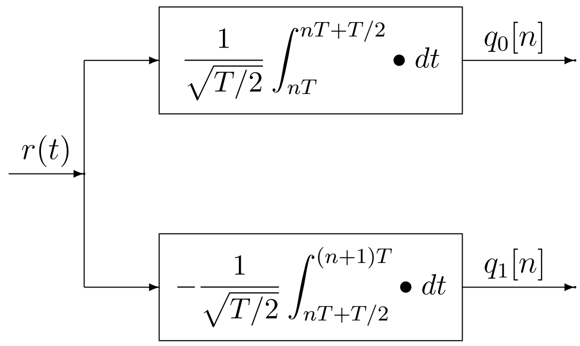
d) Transmitting at a higher bit rate for the same symbol rate means using denser constellations, and with denser constellations more energy is required to obtain the same performance.

Exercise 3.10 Solution

a) The constellation is shown in the figure



The demodulator, using correlators, is the one given in the figure



The detector follows the rule

$$q_0[n] \underset{\hat{A}[n]=a_1}{\overset{\hat{A}[n]=a_0}{\gtrless}} q_1[n].$$

Or what is the same, the decision regions are

$$I_0 = \{\mathbf{q} : q_0 > q_1\},\ I_1 = \{\mathbf{q} : q_0 \leq q_1\}$$

The probability of error is

$$P_e = Q\left(\frac{d(\mathbf{a}_0, \mathbf{a}_1)}{2\sqrt{N_0/2}}\right) = Q\left(\frac{A\sqrt{T}}{2\sqrt{N_0/2}}\right)$$

b) The detecctor is given by the following decision regions

$$I_0 = \{\mathbf{q} : q_0 + 2\alpha A\sqrt{T/2} > q_1\},\ I_1 = \{\mathbf{q} : q_0 + 2\alpha A\sqrt{T/2} \leq q_1\}$$

The probability of error is now

$$P_e = Q\left(\frac{\alpha A\sqrt{T}}{2\sqrt{N_0/2}}\right),$$

c) The optimal detector in this case follows the rule

$$q' \underset{a_1}{\overset{a_0}{\gtrless}} 0$$

i.e., the decision regions are

$$I_0 = \{q' : q' > 0\},\ I_1 = \{q' : q' \leq 0\}$$

The probability of error is

$$P_e = Q\left(\frac{A\sqrt{T}}{2\sqrt{N_0/2}}\right).$$

This probability of error coincides with that of section a). This is because what the new demodulator does is rotate the constellation to align it on the new axis q', and it has a scale factor $\sqrt{T}$ that affects both the signal and the noise.

Exercise 3.11 Solution

a) The modulator is given by the two base functions

$$\phi_0(t) = \frac{s_0(t)}{A},\ \phi_1(t) = \frac{s_1(t)}{A}$$

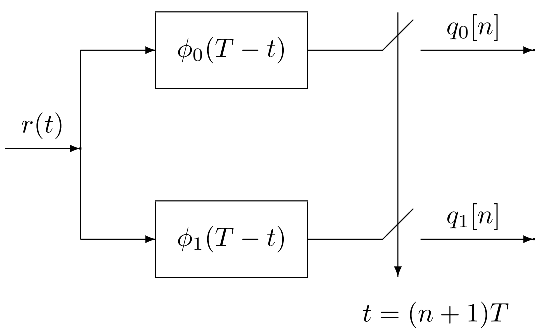
and the constellation is

$$\mathbf{a}_0 = \begin{bmatrix} A \\ 0 \end{bmatrix},\ \mathbf{a}_1 = \begin{bmatrix} 0 \\ A \end{bmatrix},\ \mathbf{a}_2 = \begin{bmatrix} \frac{2A}{3} \\ \frac{A}{3} \end{bmatrix},\ \mathbf{a}_3 = \begin{bmatrix} \frac{A}{3} \\ \frac{2A}{3} \end{bmatrix}$$

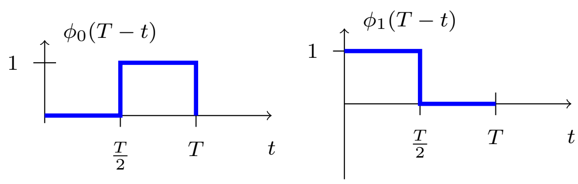
b) $E_s = \frac{7}{9}A^2$ J. Bit assignment must follow Gray's coding rule. In example (it is not the only possible one) it would be

$$\mathbf{a}_0 \equiv 10,\ \mathbf{a}_1 \equiv 00,\ \mathbf{a}_2 \equiv 11,\ \mathbf{a}_3 \equiv 01$$

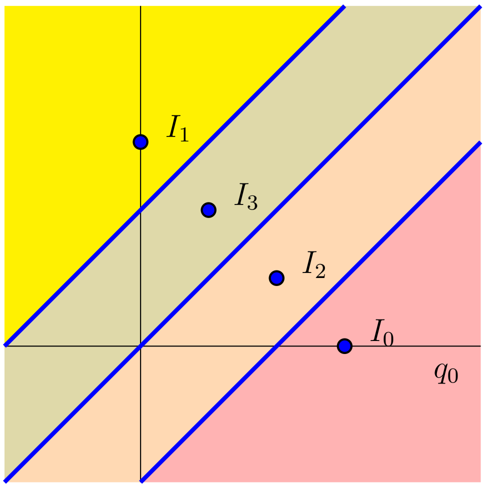
c) Demodulator



FILTROS ADAPTADOS (CAUSALES)



Detector



Expressions of the conditional distribution of the observation (general)

$$f_{\mathbf{q}|\mathbf{A}}(\mathbf{q}|\mathbf{a}_i) = \mathcal{N}^N\left(\mathbf{a}_i, \frac{N_0}{2}\right) = \frac{1}{(\pi N_0)^{N/2}} e^{-\frac{||\mathbf{q}-\mathbf{a}_i||^2}{N_0}}$$

Particularized expressions

$$f_{\mathbf{q}|\mathbf{A}}(\mathbf{q}|\mathbf{a}_0) = \frac{1}{\pi N_0} e^{-\frac{(q_0-A)^2+q_1^2}{N_0}},\ f_{\mathbf{q}|\mathbf{A}}(\mathbf{q}|\mathbf{a}_1) = \frac{1}{\pi N_0} e^{-\frac{q_0^2+(q_1-A)^2}{N_0}}$$

$$f_{\mathbf{q}|\mathbf{A}}(\mathbf{q}|\mathbf{a}_2) = \frac{1}{\pi N_0} e^{-\frac{(q_0-2A/3)^2+(q_1-A/3)^2}{N_0}},\ f_{\mathbf{q}|\mathbf{A}}(\mathbf{q}|\mathbf{a}_3) = \frac{1}{\pi N_0} e^{-\frac{(q_0-A/3)^2+(q_1-2A/3)^2}{N_0}}$$

Error probability

$$P_e = \frac{3}{2} Q\left(\frac{A}{3\sqrt{N_0}}\right)$$

d) Detector

$$I_0 = \left\{q : q > A\frac{\sqrt{2}}{3}\right\},\ I_1 = \left\{q : q < -A\frac{\sqrt{2}}{3}\right\}$$

$$I_2 = \left\{q : 0 < q \leq A\frac{\sqrt{2}}{3}\right\},\ I_3 = \left\{q : -A\frac{\sqrt{2}}{3} \leq 0\right\}$$

Conditional distributions of the observation given a transmitted symbol

$$f_{\mathbf{q}|\mathbf{A}}(q|\mathbf{a}_0) = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(q-A/\sqrt{2})^2}{N_0}},\ f_{\mathbf{q}|\mathbf{A}}(q|\mathbf{a}_1) = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(q+A/\sqrt{2})^2}{N_0}}$$

$$f_{\mathbf{q}|\mathbf{A}}(q|\mathbf{a}_2) = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(q-A/(3\sqrt{2}))^2}{N_0}},\ f_{\mathbf{q}|\mathbf{A}}(q|\mathbf{a}_3) = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(q+A/(3\sqrt{2}))^2}{N_0}}$$

Exercise 3.12 Solution

a) In the first section:

   i) The encoder is given by four symbols with coordinates

$$\mathbf{a}_0 = -1.05,\ \mathbf{a}_1 = -0.35,\ \mathbf{a}_2 = +0.35,\ \mathbf{a}_3 = +1.05.$$

ii) The assignment must follow the Gray coding rule, that is, between symbols at a minimum distance the binary assignment must differ by only one bit. An example (not the only possible one) of such an assignment would be

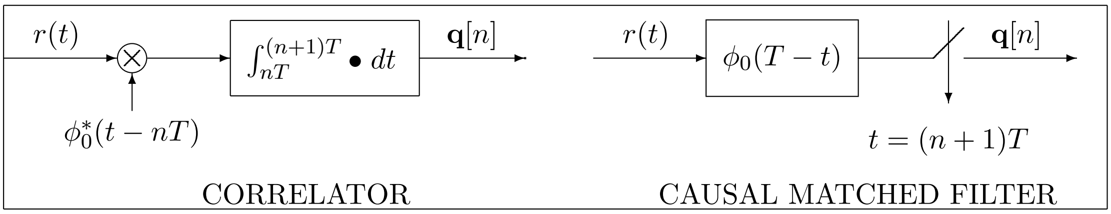
$$\mathbf{a}_0 \equiv 00,\ \mathbf{a}_1 \equiv 01,\ \mathbf{a}_2 \equiv 11,\ \mathbf{a}_3 \equiv 10.$$

The approximate bit error probability, for sufficiently high signal-to-noise ratios, is

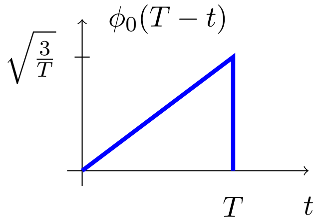
$$BER \approx 10^{-4}.$$

b) In the second section

i) Structure of the demodulators by correlation and by adapted filters



The matched filter is shown in the figure below



ii) The detector is given by the following decision regions

$$I_0 = \{q : q < -2T\},\ I_1 = \{q : -2T \leq q < 0\},\ I_2 = \{q : 0 \leq q < +2T\},\ I_3 = \{q : q \geq +2T\}$$

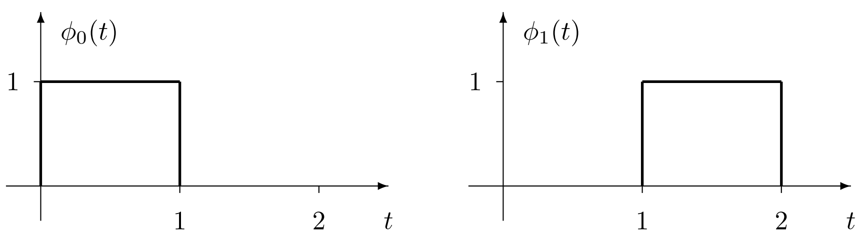
The probability of error is

$$P_e = \frac{3}{2} Q\left(\sqrt{\frac{T}{2N_0}}\right).$$

The performance obtained with this receiver can never be better than that of the optimal receiver, which by definition is the one that provides the minimum probability of symbol error. Therefore, if a demodulator different from the usual ones based on correlators or matched filters is used, as in this case, the performances will be at best the same (if the receiver performs an equivalent operation) and in general they will be worse (if the operation is not equivalent to an optimal demodulator).

Exercise 3.13 Solution

The basis and constellation of the system are shown below

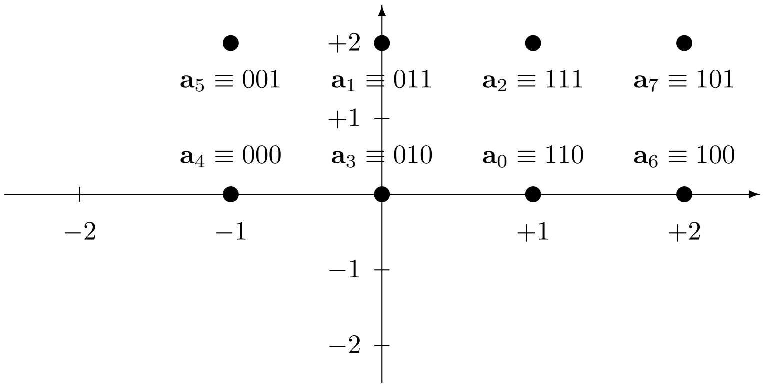


$$\mathbf{a}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ \mathbf{a}_1 = \begin{bmatrix} 0 \\ 2 \end{bmatrix},\ \mathbf{a}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix},\ \mathbf{a}_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathbf{a}_4 = \begin{bmatrix} -1 \\ 0 \end{bmatrix},\ \mathbf{a}_5 = \begin{bmatrix} -1 \\ 2 \end{bmatrix},\ \mathbf{a}_6 = \begin{bmatrix} 2 \\ 0 \end{bmatrix},\ \mathbf{a}_7 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

a) Binary assignment must follow Gray coding, since this is the one that minimizes the bit error rate for a constellation An example of this type of coding, in which symbols at minimum distance differ by a single bit in the assignment, it would be

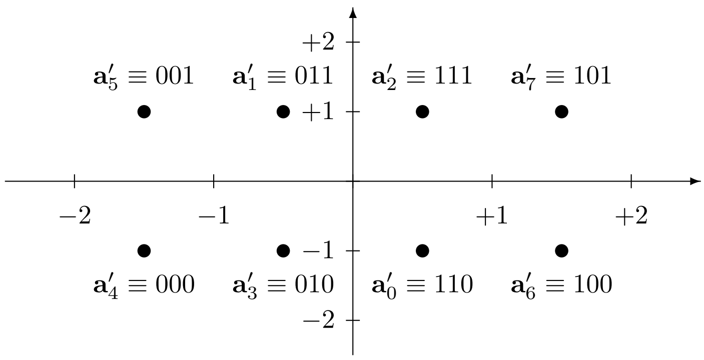
$$\mathbf{a}_0 \equiv 110,\ \mathbf{a}_1 \equiv 011,\ \mathbf{a}_2 \equiv 111,\ \mathbf{a}_3 \equiv 010,\ \mathbf{a}_4 \equiv 000,\ \mathbf{a}_5 \equiv 001,\ \mathbf{a}_6 \equiv 100,\ \mathbf{a}_7 \equiv 101$$



b) The symbols of the modified constellation would be:

$$\mathbf{a}'_0 = \begin{bmatrix} +1/2 \\ -1 \end{bmatrix},\ \mathbf{a}'_1 = \begin{bmatrix} -1/2 \\ +1 \end{bmatrix},\ \mathbf{a}'_2 = \begin{bmatrix} +1/2 \\ +1 \end{bmatrix},\ \mathbf{a}'_3 = \begin{bmatrix} -1/2 \\ -1 \end{bmatrix}$$

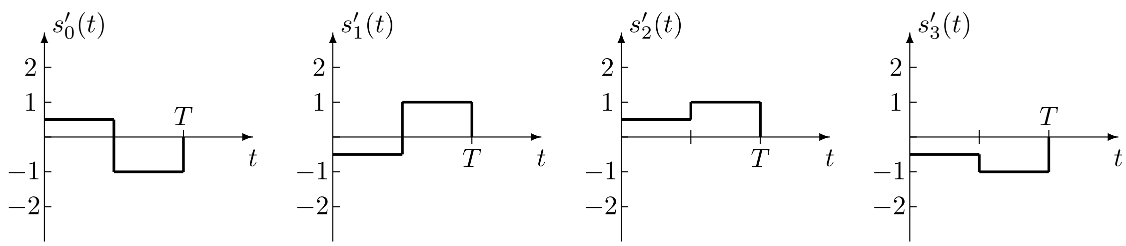
$$\mathbf{a}'_4 = \begin{bmatrix} -3/2 \\ -1 \end{bmatrix},\ \mathbf{a}'_5 = \begin{bmatrix} -3/2 \\ +1 \end{bmatrix},\ \mathbf{a}'_6 = \begin{bmatrix} +3/2 \\ -1 \end{bmatrix},\ \mathbf{a}'_7 = \begin{bmatrix} +3/2 \\ +1 \end{bmatrix}$$

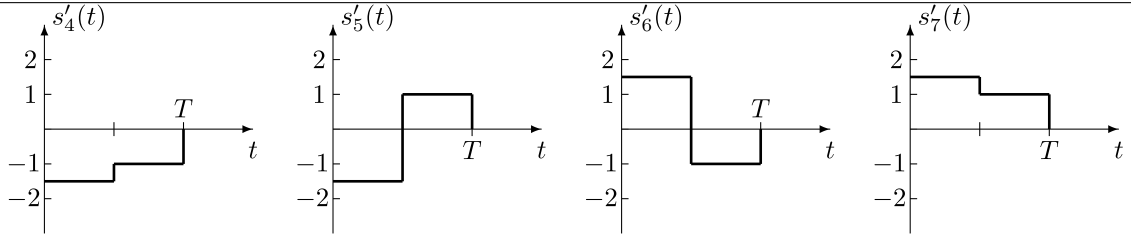


The new signals would have as an analytical expression

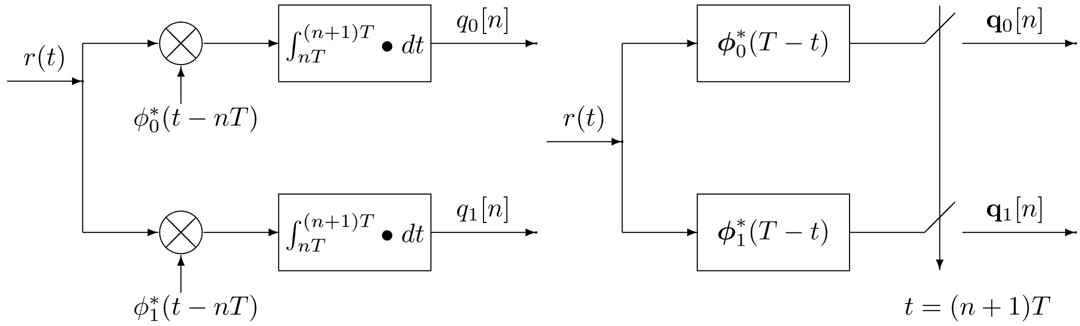
$$s'_i(t) = a'_{i,0} \cdot \phi_o(t) + a'_{i,1} \cdot \phi_1(t),$$

and can be seen represented in the figure



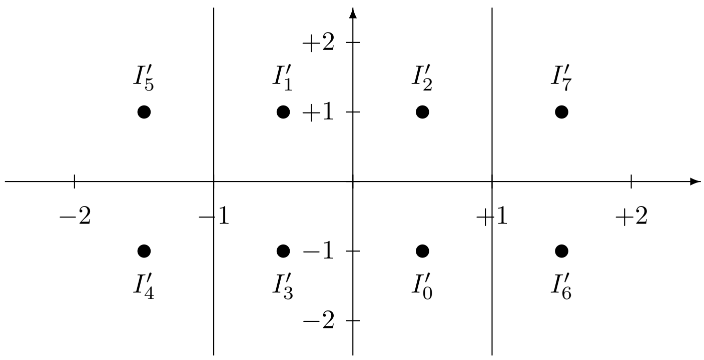


c) The optimal demodulator is based on matched filters or on correlators with the functions that form the orthonormal basis, either of the two options is valid.



CORRELATOR MATCHED FILTERS (CAUSAL)

The optimal detector (given by the decision regions) is shown below



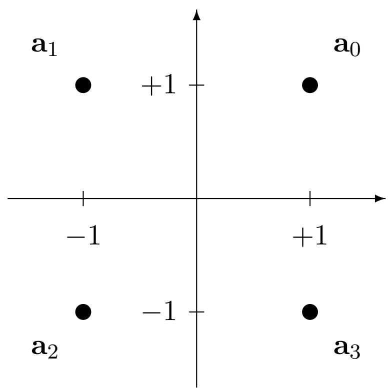
And the symbol error rate is now

$$P_e = \frac{3}{2} \cdot Q\left(\frac{1}{2\sqrt{N_0/2}}\right) + Q\left(\frac{1}{\sqrt{N_0/2}}\right) - \frac{3}{2} \cdot Q\left(\frac{1}{2\sqrt{N_0/2}}\right) \cdot Q\left(\frac{1}{\sqrt{N_0/2}}\right)$$

Exercise 3.14 Solution

a) Binary assignment, to minimize the probability of error of bit, it must be a Gray encoding, since in this way the Most likely errors at the symbol level involve a single error at bit level.

In this case the constellation is the one shown in the figure



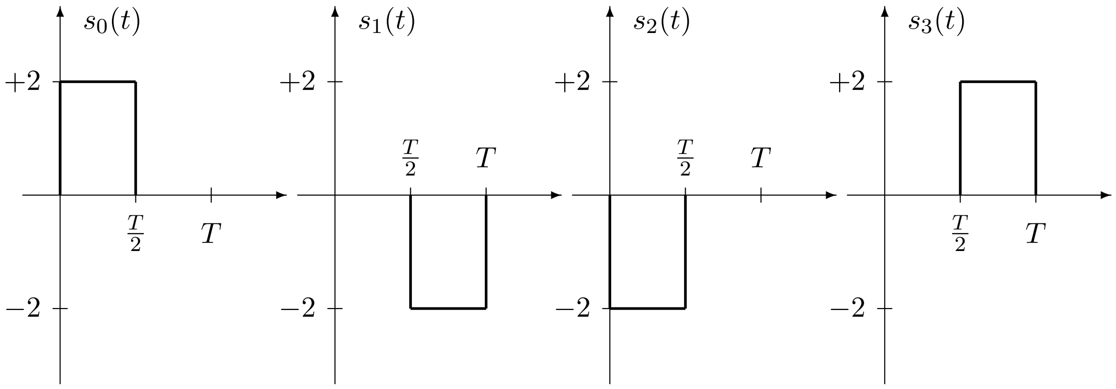
A possible Gray assignment (not the only possible one) would be

$$\mathbf{a}_0 = 00,\ \mathbf{a}_1 = 01,\ \mathbf{a}_2 = 11,\ \mathbf{a}_3 = 10.$$

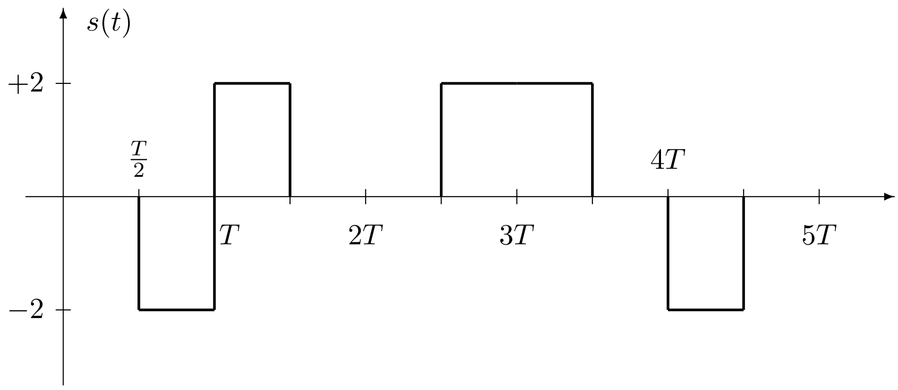
The symbol and bit rates are

$$R_s = \frac{1}{T} = 1 \text{ symbols/s (bauds), and } R_b = 2 \text{ bits/s.}$$

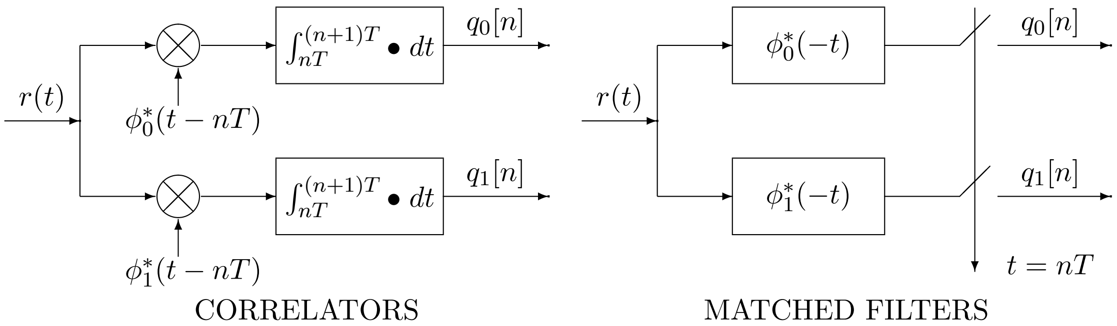
b) The four signals are shown below



The signal resulting from transmitting the given sequence is



c) For the optimal demodulator any of the two typical structures, using correlators or using matched filters filters, can be used.



Detector: given by the following decision regions

$$\mathbf{q} \in I_0 \text{ if } \begin{cases} q_0 \geq 0 \\ q_1 \geq 0 \end{cases},\ \mathbf{q} \in I_1 \text{ if } \begin{cases} q_0 < 0 \\ q_1 \geq 0 \end{cases},\ \mathbf{q} \in I_2 \text{ if } \begin{cases} q_0 < 0 \\ q_1 < 0 \end{cases},\ \mathbf{q} \in I_3 \text{ if } \begin{cases} q_0 \geq 0 \\ q_1 < 0 \end{cases}.$$

The error probability is

$$P_e = 2 \cdot Q\left(\frac{1}{\sqrt{N_0/2}}\right) - \left(Q\left(\frac{1}{\sqrt{N_0/2}}\right)\right)^2.$$

d) The optimal detector in this case is given by the following decision regions

$$q \in I_0 \text{ si } q \geq \frac{3}{2},\ q \in I_1 \text{ si } -\frac{3}{2} < q \leq 0,\ q \in I_2 \text{ si } q \leq \frac{3}{2},\ q \in I_3 \text{ if } 0 < q < \frac{3}{2},$$

Finally, the probability of error is

$$P_e = Q\left(\frac{1}{\sqrt{5N_0}}\right) + \frac{1}{2} \cdot Q\left(\frac{2}{\sqrt{5N_0}}\right).$$

Exercise 3.15 Solution

a) In this case we are looking for an approximate error probability of $P_e \approx 2 \times 10^{-6}$

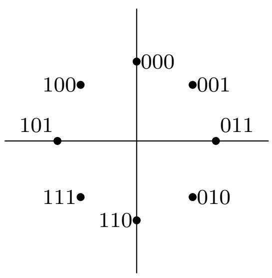
i) The radius is valid

$$R = 1.5202.$$

ii) The average energy per symbol

$$E_s = R^2 = 2.311 \text{ J}.$$

iii) To carry out the binary assignment ($m = \log_2 M = 3$ bits/symbol) a Gray encoding must be used, so that symbols that are at a minimum distance have an assignment that differs by only one bit. Thus, since most errors occur with symbols at a minimum distance, the probability of bit error is minimized.



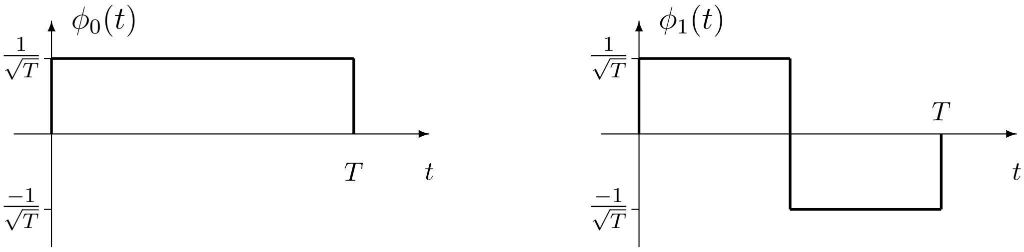
$$\mathbf{a}_0 \equiv 011,\ \mathbf{a}_1 \equiv 001,\ \mathbf{a}_2 \equiv 000,\ \mathbf{a}_3 \equiv 100,\ \mathbf{a}_4 \equiv 101,\ \mathbf{a}_5 \equiv 111,\ \mathbf{a}_6 \equiv 110,\ \mathbf{a}_7 \equiv 010.$$

With a Gray code, if the signal-to-noise ratio is high

$$BER \approx \frac{1}{m} \cdot P_e = \frac{1}{3} \cdot P_e = \frac{2}{3} \times 10^{-6}.$$

b) In this case, transmission at the bit rate $R_b = 1500$ bits/s is sought.

i) The dimension of the signal space, given this constellation, is $N = 2$, therefore two functions that form an orthonormal basis (their inner product is zero and each has unit energy) must be found. For baseband transmission, square pulse type waveforms are appropriate, since their spectrum is in baseband. There are several options. One possibility (not the only one) would be the basis given in the figure



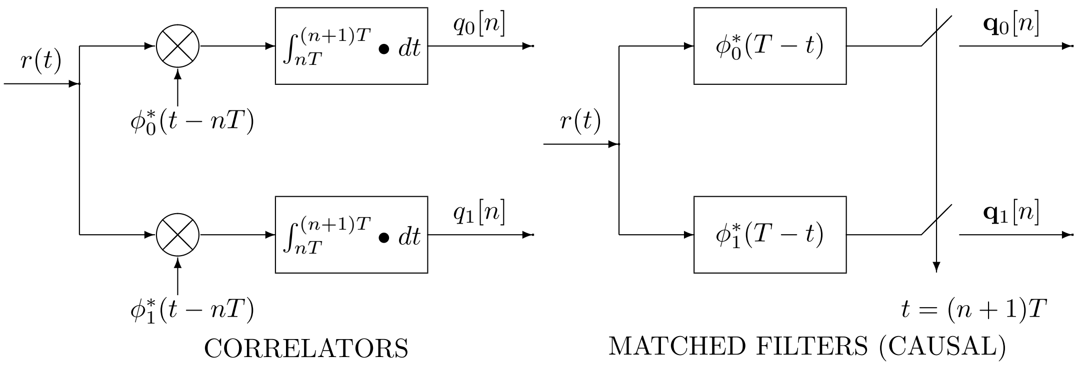
To have them completely defined, the value of T must be determined, and this is related to the transmission rate, since the symbol rate is $R_s = \frac{1}{T}$ bauds. Symbol rate is related to bit rate through the number of bits per symbol

$$R_s = \frac{R_b}{m} = \frac{1500}{3} = 500 \text{ bauds.}$$

This implies that the symbol duration time is

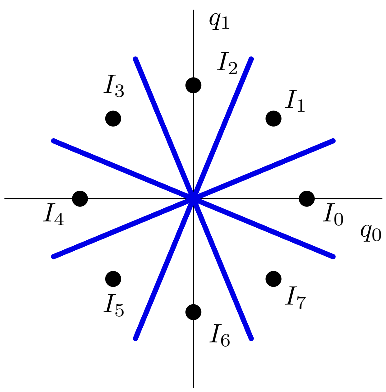
$$T = \frac{1}{R_s} = \frac{1}{500} = 0.002 \text{ seconds} = 2 \text{ ms.}$$

ii) The optimal demodulator can be based on causal matched filters or on correlators with the functions that form the orthonormal basis, either of the two options is valid.



In this case, since the base functions are real, the conjugates are not relevant.

c) This section considers the design of the detector



i) For the design of the detector, since the symbols are equiprobable and the noise is Gaussian, the minimum Euclidean distance criterion is applied: the decision region of a symbol are those points in space closer to that symbol than to the rest of the symbols of the constellation, which in this case would be those shown in the previous figure.

ii) The union bound is obtained as

$$P_e \leq \sum_{i=0}^{M-1} p_{\mathbf{A}}(\mathbf{a}_i) \sum_{\substack{j=0\\ j\neq i}}^{M-1} P_e(\mathbf{a}_i,\mathbf{a}_j) = \sum_{i=0}^{M-1} p_{\mathbf{A}}(\mathbf{a}_i) \sum_{\substack{j=0\\ j\neq i}}^{M-1} Q\left(\frac{d(\mathbf{a}_i,\mathbf{a}_j)}{2\sqrt{N_o/2}}\right).$$

For this constellation, each symbol has the rest of the symbols at the following distances:

- Two symbols at a distance $d_{min} = 1.1635$ (angle $\theta = 45$)
- Two symbols at a distance $d_a = 2.15$ (angle $\theta = 90$)
- Two symbols at a distance $d_b = 2.8$ (angle $\theta = 135$)
- A symbol at a distance $d_c = 2R = 3.04$ (angle $\theta = 180$)

Therefore, since the bound for the conditional error probability for each symbol is the same

$$P_e \leq 2\,Q\left(\frac{d_{min}}{2\sqrt{N_o/2}}\right)+2\,Q\left(\frac{d_a}{2\sqrt{N_o/2}}\right)+2\,Q\left(\frac{d_b}{2\sqrt{N_o/2}}\right)+Q\left(\frac{d_c}{2\sqrt{N_o/2}}\right) = 2.0345\times 10^{-6}.$$

d) The loose bound is

$$P_e \leq 7\,Q\left(\frac{1.1635}{2\sqrt{N_0/2}}\right) = 7.12\times 10^{-6}.$$

Expressed as a function of the E_s/N_0 ratio, the error probability becomes

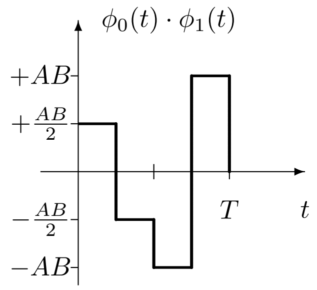
$$P_e \leq 7\,Q\left(0.5412\,\sqrt{\frac{E_s}{N_0}}\right).$$

Exercise 3.16 Solution

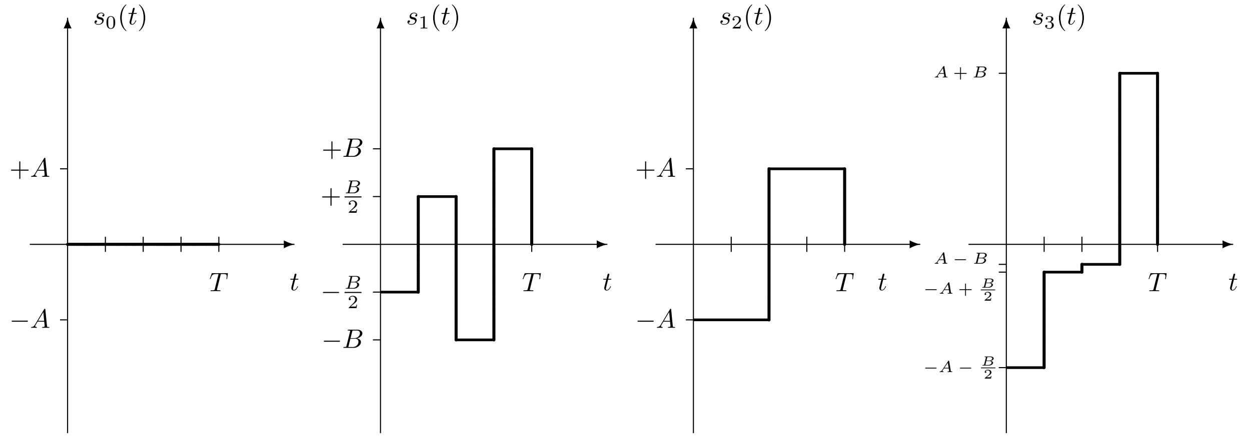
a) Since the energy of both signals is one, the only thing missing to demonstrate that they form an orthonormal basis is that the dot product between both signals is zero, which for real signals is defined as

$$\langle \phi_0(t), \phi_1(t)\rangle = \int_{-\infty}^{+\infty} \phi_0(t)\cdot\phi_1(t)\, dt = 0$$

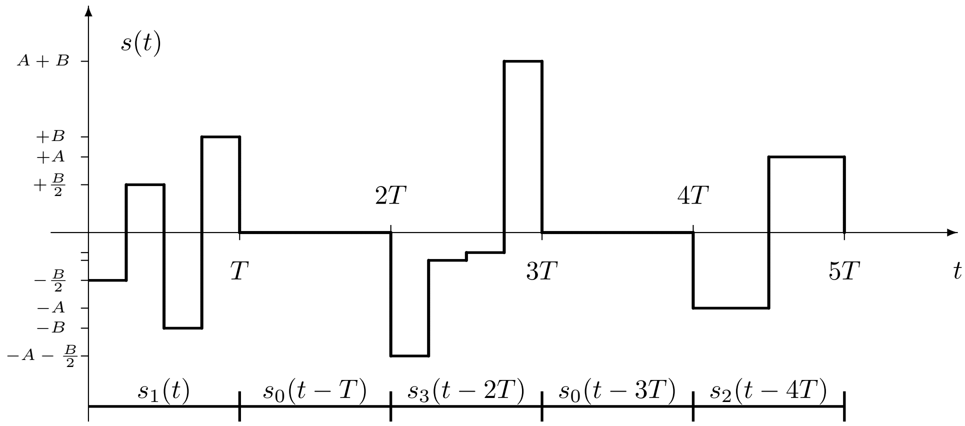
The integrand, the signal resulting from the product between $\phi_0(t)\cdot\phi_1(t)$, is the one shown in the figure, and it can be clearly seen that its integral is zero.



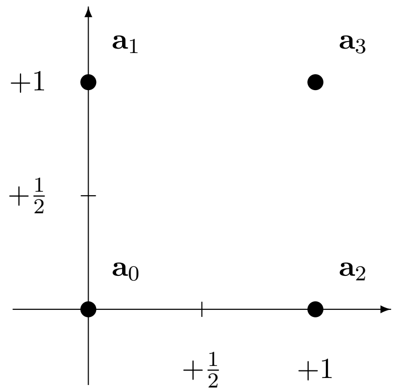
b) The four signals are shown below



And the signal that is generated by the given data sequence is



c) To carry out the binary assignment, a Gray code must be used, so that symbols that are at a minimum distance have an assignment that differs by only one bit. This is the assignment that minimizes the bit error rate (BER).



Given the above constellation, a possible Gray code is

$$\mathbf{a}_0 = 00,\ \mathbf{a}_1 = 01,\ \mathbf{a}_2 = 10,\ \mathbf{a}_3 = 11.$$

Symbol and bit transmission rates are $R_s = \frac{1}{T} = 1$ symbols/s (baud) and $R_b = 2$ bits/s, respectively.

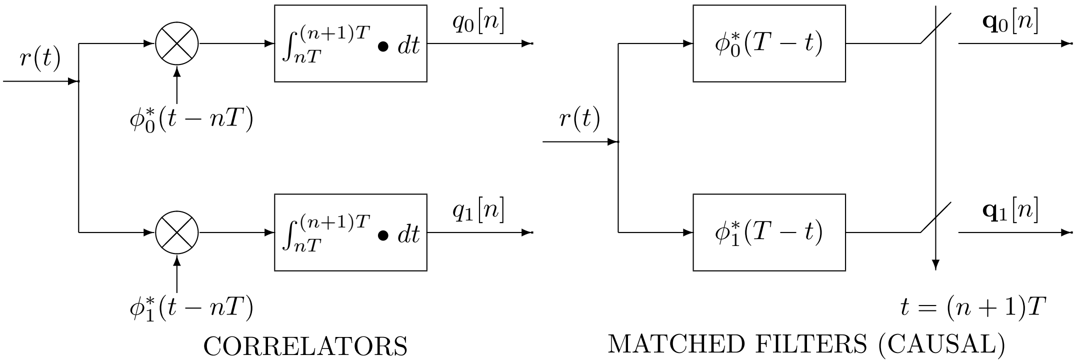
Average energy per symbol is

$$E_s = 1.$$

This constellation is not the most appropriate, since it does not meet the condition that the mean of the constellation is zero, which is what guarantees a minimum average energy per symbol for the relative distances given by the constellation, and in this case the mean is

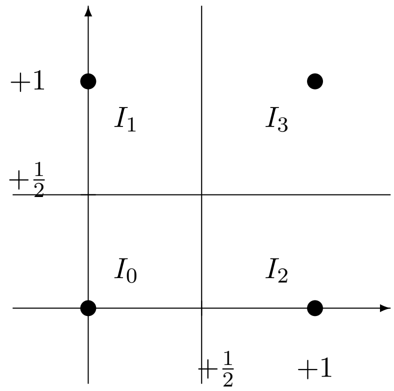
$$E[\mathbf{a}_i] = \begin{bmatrix} +\frac{1}{2} \\ +\frac{1}{2} \end{bmatrix}.$$

d) The optimal demodulator is based on adapted filters or on correlators with the functions that form the orthonormal basis, either of the two options is valid.



In this case, since the base functions are real, the conjugates are not relevant.

The optimal decision maker is the one shown in the figure



The symbol error rate is

$$P_e = 2\cdot Q\left(\frac{1}{2\sqrt{N_0/2}}\right) - \left[Q\left(\frac{1}{2\sqrt{N_0/2}}\right)\right]^2$$

e) The usual approximation for the probability of error is

$$P_e \approx 2\cdot Q\left(\frac{1}{2\sqrt{N_0/2}}\right).$$

Regarding the bounds for the error probability, the union bound is

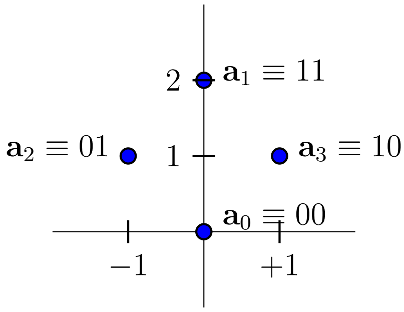
$$P_e \leq 2\cdot Q\left(\frac{1}{2\sqrt{N_0/2}}\right) + Q\left(\frac{\sqrt{2}}{2\sqrt{N_0/2}}\right) = 2\cdot Q\left(\frac{1}{\sqrt{2N_0}}\right) + Q\left(\frac{1}{\sqrt{N_0}}\right).$$

And the loose bound is

$$P_e \leq 3\cdot Q\left(\frac{1}{2\sqrt{N_0/2}}\right).$$

Exercise 3.17 Solution

a) First we represent the 4-point constellation of the system



i) Binary assignment has to follow Gray's rule. Given the constellation, a possible binary assignment that meets this rule (it is not the only possible one) would be the one in the figure

$$\mathbf{a}_0 \equiv 00,\ \mathbf{a}_1 \equiv 11,\ \mathbf{a}_2 \equiv 01,\ \mathbf{a}_3 \equiv 10.$$

ii) The average energy per symbol is obtained by averaging the energy of each symbol. For this constellation of equiprobable symbols, we have

$$E_s = 2 \text{ J}$$

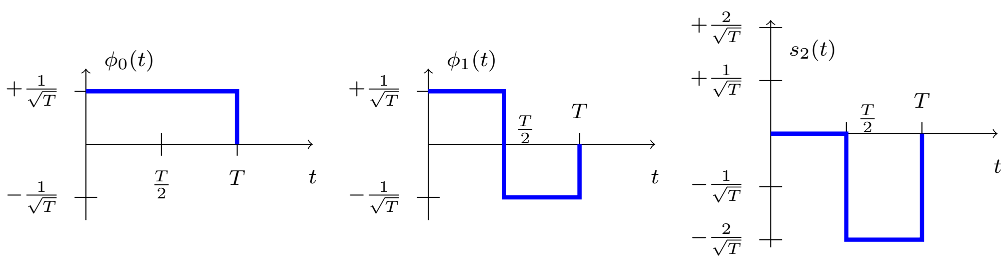
The constellation, since it does not have a zero mean, does not have a good trade-off between performance and energy use. For the same distances between symbols, a constellation with zero mean (the same constellation displaced by 1 on the ordinate axis) would be more efficient (lower energy), which would have half the average energy per symbol.

b) Now a transmission over a channel with low pass response is considered.

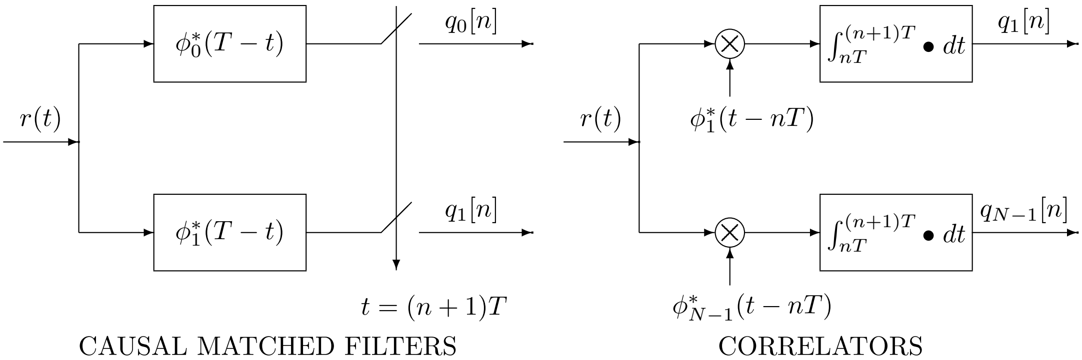
i) In this case, an orthonormal basis of the dimension of the signal space is needed, which for the given constellation is $N = 2$, and whose frequency response is appropriate to the channel response, which in this case means that It must be a response with low frequency components (low pass). For example, rectangular pulses can be used, which have this type of response. Two normalized and orthogonal pulses are necessary. An example (not the only possible one) would be

$$\phi_0(t) = \begin{cases} \frac{1}{\sqrt{T}}, & \text{if } 0 \le t < T \\ 0, & \text{otherwise} \end{cases}, \quad \phi_1(t) = \begin{cases} +\frac{1}{\sqrt{T}}, & \text{if } 0 \le t < \frac{T}{2} \\ -\frac{1}{\sqrt{T}}, & \text{if } \frac{T}{2} \le t < T \\ 0, & \text{otherwise} \end{cases}.$$

ii) Using the base proposed in the previous section, the signal $s_2(t)$, together with the elements of the basis, is represented in the figure

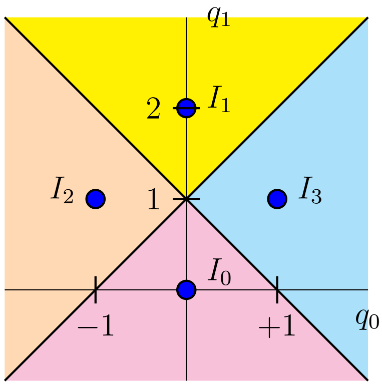


iii) For the demodulator, a bank of two matched filters (in this case causal) or a bank of two correlors can be used, as shown in the figure



In this case, since the orthonormal basis functions are real, the complex conjugate operators are not relevant.

iv) The decision regions are those shown in the figure



c) In this section, several expressions related to the probability of error will be obtained.

i) The expressions to approximate the error probability and for the loose bound are

$$\text{Approximation: } P_e \approx 2\,Q\left(\frac{1}{\sqrt{N_0}}\right) \quad \text{Loose bound: } P_e \leq 3\,Q\left(\frac{1}{\sqrt{N_0}}\right)$$

ii) The exact error probability is

$$P_e = 2Q\left(\frac{1}{\sqrt{N_0}}\right) - \left[Q\left(\frac{1}{\sqrt{N_0}}\right)\right]^2.$$

Expressed as a function of the E_s/N_0 ratio, the error probability becomes

$$P_e = 2Q\left(\sqrt{\frac{1}{2}\times\frac{E_s}{N_0}}\right) - \left[Q\left(\sqrt{\frac{1}{2}\times\frac{E_s}{N_0}}\right)\right]^2.$$

Exercise 3.18 Solution

a) Each of the three sections is answered:

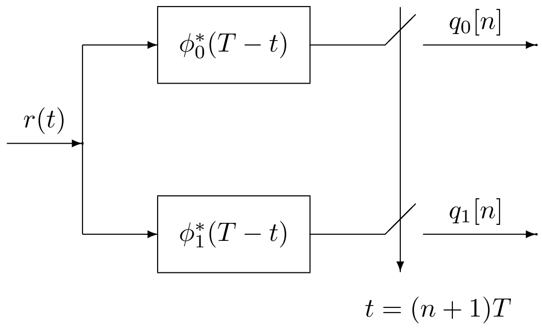
i) The binary rate is

$$R_b = 12 \text{ Mbits/s}.$$

Average energy per symbol

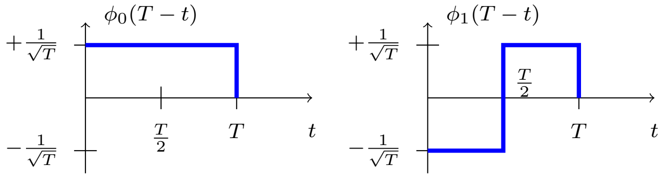
$$E_s = 3+\sqrt{3} \text{ J}$$

ii) The optimal demodulator with causal matched filters

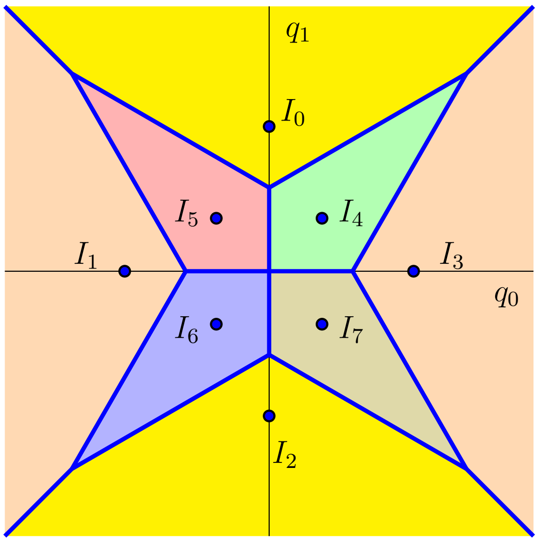


CAUSAL MATCHED FILTERS

The causal matched filters are



The optimal detector is shown below (the decision regions)



iii) The union bound is

$$P_e \leq 3Q\left(\frac{1}{\sqrt{N_0/2}}\right)+2Q\left(\frac{1.93}{\sqrt{N_0/2}}\right)+Q\left(\frac{2.64}{\sqrt{N_o/2}}\right)+\frac{1}{2}Q\left(\frac{1+\sqrt{3}}{\sqrt{N_o/2}}\right)+\frac{1}{2}Q\left(\frac{\sqrt{2}}{2\sqrt{N_o/2}}\right)$$

And the approximated error probability

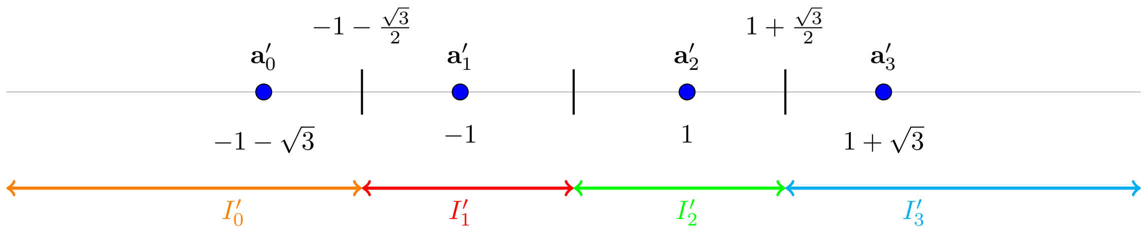
$$P_e \approx 4\,Q\left(\frac{1}{\sqrt{N_0/2}}\right)$$

b) In this section the system is considered with the demodulator partially functioning;

i) You could choose, for example, the symbols

$$\mathbf{a}_1 \rightarrow \mathbf{a}'_0 = -A,\ \mathbf{a}_5 \rightarrow \mathbf{a}'_1 = -B,\ \mathbf{a}_4 \rightarrow \mathbf{a}'_2 = +B,\ \mathbf{a}_3 \rightarrow \mathbf{a}'_3 = +A.$$

ii) The decision regions would be those shown in the figure



iii) The exact error probability is now

$$P_e = \left(\frac{\sqrt{3}/2}{\sqrt{N_0/2}}\right) + \frac{1}{2}\left(\frac{1}{\sqrt{N_0/2}}\right)$$