uc3m | Universidad Carlos III de Madrid

OpenCourseWare

Matemáticas para la Economía II (Grados Empresa)

Paula Rosado Jiménez

Ejemplo de examen con solución

Mayo 2024

University Carlos III
Department of Economics
Mathematics II. Final Exam. May 23rd 2024

Last Name:		**Name:**
ID number:	**Degree:**	**Group:**

IMPORTANT

- **DURATION OF THE EXAM: 2h**
- Calculators are **NOT** allowed.
- **Scrap paper:** You may use the last two pages of this exam and the space behind this page.
- **Do NOT UNSTAPLE** the exam.
- You must show a valid ID to the professor.

Problem	Points
1	
2	
3	
4	
5	
Total	

(1) Given the following system of linear equations,

$$\begin{cases} x + 3y - az &= 4 \\ 2x - 3y + 2z &= 2 \\ 3x + az &= b \end{cases}$$

where $a, b \in \mathbb{R}$.

(a) **(20 points)** Classify the system according to the values of a and b.
Solution: *The matrix associated with the system is*

$$\begin{pmatrix} 1 & 3 & -a & 4 \\ 2 & -3 & 2 & 2 \\ 3 & 0 & a & b \end{pmatrix}$$

We perform the following operations

$$\textit{row 2} \mapsto \textit{row 2} - 2 \times \textit{row 1}$$

$$\textit{row 3} \mapsto \textit{row 3} - 3 \times \textit{row 1}$$

And we obtain that the original system is equivalent to another one whose augmented matrix is the following

$$\begin{pmatrix} 1 & 3 & -a & 4 \\ 0 & -9 & 2a+2 & -6 \\ 0 & -9 & 4a & b-12 \end{pmatrix}$$

Now, we perform the operation row 3 $\mapsto$ *row 3 − row 2 and we obtain*

$$\begin{pmatrix} 1 & 3 & -a & 4 \\ 0 & -9 & 2a+2 & -6 \\ 0 & 0 & 2a-2 & b-6 \end{pmatrix}$$

We see that

(i) *if* $a \neq 1$*, then* $\operatorname{rank} A = 3 = \operatorname{rank}(A|b)$*. The system is consistent with a unique solution.*

(ii) *If* $a = 1$ *and* $b = 6$*, then* $\operatorname{rank} A = \operatorname{rank}(A|b) = 2$*. The system is consistent with* $3 - 2 = 1$ *parameters.*

(iii) *If* $a = 1$ *and* $b \neq 6$*, then* $\operatorname{rank} A = 2 < \operatorname{rank}(A|b) = 3$*. The system is not consistent.*

(b) **(10 points)** Solve the above system for the values of a and b for which the system has infinitely many solutions.
Solution: *We need* $a = 1$ *and* $b = 6$*. The proposed system of linear equations is equivalent to the following one*

$$\begin{cases} x + 3y - z &= 4 \\ -9y + 4z &= -6 \end{cases}$$

The solution is

$$z \in \mathbb{R}, \quad x = 2 - \frac{z}{3}, \quad y = \frac{2}{9}(2z + 3)$$

(2) Consider the set

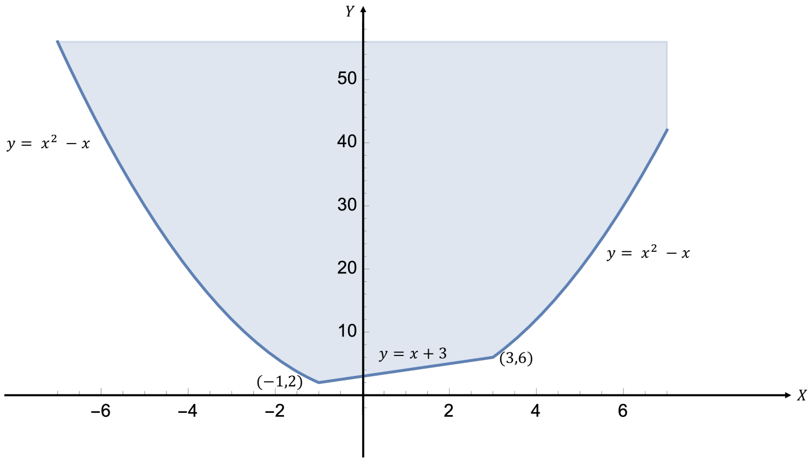
$$A = \{(x,y) \in \mathbb{R}^2 : \quad y - x^2 + x \geq 0, \quad y - x - 3 \geq 0\}$$

and the function

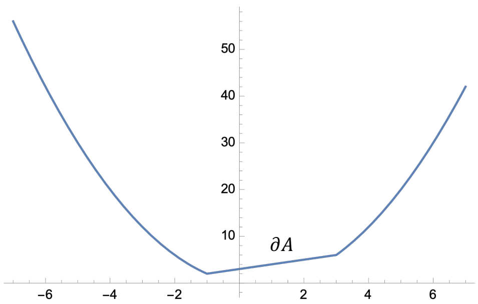
$$f(x,y) = y - 2x$$

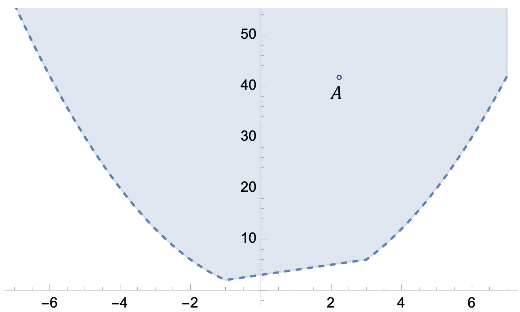
(a) **(20 points)** Sketch the graph of the set A, its boundary and its interior and justify if it is open, closed, bounded, compact or convex.

Solution: *The set A is approximately as indicated in the picture.*



The interior and the boundary are





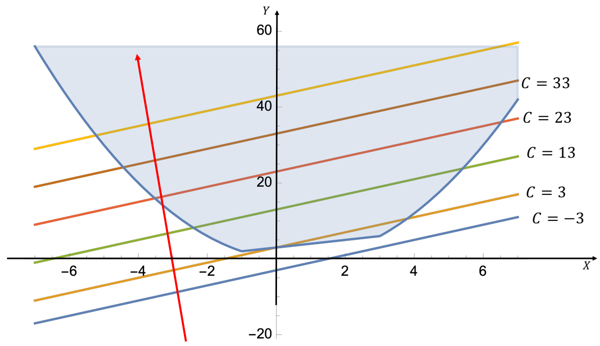
The functions $h_1(x,y) = y - x^2 + x$ and $h_2(x,y) = y - x - 3$ are continuous (since, they are polynomials) and $A = \{(x,y) \in \mathbb{R}^2 : 1 \leq h_1(x,y) \geq 0, \; h_2(x,y) \geq 0\}$. Hence, the set A closed (Note also that $\partial A \subset A$). It is not open because $A \cap \partial A \neq \emptyset$.
We see that any point of the form $(0,y)$ with $y \geq 10$ is in the set A. Hence, the set A is not bounded. Therefore, the set A is not compact.

We show next that A is also convex. The function $x^2 + x$ is convex. Hence the set $A_1 = \{(x,y) \in \mathbb{R}^2 : y \geq y - x^2 + x\}$ is convex. On he other hand, the function $x + 3$ is convex. Therefore, the set $A_2 = \{(x,y) \in \mathbb{R}^2 : y \geq x + 3\}$ is also convex. We conclude now that the set $A = A_1 \cap A_2$ is also convex.

(b) **(10 points)** State Weierstrass' Theorem. Determine if it is possible to apply Weierstrass' Theorem to the function f defined on A.

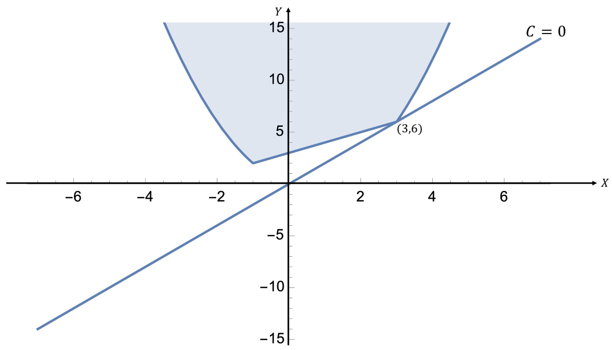
Solution: *The set A is not compact. Therefore, Weierstrass Theorem may not be applied.*

(c) **(10 points)** Draw the level curves of f, indicating the direction of growth of the function.
Solution:
The level curves $f(x, y) = y - 2x = C$ are straight lines of the form $y = 2x + C$ Graphically,



The red arrow represents the direction of growth of the function f.

(d) **(20 points)** Using the level curves of f, determine (if they exist) the extreme global points of f on the set A.
Solution: *Since, any point of the form $(0, y)$ with $y \geq 10$ is in the set A, the function f does not attain a maximum in A. Graphically, we wee that the minimum value is attained at the point $(3, 6)$.*



The minimum value is $f(3, 6) = 0$.

(3) Consider the set of equations

$$\begin{aligned} 3xy + y^2 + z^2 &= 1 \\ x^2 + yz &= 1 \end{aligned}$$

(a) **(10 points)** Prove that the above system of equations determines implicitly two differentiable functions $y(x)$ and $z(x)$ in a neighborhood of the point $(x_0, y_0, z_0) = (1, 0, -1)$.
Solution: *We first remark that* $(x_0, y_0, z_0) = (1, 0, -1)$*. is a solution of the system of equations. The functions* $f_1(x, y, z) = 3xy + y^2 + z^2 - 1$ *and* $f_2(x, y, z) = x^2 + yz - 1$ *are of class* C^∞*. We compute*

$$\left| \begin{matrix} \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{matrix} \right|_{(x,y,z)=(1,0,-1)} = \left| \begin{matrix} 3x + 2y & 2z \\ z & y \end{matrix} \right|_{(x,y,z)=(1,0,-1)} = \left| \begin{matrix} 3 & -2 \\ -1 & 0 \end{matrix} \right| = -2$$

By the implicit function theorem, the above system of equations determines implicitly two differentiable functions $y(x)$ *and* $z(x)$ *in a neighborhood of the point* $(x_0, y_0, z_0) = (1, 0, -1)$.

(b) **(20 points)** Compute

$$y'(x), \quad z'(x)$$

at the point $x_0 = 1$.
Solution: *Differentiating implicitly with respect to* x,

(1) $$2y(x)y'(x) + 3xy'(x) + 3y(x) + 2z(x)z'(x) = 0$$
(2) $$z(x)y'(x) + y(x)z'(x) + 2x = 0$$
(3)

We plug in the values $x = 1$, $y(1) = 0$, $z(1)) = -1$ *to obtain the following*

$$\begin{aligned} 3y'(1) - 2z'(1) &= 0 \\ 2 - y'(1) &= 0 \end{aligned}$$

So,

$$y'(1) = 2, \quad z'(1) = 3$$

(c) **(20 points)** Compute

$$y''(x), \quad z''(x)$$

at the point $x_0 = 1$.
Solution: *Differentiation equation 1 with respect to* x *we obtain*

$$\begin{aligned} 3xy''(x) + 2y(x)y''(x) + 2y'(x)^2 + 6y'(x) + 2z(x)z''(x) + 2z'(x)^2 &= 0 \\ z(x)y''(x) + 2y'(x)z'(x) + y(x)z''(x) + 2 &= 0 \end{aligned}$$

We plug in the values $x = 1$, $y(1) = 0$, $z(1)) = -1$, $y'(1) = 2$, $z'(1) = 3$ *to obtain the following*

$$\begin{aligned} 3y''(1) - 2z''(1) + 38 &= 0 \\ 14 - y''(1) &= 0 \end{aligned}$$

So,

$$y''1) = 14, \quad z''(1) = 40$$

(4) Classify the following quadratic form $Q(x,y,z) = c^2x^2 - 2cxz + x^2 - 2xy - 2xz + y^2 + 2yz + 2z^2$ according to the values of $c \in \mathbb{R}$. **(30 points)**

Solution: *The associated matrix is*

$$A = \begin{pmatrix} c^2+1 & -1 & -c-1 \\ -1 & 1 & 1 \\ -c-1 & 1 & 2 \end{pmatrix}$$

We have $D_1 = c^2+1 > 0$. $D_2 = \begin{vmatrix} c^2+1 & -1 \\ -1 & 1 \end{vmatrix} = c^2 \geq 0$. *To compute* D_3 *we note that*

$$|A| = \begin{vmatrix} c^2+1 & -1 & -c-1 \\ -1 & 1 & 1 \\ -c-1 & 1 & 2 \end{vmatrix} \overset{r3\mapsto r2-r3}{=} - \begin{vmatrix} c^2+1 & -1 & -c-1 \\ -1 & 1 & 1 \\ c & 0 & -1 \end{vmatrix} \overset{r2\mapsto r2+r1}{=} - \begin{vmatrix} c^2+1 & -1 & -c-1 \\ c^2 & 0 & -c \\ c & 0 & -1 \end{vmatrix} = - \begin{vmatrix} c^2 & -c \\ c & -1 \end{vmatrix} = 0$$

So, $D_3 = |A| = 0$. *We see immediately that if* $c \neq 0$, *the quadratic form* Q *is positive semidefinite. If* $c = 0$, *The associated matrix is*

$$\begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix}$$

with $D_1 = 1 > 0$. $D_2 = D_3 = 0$. *However, if look at the chain of principal minors*

$$D_1 = a_{33} = 2 > 0, \quad D_2 = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1 > 0, \quad D_3 = 0$$

We see that the quadratic form is positive semidefinite.

(5) Consider the extreme points of the function

$$f(x, y) = x^2 - xy + y^2 - 3y$$

in the set

$$S = \{(x, y) \in \mathbb{R}^2 : 2x - y = 4\}$$

(a) **(10 points)** Write the Lagrangian function and the Lagrange equations.
Solution: *The Lagrangian is*

$$L(x, y) = x^2 - xy + y^2 - 3y + \lambda(2x - y - 4)$$

The Lagrange equations are

$$\begin{aligned} 2\lambda + 2x - y &= 0 \\ -\lambda - x + 2y - 3 &= 0 \\ 2x - y &= 4 \end{aligned}$$

(b) **(20 points)** Compute the solution(s) of the Lagrange equations.
Solution: *Plugging* $2x - y = 4$ *into the first equation we obtain* $\lambda = -2$. *Plugging now* $\lambda = -2$ *into the second equation we obtain the linear system*

$$\begin{aligned} -x + 2y &= 1 \\ 2x - y &= 4 \end{aligned}$$

whose solution is $x = 3,\ y = 2$.

(c) **(20 points)** Use the second order conditions to determine if the solution(s) of the Lagrange equations correspond to a local maximum or minimum value of f in S.
Solution: *The Hessian matrix associated with the Lagrangian is*

$$HL(x, y; \lambda) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

which is definite positive, since $D_1 = 2 > 0$ *and* $D_2 = 3 > 0$. *Hence the point* $(3, 2)$ *corresponds to a local minimum.*

(d) **(20 points)** Does any of the solutions of the Lagrange equations correspond to global maximum or minimum of the function f in the set S?
Solution: *The set* S *is not compact. Therefore, Weiestrass' Theorem does not apply. However, we can check easily that the Hessian matrix of the function* $f(x, y) = x^2 - xy + y^2 - 3y$ *is also*

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

which, as seen above, is definite positive. Hence the function f *is convex in the (convex) set* S*.Therefore, the function* f *attains a minimum value on* S*, which must be a solution of the Lagrange equations. We conclude that the point* $(3, 2)$ *corresponds to a global minimum of* f *on* S*. Since, we have seen in the previous part that the function* f *does not have a local maximum in* S*, we conclude immediately that it does not have neither a global maximum in* S*. Another way to obtain the same conclusion is to note that* $\lim_{y\to\infty} f(x, 2x - 4) = \lim_{y\to\infty} \left(3x^2 - 18x + 28\right) = \infty$*, which also proves that the function* $f(x, 2x - 4)$ *does not have a global maximum in* S*.*