uc3m | Universidad Carlos III de Madrid
OpenCourseWare

Control Engineering II

Topic 1:
Z-Transform of Discrete-Time Systems

C.A. Monje, D. Copaci, M. Malfaz, J. Muñoz,
L. Mena, L. Moreno, S. Garrido

2026

Contents

Introduction to Digital Control

- In Control Engineering I, we studied how to design controllers so that the system output satisfies certain specifications such as overshoot, rise time, steady-state error, etc.
- Controllers that process the error signal to compute the desired control action have historically been analog devices: electrical, pneumatic, or mechanical.
- All these systems had analog inputs and outputs that were continuously defined over time within a given range of values.
- However, over the last 25 years, analog controllers have been replaced by digital controllers whose inputs and outputs are defined only at specific time instants (discrete time). These digital controllers may be implemented using digital circuits, computers, or microprocessors.

Reasons for the Transition

- Accuracy.
- Implementation Errors.
- Flexibility.
- Complexity.
- Speed.
- Cost.

Example

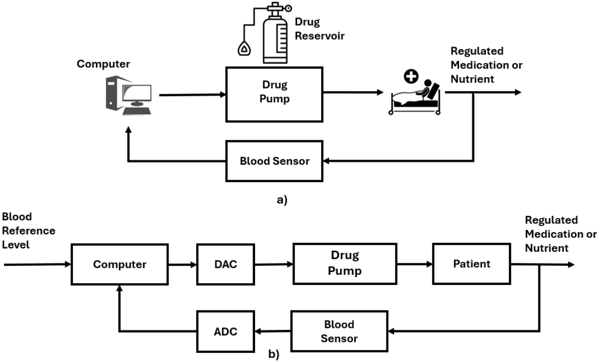


Figure 1: Digital drug delivery control system:
a) Schematic representation of a drug delivery system.
b) Block diagram of a drug delivery system.

Z Transform

Definition 1

- Given a causal sequence of numbers:

$$x(k) = x_0, x_1, x_2, \ldots, x_k, \ldots \tag{1}$$

the Z transform of this sequence is defined as:

$$X(z) = x_0 + x_1 z^{-1} + x_2 z^{-2} + \ldots + x_k z^{-k} + \ldots = \sum_{k=0}^{\infty} x(k) z^{-k} \tag{2}$$

- The Z transform defined by equation (2) is known as the **unilateral Z transform**.
- The term z^{-1} is called the delay operator.
- Where $z = re^{-j\Omega}$, with r being the magnitude and Ω the angle of z.

Example

- Obtain the Z transform of the following sequence:

$$x(k) = \{7, 4, 3, 0, 4, 0, 0, 2, 0 \ldots\}$$

- Solution:
According to the definition:

$$X(z) = \sum_{k=0}^{\infty} x(k) z^{-k} = 7 + 4z^{-1} + 3z^{-2} + 4z^{-4} + 2z^{-7} + \ldots$$

Z Transform of an Impulse Train

Definition 2

- An impulse train representing a sampled signal can be expressed as:

$$u^*(t) = u_0\delta(t) + u_1\delta(t-T) + \ldots + u_k\delta(t-kT) + \ldots = \sum_{k=0}^{\infty} u_k\delta(t-kT)$$

- The Laplace transform of this signal is:

$$U^*(s) = u_0 + u_1 e^{-sT} + \ldots + u_k e^{-skT} + \ldots = \sum_{k=0}^{\infty} u_k e^{-skT}$$

- And the Z transform is

$$U(z) = \sum_{k=0}^{\infty} u_k z^{-k}$$

- Where

$$z = e^{sT}$$

Expressions We Will Use

- Sum of a geometric series with n terms

$$\sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}, \quad a \neq 1$$

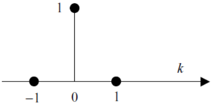
- Sum of an infinite geometric series

$$\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}, \quad |a| < 1$$

If $|a| \geqslant 1$, the sum is infinite.

- **Unit impulse** or unit sample, $\delta(k)$:

$$\delta(k) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases} \quad (3)$$



- The Z transform of this signal is

$$U(z) = \mathcal{Z}[\delta(k)] = \sum_{k=0}^{\infty} \delta(k) z^{-k} = z^{-0} = 1$$

- The Laplace transform of this signal ($u^*(t) = \delta(t)$) is

$$U^*(s) = 1$$

- The impulse signal can also be written as the difference of two step functions:

$$\delta(k) = u(k) - u(k-1)$$

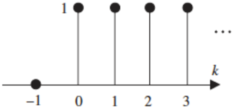
Z Transform of Some Basic Signals
Sampled Unit Step

- **Unit step**:

$$u(k) = \begin{cases} 1 & \text{if } k \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

has the Z transform

$$U(z) = 1 + z^{-1} + z^{-2} + \ldots + z^{-k} + \ldots = \sum_{k=0}^{\infty} z^{-k}$$



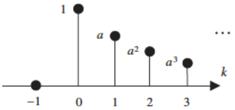
$$U(z) = \frac{1}{1 - z^{-1}} = \frac{z}{z - 1}$$

- It can be interpreted as a sum of unit impulses shifted in time:

$$u(k) = \sum_{n=-\infty}^{\infty} u(n)\, \delta(k - n)$$

- **Exponential**. The exponential sequence is defined as:

$$u(k) = \begin{cases} a^k & \text{if } k \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$



where a may be a real or complex number. When $a = e^{j\omega_0}$, where ω_0 is a real number, we have a complex exponential $e^{jk\omega_0} = \cos(k\omega_0) + j\sin(k\omega_0)$

- Its Z transform is:

$$U(z) = 1 + az^{-1} + a^2z^{-2} + \ldots + a^kz^{-k} + \ldots = \sum_{k=0}^{\infty}\left(\frac{a}{z}\right)^k$$

$$U(z) = \frac{1}{1-(a/z)} = \frac{z}{z-a}$$

The Z transform of the exponential function $f(k) = e^{-\alpha k}$ for $k \geq 0$ is obtained as

$$
\begin{aligned}
F(z) &= \mathcal{Z}[e^{-\alpha k}] = \sum_{k=0}^{\infty} e^{-\alpha k T} z^{-k} \\
&= 1 + e^{-\alpha T} z^{-1} + e^{-\alpha 2T} z^{-2} + \ldots \\
&= \frac{1}{1 - e^{-\alpha T} z^{-1}} \\
&= \frac{z}{z - e^{-\alpha T}}
\end{aligned} \tag{4}
$$

The table below shows the Z transforms of some of the most commonly used functions.

$f(k)$	$F(z)$
$\delta(k)$	1
$\delta(k-n)$	z^{-n}
$u_0(k)$	$\frac{z}{z-1}$
a^k	$\frac{z}{z-a}$
k	$\frac{z}{(z-1)^2}$
$k+1$	$\frac{z^2}{(z-1)^2}$
$\sin(\alpha k)$	$\frac{(\sin\alpha)z}{z^2-(2\cos\alpha)z+1}$
$\cos(\alpha k)$	$\frac{z^2-(\cos\alpha)z}{z^2-(2\cos\alpha)z+1}$

Signal Decomposition

- Using the unit impulse signal, a signal can be decomposed into a sum of time-shifted and weighted impulses:

$$x(k) = \ldots + x(-1)\delta(k+1) + x(0)\delta(k) + x(1)\delta(k-1) + x(2)\delta(k-2) + \ldots$$

- In a more compact notation:

$$x(k) = \sum_{n=-\infty}^{\infty} x(n)\delta(k-n) \tag{5}$$

Z Transform Operator

Definition

It is very common to consider the Z transform as an operator that transforms a sequence $x(k)$ into a function $X(z)$, which is symbolically represented as

$$X(z) = \mathcal{Z}[x(k)] \tag{6}$$

Properties of the Unilateral Z Transform

- **Linearity**

  A very important property of the Z transform is that it is a linear operator. Given two functions (sequences) $f(k)$ and $g(k)$ and an arbitrary constant α, the following holds:

$$\mathcal{Z}[f(k)+g(k)] = \mathcal{Z}[f(k)] + \mathcal{Z}[g(k)] \tag{7}$$

  and

$$\mathcal{Z}[\alpha f(k)] = \alpha \mathcal{Z}[f(k)] = \alpha F(z) \tag{8}$$

- Example:

$$f(k) = 2u(k) + 4\delta(k)$$

$$\begin{aligned} F(z) &= \mathcal{Z}[2u(k)] + \mathcal{Z}[4\delta(k)] \\ &= 2\mathcal{Z}[u(k)] + 4\mathcal{Z}[\delta(k)] \\ &= \frac{2z}{z-1} + 4 = \frac{6z-4}{z-1} \end{aligned}$$

Properties of the Unilateral Z Transform

- **Time Shift**
  The Z transform of a sequence $f(k)$ delayed by n samples, $f(k-n)$, can be expressed as:
  $$\mathcal{Z}[f(k-n)] = z^{-n}F(z) \tag{9}$$
  and when it is advanced:
  $$\mathcal{Z}[f(k+1)] = zF(z) - zf(0) \tag{10}$$
  $$\mathcal{Z}[f(k+n)] = \left[z^nF(z) - \sum_{k=0}^{n-1} f(k)z^{n-k}\right] \tag{11}$$
- Example (time delay):
  $$f(k) = 4, \quad k = 2, 3, 4, 5, \ldots$$
  $$\begin{aligned} F(z) &= \mathcal{Z}[4u(k-2)] \\ &= 4z^{-2}\mathcal{Z}[u(k)] \\ &= 4z^{-2}\frac{z}{z-1} = \frac{4}{z(z-1)} \end{aligned}$$

Properties of the Unilateral Z Transform
Cont.

- Example (time advance):
  Given the sequence $f(k)$, $\{f(k)\} = \{4, 8, 16, \ldots\}$, its Z transform is:

$$\begin{aligned} f(k) &= 2^{k+2} = g(k+2) \\ g(k) &= 2^k, \quad k = 0, 1, 2, \ldots \end{aligned}$$

$$\begin{aligned} F(z) &= \mathcal{Z}[f(k)] = z^2 G(z) - z^2 g(0) - zg(1) \\ &= z^2 \frac{z}{z-2} - z^2 - 2z = \frac{4z}{z-2} \end{aligned}$$

- A simpler solution:
  - Write the sequence as $\{f(k)\} = 4\{1, 2, 4, \ldots\} = 4 \times \{2^k\}$
  - Use the linearity property.

- **Complex Shift**

  If $f(t)$ has the Z transform $F(z)$, then the Z transform of $e^{-at}f(t)$ is defined as $F(ze^{aT})$, which is known as the complex shift theorem.

$$\mathcal{Z}[e^{-at}f(t)] = \sum_{k=0}^{n-1} f(kT)e^{-akT}z^{-k} = \sum_{k=0}^{n-1} f(kT)(ze^{aT})^{-k} = F(ze^{aT}) \quad (12)$$

Properties of the Unilateral Z Transform

- **Initial Value Theorem**
  If the Z transform of the function $f(k)$ is $F(z)$, and if $\lim_{z \to \infty} F(z)$ exists, then the initial value $f(0)$ of $f(k)$ is given by:

$$f(0) = \lim_{k \to 0} f(k) = \lim_{z \to \infty} F(z) \tag{13}$$

Properties of the Unilateral Z Transform

- **Final Value Theorem**

  If the Z transform of the function $f(k)$, with $f(k) = 0$ for $k < 0$, is $F(z)$, and if $F(z)$ *has all its poles inside the unit circle*, with the possible exception of a simple pole at $z = 1$, then the final value of $f(k)$ can be obtained as:

$$\begin{aligned} \lim_{k \to \infty} f(k) &= \lim_{z \to 1} [(1 - z^{-1}) F(z)] \\ &= \lim_{z \to 1} \left[\frac{z - 1}{z} F(z) \right] \end{aligned} \tag{14}$$

Properties of the Unilateral Z Transform

- **Multiplication by a^k**

  If the Z transform of the function $f(k)$ is $F(z)$, then the Z transform of $a^k f(k)$ is obtained as

$$\mathcal{Z}[a^k f(k)] = F(a^{-1}z) = F(z/a) \tag{15}$$

- **Discrete-Time (Real) Convolution**

  Let $F_1(z)$ and $F_2(z)$ be the Z transforms of the functions $f_1(t)$ and $f_2(t)$, then

$$
\begin{aligned}
F_1(z)F_2(z) &= \mathcal{Z}[f_1(k) * f_2(k)] \\
&= \mathcal{Z}\left[\sum_{k=0}^{N} f_1(k)f_2(N-k)\right] \\
&= \mathcal{Z}\left[\sum_{k=0}^{N} f_2(k)f_1(N-k)\right]
\end{aligned}
$$

  In this expression, the symbol $*$ denotes the convolution operation in the discrete-time domain.

  An exception occurs when one of the two functions is the integer delay e^{-NTs}, since in this case

$$
\mathcal{Z}[e^{-NTs}F(s)] = \mathcal{Z}[e^{-NTs}]\mathcal{Z}[F(s)] = z^{-N}F(z) \tag{16}
$$

Properties of the Unilateral Z Transform
Cont.

- Example (discrete convolution):

  Given two sampled unit-step signals $u(k)$, $\{u(k)\} = \{1, 1, 1, \ldots\}$, obtain the Z transform of their discrete convolution:

$$U(z) = \frac{z}{z-1}$$

  Applying the convolution theorem:

$$\begin{aligned} F(z) &= \mathcal{Z}[u(k) * u(k)] \\ &= U(z)\,U(z) = \frac{z}{z-1}\,\frac{z}{z-1} \\ &= \left(\frac{z}{z-1}\right)^2 \end{aligned}$$

Properties of the Unilateral Z Transform

- **Differentiation in the Complex Domain**

  The derivative with respect to z of the unilateral Z transform $F(z)$, multiplied by z, is equal to:

$$\mathcal{Z}[k\,x(k)] = -z\frac{dF(z)}{dz} \tag{17}$$

Properties of the Unilateral Z Transform

Cont.

- Example:
  Given a sampled ramp signal $f(k) = k$, $\{f(k)\} = \{0, 1, 2, \ldots\}$, obtain its Z transform.
- Solution: the Z transform of the sampled unit-step signal is $U(z) = \frac{z}{z-1}$.
  If we express the signal $f(k)$ as

$$f(k) = k\,u(k), \qquad k = 0, 1, 2, \ldots$$

  Applying the complex differentiation property:

$$\begin{aligned} F(z) &= \mathcal{Z}[k\,u(k)] \\ &= -z\frac{d}{dz}\left(\frac{z}{z-1}\right) \\ &= -z\frac{(z-1)-z}{(z-1)^2} = \frac{z}{(z-1)^2} \end{aligned}$$

Properties of the Unilateral Z Transform

- **First Difference**

  The unilateral Z transform of the first difference between two signals, $f(k) - f(k-1)$, is

$$\mathcal{Z}[f(k) - f(k-1)] = (1 - z^{-1})F(z) = \frac{z-1}{z}F(z) \tag{18}$$

- **Summation**

  The unilateral Z transform of the cumulative sum of a sequence $f(k)$ is given by

$$\mathcal{Z}\left[\sum_{n=0}^{N} f(k)\right] = \frac{1}{1-z^{-1}}F(z) = \frac{z}{z-1}F(z) \tag{19}$$

Inverse Z Transform

- The Z transform in discrete-time systems plays a role equivalent to that of the Laplace transform in continuous-time systems. Similarly, after performing operations in the Z-plane to solve a difference equation, it is necessary to convert the solution back to the time domain. In this case, returning to the time domain yields the corresponding discrete-time sequence $f(k)$.
- In the case of the inverse Z transform, the resulting sequence is defined only at the sampling instants. Therefore, a unique sequence $f(k)$ is obtained, but it may correspond to the sampling of an infinite number of continuous-time functions $f(t)$.

Inverse Z Transform

Inversion Formula

- As in the case of the Laplace transform, there exists an expression that provides the inverse Z transform as an integral expression in the Z-plane:

$$x(k) = \frac{1}{2\pi j} \oint_C X(z) z^{k-1} \, \mathrm{d}z$$

  where C is a counterclockwise integration contour that encloses the origin (complex variable theory).

- It is of limited practical use because it is difficult to evaluate.

- It is possible to compute the inverse Z transform using its integral expression, but this requires evaluating the integral. There are several alternative methods that are much simpler:
  - **Long division** of polynomials.
  - **Transform tables**.
  - **Partial fraction expansion** followed by a lookup in transform tables.

- By performing polynomial long division, $F(z)$ can be expanded into a series:

$$
\begin{aligned}
F_t(z) &= f_0 + f_1 z^{-1} + f_2 z^{-2} + \ldots + f_i z^{-i} \\
&= \sum_{k=0}^{i} f_k z^{-k}
\end{aligned}
$$

- The inverse transform is then obtained as the sequence

$$\{f_0, f_1, f_2, \ldots, f_i, \ldots\}$$

Inverse Z Transform

Example: Long Division

- Obtain the inverse Z transform of $F(z) = \frac{z+1}{z^2+0.2z+0.1}$ using polynomial division.
- Solution:

$$
\begin{array}{ll}
z+1 & z^2+0.2z+0.1 \\
-(z+0.2+0.1z^{-1}) & z^{-1} + 0.8\,z^{-2} - 0.26\,z^{-3}\ \ldots\ldots \\
\text{-----------------} & \\
\quad 0.8 - 0.1z^{-1} & \\
\quad -(0.8+0.16z^{-1}+0.08z^{-2}) & \\
\quad \text{--------------------------} & \\
\qquad -0.26z^{-1}-0.08z^{-2} & \\
\qquad -(-0.26z^{-1}-0.052z^{-2}-0.026z^{-3}) & \\
\qquad \text{-----------------------------------} & \\
\qquad\qquad -0.028z^{-2}+0.02z^{-3} & \\
\qquad\qquad \ldots\ldots &
\end{array}
$$

$$F_t(z) = 0 + z^{-1} + 0.8z^{-2} + (-0.26)z^{-3} + \ldots$$

- The inverse transform is the sequence

$$\{f(k)\} = \{0, 1, 0.8, -0.26, \ldots\}$$

- It is possible to compute the inverse Z transform using its integral expression, but this requires evaluating the integral.
- A much simpler alternative is to use transform tables. This can be done when $X(z)$ can be expressed as a sum of Z transforms of tabulated signals:

$$X(z) = X_1(z) + \ldots + X_n(z)$$

  whose inverse transforms are known:

$$x_1(k), \ldots, x_n(k)$$

  that is,

$$x(k) = x_1(k) + \ldots + x_n(k)$$

- **But how can we do this?**

- In most control applications, both difference equations and their solutions usually have the form of rational functions in z, of the type

$$F(z) = \frac{N(z)}{D(z)} \tag{20}$$

where $N(z)$ and $D(z)$ are polynomials in z.

- It is assumed that the degree of the numerator polynomial $N(z)$ in z is less than or equal to the degree of the denominator polynomial $D(z)$.
- This expression can be written in the following form:

$$F(z) = \frac{N(z)}{z^n + a_1 z^{n-1} + \ldots + a_{n-1} z + a_n} \tag{21}$$

where the coefficients a_i are real coefficients.

- **Method**:
  1. Find the partial fraction expansion of the expression $F(z)/z$.
  2. Obtain the inverse transform $f(k)$ using Z transform tables.
- There are three types of basic functions $F(z)$ in the z domain:
  1. $F(z)$ has simple real poles, not repeated.
  2. $F(z)$ has complex conjugate and real poles.
  3. $F(z)$ has repeated poles.

Case I: Simple Real Poles

- If all the poles of $D(z)$ are simple and real, and there is at least one zero at the origin, then $\frac{F(z)}{z}$ is expanded as follows:

$$\frac{F(z)}{z} = \frac{A_0}{z} + \frac{A_1}{(z-p_1)} + \frac{A_2}{(z-p_2)} + \ldots + \frac{A_n}{(z-p_n)} \tag{22}$$

- The coefficients $A_1, \ldots, A_n$ are obtained as the residues of the complex function $F(z)/z$ at the simple pole $z = p_i$:

$$A_i = \left[(z-p_i)\frac{F(z)}{z} \right] \bigg|_{z=p_i}$$

  and the coefficient A_0 as

$$A_0 = F(z)\Big|_{z=0}$$

- The function $F(z)/z$ is expanded instead of $F(z)$ so that the simple fractions in the expansion have the same form as in partial fraction expansions in s.

Case I: Simple Real Poles

Example

- Obtain the inverse Z transform of the function

$$F(z) = \frac{z+1}{z^2 + 0.3z + 0.02}$$

- **Solution:** We will solve it using two different methods.
  - Partial fraction expansion (dividing by z):

$$\frac{F(z)}{z} = \frac{z+1}{z(z^2 + 0.3z + 0.02)} = \frac{A}{z} + \frac{B}{z+0.1} + \frac{C}{z+0.2}$$

    that is,

$$F(z) = \frac{Az}{z} + \frac{Bz}{z+0.1} + \frac{Cz}{z+0.2}$$

Case I: Simple Real Poles

Example (Cont.)

- Calculation of the residues:

$$A = \left[z\frac{F(z)}{z} \right] \Big|_{z=0} = F(0) = \frac{1}{0.02} = 50$$

$$B = \left[(z+0.1)\frac{F(z)}{z} \right] \Big|_{z=-0.1} = \frac{1-0.1}{(-0.1)(0.1)} = -90$$

$$C = \left[(z+0.2)\frac{F(z)}{z} \right] \Big|_{z=-0.2} = \frac{1-0.2}{(-0.2)(-0.1)} = 40$$

- Therefore, we obtain

$$F(z) = 50 - \frac{90z}{z+0.1} + \frac{40z}{z+0.2}$$

- Looking up the transform table, we obtain

$$f(k) = \begin{cases} 50\delta(k) - 90(-0.1)^k + 40(-0.2)^k & \text{if } k \geqslant 0 \\ 0 & \text{if } k < 0 \end{cases}$$

- Note: Since $f(0) = 0$, the time sequence can be rewritten as

$$f(k) = \begin{cases} -90(-0.1)^k + 40(-0.2)^k & \text{if } k \geqslant 1 \\ 0 & \text{if } k < 1 \end{cases}$$

Case I: Simple Real Poles

Example

- Partial fraction expansion (without dividing by z):

$$F(z) = \frac{z+1}{z^2+0.3z+0.02} = \frac{A}{z+0.1} + \frac{B}{z+0.2}$$

- Calculation of the residues:

$$A = [(z+0.1)F(z)]\Big|_{z=-0.1} = \frac{1-0.1}{0.1} = 9$$

$$B = [(z+0.2)F(z)]\Big|_{z=-0.2} = \frac{1-0.2}{(-0.1)} = -8$$

Case I: Simple Real Poles

Example (Cont.)

- Rewriting $F(z)$

$$\begin{aligned} F(z) &= \frac{9}{z+0.1} - \frac{8}{z+0.2} \\ &= \frac{9z}{z+0.1} z^{-1} - \frac{8z}{z+0.2} z^{-1} \end{aligned}$$

- Looking up the transform table and using the delay theorem, we obtain

$$f(k) = \begin{cases} 9(-0.1)^{k-1} - 8(-0.2)^{k-1} & \text{if } k \geqslant 1 \\ 0 & \text{if } k < 1 \end{cases}$$

which gives the same result as before, although the expression is different.

Case II: Multiple Real Poles

- If the poles of $D(z)$ are multiple, then equation (21) takes the following form:

$$F(z) = \frac{N(z)}{(z - p_1)^2} \tag{23}$$

and in it, p_1 has multiplicity order 2.

Decomposing expression (23) into partial fractions, we obtain

$$\frac{F(z)}{z} = \frac{A_1}{(z - p_1)^2} + \frac{A_2}{(z - p_1)} \tag{24}$$

where the coefficients A_1 and A_2 corresponding to the multiple pole are obtained as

$$A_1 = \left[(z - p_1)^2 \frac{F(z)}{z}\right]\Big|_{z=p_1} \qquad A_2 = \frac{\mathrm{d}}{\mathrm{d}z}\left[(z - p_1)^2 \frac{F(z)}{z}\right]\Big|_{z=p_1} \tag{25}$$

- In general, if $F(z)$ has poles of multiple order, for example, if the pole p_i is multiple with order r, the expansion has the form

$$\frac{F(z)}{z} = \frac{N(z)}{(z-p_1)^r \prod_{j=r+1}^{n} z-p_j} = \sum_{i=1}^{r} \frac{A_{1i}}{(z-p_1)^{r+1-i}} + \sum_{j=r+1}^{n} \frac{A_j}{z-p_j}$$

- The expansion of the term corresponding to this multiple pole has the form

$$\frac{A_{11}}{z-p_1} + \frac{A_{12}}{(z-p_1)^2} + \ldots + \frac{A_{1r}}{(z-p_1)^r}$$

where

$$A_{1,r-k} = \frac{1}{k!}\frac{\mathrm{d}^k}{\mathrm{d}z^k}\left\{(z-p_1)^r \frac{F(z)}{z}\right\}\Big|_{z=p_1}$$

- Obtain the inverse Z transform of the function

$$F(z) = \frac{1}{z^2(z-0.5)}$$

- **Solution:**
  - Partial fraction expansion (dividing by z):

$$\frac{F(z)}{z} = \frac{1}{z^3(z-0.5)} = \frac{A_{11}}{z^3} + \frac{A_{12}}{z^2} + \frac{A_{13}}{z} + \frac{A_4}{z-0.5}$$

Case II: Multiple Real Poles

Example (Cont.)

- Calculation of the residues of $\frac{F(z)}{z} = \frac{1}{z^3(z-0.5)}$

  For the simple pole:

$$A_4 = \left[(z-0.5)\frac{F(z)}{z} \right] \Big|_{z=0.5} = \left[\frac{1}{z^3} \right] \Big|_{z=0.5} = 8$$

  For the multiple pole:

$$A_{11} = \left[z^3 \frac{F(z)}{z} \right] \Big|_{z=0} = \left[\frac{1}{z-0.5} \right] \Big|_{z=0} = -2$$

$$A_{12} = \frac{1}{1!}\frac{\mathrm{d}}{\mathrm{d}z}\left\{ z^3 \frac{F(z)}{z} \right\}\Big|_{z=0} = \frac{\mathrm{d}}{\mathrm{d}z}\left\{ \frac{1}{z-0.5} \right\}\Big|_{z=0} = \left\{ \frac{-1}{(z-0.5)^2} \right\}\Big|_{z=0} = -4$$

$$A_{13} = \frac{1}{2!}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left\{ z^3 \frac{F(z)}{z} \right\}\Big|_{z=0} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}z}\left\{ \frac{-1}{(z-0.5)^2} \right\}\Big|_{z=0} = \frac{1}{2}\left\{ \frac{(-1)(-2)}{(z-0.5)^3} \right\}\Big|_{z=0} = -8$$

- The expansion is

$$\frac{F(z)}{z} = \frac{1}{z^3(z-0.5)} = \frac{8}{z-0.5} - \frac{2}{z^3} - \frac{4}{z^2} - \frac{8}{z}$$

or equivalently,

$$F(z) = \frac{8z}{z-0.5} - 2z^{-2} - 4z^{-1} - 8$$

- Looking up the transform table, we obtain

$$f(k) = \begin{cases} 8(0.5)^k - 2\delta(k-2) - 4\delta(k-1) - 8\delta(k) & \text{if } k \geqslant 0 \\ 0 & \text{if } k < 0 \end{cases}$$

Case III: Simple Complex Conjugate Poles

- The following Z transforms will be used (ω_d in radians):

$$\begin{aligned}
\mathcal{Z}[e^{-\alpha k}\sin(k\omega_d)] &= \frac{e^{-\alpha}\sin(\omega_d)z}{z^2 - 2e^{-\alpha}\cos(\omega_d)z + e^{-2\alpha}} \\
\mathcal{Z}[e^{-\alpha k}\cos(k\omega_d)] &= \frac{z[z - e^{-\alpha}\cos(\omega_d)z]}{z^2 - 2e^{-\alpha}\cos(\omega_d)z + e^{-2\alpha}}
\end{aligned}$$

where the poles are:

$$p_{1,2} = e^{-\alpha}e^{\pm j\omega_d}$$

Case III: Simple Complex Conjugate Poles

Example

- Obtain the inverse Z transform of the function

$$F(z) = \frac{z^3 + 2z + 1}{(z - 0.1)(z^2 + z + 0.5)}$$

- **Solution:**
  - Partial fraction expansion.

$$\frac{Az}{z - p} + \frac{A^* z}{z - p^*} = \frac{2\mathrm{Re}\{K\}z - 2\mathrm{Re}\{Kp^*\}}{z^2 - (p + p^*)z + |p|^2} = \frac{Az + B}{z^2 - (p + p^*)z + |p|^2}$$

  - Dividing $F(z)$ by z, we have

$$\frac{F(z)}{z} = \frac{z^3 + 2z + 1}{z(z - 0.1)(z^2 + z + 0.5)} = \frac{A_1}{z} + \frac{A_2}{z - 0.1} + \frac{Az + B}{z^2 + z + 0.5}$$

- Calculation of the residues of $\frac{F(z)}{z} = \frac{z^3+2z+1}{z(z-0.1)(z^2+z+0.5)}$

  For the simple poles:

$$A_1 = \left[z\frac{F(z)}{z} \right] \Big|_{z=0} = \left[\frac{z^3 + 2z + 1}{(z - 0.1)(z^2 + z + 0.5)} \right] \Big|_{z=0} = -20$$

$$A_2 = \left[(z - 0.1)\frac{F(z)}{z} \right] \Big|_{z=0.1} = \left[\frac{z^3 + 2z + 1}{z(z^2 + z + 0.5)} \right] \Big|_{z=0.1} \approx 19.689$$

- For the complex conjugate poles, operating algebraically (the denominator is multiplied out and the system is obtained):

$$\begin{aligned} z^3 &: \quad A_1 + A_2 + A = 1 \\ z^1 &: \quad 0.4A_1 + 0.5A_2 - 0.1B = 2 \end{aligned}$$

Solving the system, since the values of A_1 and A_2 are already known, we obtain A and B: $A \approx 1.311, \quad B \approx -1.557$

The other two equations can be used to check the validity:

$$\begin{aligned} z^0 &: \quad -0.05A_1 = -0.05(-20) = 1 \\ z^2 &: \quad 0.9A_1 + A_2 - 0.1A + B \\ &\quad\; = 0.9(-20) + 19.689 - 0.1(1.311) - 1.557 = 0 \end{aligned}$$

Thus, the expansion becomes, after multiplying by z:

$$F(z) = -20 + \frac{19.689z}{z - 0.1} + \frac{1.311z^2 - 1.557z}{z^2 + z + 0.5}$$

Case III: Simple Complex Conjugate Poles

Example (Cont.)

- Looking up the transform table, the first two terms are straightforward:

$$\frac{1.311z^2 - 1.557z}{z^2 - 2(-0.5)z + 0.5} = \frac{1.311z[z - e^{-\alpha}\cos(\omega_d)] - Cze^{-\alpha}\sin(\omega_d)}{z^2 - 2e^{-\alpha}\cos(\omega_d)z + e^{-2\alpha}}$$

Equating the coefficients of the denominators of both expressions:

$$e^{-\alpha} = \sqrt{0.5} = 0.707$$

$$\cos(\omega_d) = -0.5/e^{-\alpha} = -\sqrt{0.5} = -0.707$$

and therefore $\omega_d = 3\pi/4$, which is an angle in the second quadrant, and $\sin(\omega_d) = 0.707$.

Equating the coefficients of z^1 in the numerator, we have:

$$-1.311e^{-\alpha}\cos(\omega_d) - Ce^{-\alpha}\sin(\omega_d) = -0.5(C - 1.311) = -1.557$$

$$C = 4.426$$

Case III: Simple Complex Conjugate Poles

Example (Cont.)

- Substituting the coefficients, we obtain

$$F(z) = -20 + \frac{19.689z}{z-0.1} + \frac{1.311z\left[z - 0.707\cos\left(\frac{3\pi}{4}\right)\right] - 3.129z\sin\left(\frac{3\pi}{4}\right)}{z^2 - 2(0.797)\cos\left(\frac{3\pi}{4}\right)z + 0.5}$$

  and therefore

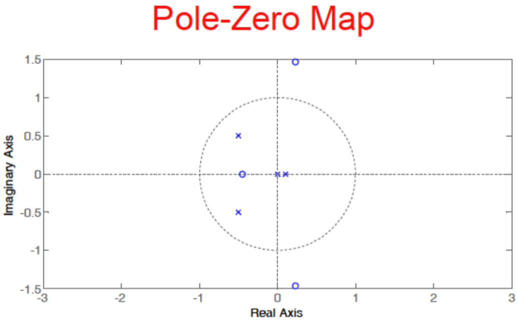
$$\mathcal{Z}[e^{-\alpha k}\sin(k\omega_d)] = \frac{e^{-\alpha}\sin(\omega_d)z}{z^2 - 2e^{-\alpha}\cos(\omega_d)z + e^{-2\alpha}}$$

$$\mathcal{Z}[e^{-\alpha k}\cos(k\omega_d)] = \frac{z\left[z - e^{-\alpha}\cos(\omega_d)\right]}{z^2 - 2e^{-\alpha}\cos(\omega_d)z + e^{-2\alpha}}$$

- Therefore, the inverse transform is obtained (from transform tables) as $f(k) =$

$$-\delta(k)+19.689(0.1)^k+(0.707)^k\left[1.311\cos\left(\frac{3\pi k}{4}\right)-4.426\sin\left(\frac{3\pi k}{4}\right)\right],$$

for $k \geqslant 0$.



- $G(z) = \dfrac{5(z+3)}{z^3 + 0.1z^2 + 0.4z}$

```
>> num = 5*[1 3]
>> den = [1 0.1 0.4 0]
>> denp = conv(den1, den2)
```

- Calculation of the partial fraction coefficients

```
>> [r, p, k] = residue(num,den)
```

where:

- p = poles
- r = residues
- k = coefficients of the resulting polynomials obtained by dividing the numerator by the denominator

- ```
  r =
  0.6557 + 2.2131i
  0.6557 - 2.2131i
  19.6885
  -20.0000
  ```
- ```
  p =
  -0.5000 + 0.5000i
  -0.5000 - 0.5000i
  0.1000
  0
  ```
- $$\frac{F(z)}{z} = \frac{z^3 + 2z + 1}{z(z-0.1)[(z+0.5)^2 + 0.5^2]}$$

- When we have

$$F(z) = \frac{N(z)}{D(z)} = k\frac{(z-z_1)(z-z_2)\dots(z-z_m)}{(z-p_1)(z-p_2)\dots(z-p_n)}$$

with $m > n$, although this case does not usually occur in control because it would imply system noncausality, from a theoretical point of view the treatment consists of developing the partial fraction expansion of $F(z)$ as follows:

$$F(z) = \sum_{q=0}^{m-n} b_q z^q + \sum_{k=1}^{n} c_k \frac{z}{z-p_k} \tag{26}$$

where the coefficients $b_0, \dots, b_q$ are obtained as the result of direct polynomial division of the numerator by the denominator.

- Suppose a certain system has the following difference equation:

$$x(k+2) + 2x(k+1) + x(k) = 0 \tag{27}$$

We also know that, for this system, the values of x for $k = 0$ and $k = 1$ are respectively $x(0) = 1$ and $x(1) = 0$, and we want to obtain the output $x(k)$ produced by the system (27).

- Applying the real-shift property of the Z transform, we obtain:

$$[z^2X(z) - z^2x(0) - zx(1)] + 2[zX(z) - zx(0)] + X(z) = 0$$

- Substituting $x(0)$ and $x(1)$ by their values, we get:

$$\begin{aligned}
[z^2X(z) - z^2] + [2zX(z) - 2z] + X(z) &= 0 \\
z^2X(z) + 2zX(z) + X(z) &= z^2 + 2z \\
X(z) &= \frac{z^2 + 2z}{z^2 + 2z + 1} = \frac{z^2 + 2z}{(z+1)^2}
\end{aligned}$$

Solving Difference Equations

Example (Cont.)

- Expanding this expression into partial fractions:

$$\frac{X(z)}{z} = \frac{A_1}{(z+1)^2} + \frac{A_2}{(z+1)} \tag{28}$$

The coefficients are determined by

$$\begin{aligned} A_1 &= \left[(z+1)^2 \frac{X(z)}{z}\right]\Big|_{z=-1} = z\Big|_{z=-1} = 1 \\ A_2 &= \frac{\mathrm{d}}{\mathrm{d}z}\left[(z+1)^2 \frac{X(z)}{z}\right]\Big|_{z=-1} = 1 \end{aligned}$$

- Its inverse transform will be the solution of the difference equation for the assumed initial conditions, namely

$$x(k) = (-1)^{k-1} + (-1)^k \tag{29}$$

Delay Operator

- As in continuous-time systems, it is often useful to employ an operator that allows the relationship between the input and output signals to be expressed in polynomial form. In discrete-time systems this operator is the so-called *advance operator*, usually denoted by the symbol q:

$$qu(k) = u(k+1) \tag{30}$$

or its inverse, the *delay operator*:

$$q^{-1}u(k) = u(k-1) \tag{31}$$

Delay Operator

- Using this operator, an nth-order difference equation

$$\begin{aligned} y(k) &+ a_1 y(k-1) + \ldots + a_{n-1} y(k-(n-1)) + a_n y(k-n) \\ &= b_1 u(k-1) + \ldots + b_{n-1} u(k-(n-1)) + b_n u(k-n) \end{aligned}$$

can be expressed in polynomial form:

$$\begin{aligned} y(k) &+ a_1 q^{-1} y(k) + \ldots + a_{n-1} q^{-(n-1)} y(k) + a_n q^{-n} y(k) \\ &= b_1 q^{-1} u(k) + \ldots + b_{n-1} q^{-(n-1)} u(k) + b_n q^{-n} u(k) \end{aligned}$$

that is,

$$\begin{aligned} & [1 + a_1 q^{-1} + \ldots + a_{n-1} q^{-(n-1)} + a_n q^{-n}] y(k) \\ &= [b_1 q^{-1} + \ldots + b_{n-1} q^{-(n-1)} + b_n q^{-n}] u(k) \\ A(q^{-1}) y(k) &= B(q^{-1}) u(k) \\ y(k) &= \frac{B(q^{-1})}{A(q^{-1})} u(k) \end{aligned}$$

- The delay operator is equivalent to z^{-1}, but in the time domain.

Relationship Between the Z and Laplace Transforms

- Definition 1:
  Given a sequence $u(k) = \{u(0), u(1), \ldots, u(k), \ldots\}$, its Z transform is defined as

$$\begin{aligned} U(z) &= u(0) + u(1)z^{-1} + u(2)z^{-2} + \ldots + u(k)z^{-k} + \ldots \\ &= \sum_{n=0}^{\infty} u(n)z^{-n} \end{aligned} \quad (32)$$

- Definition 2:
  Given a sequence $u(k) = \{u(0), u(1), \ldots, u(k), \ldots\}$, its representation as an impulse train (sampling of a continuous-time signal with period T) is defined as

$$\begin{aligned} u^*(t) &= u(0)\delta(t) + u(1)\delta(t-T) + \ldots + u(k)\delta(t-kT) + \ldots \\ &= \sum_{n=0}^{\infty} u(n)\delta(t-nT) \end{aligned} \quad (33)$$

Relationship Between the Z and Laplace Transforms

- If we take the Laplace transform of this expression:

$$\begin{aligned} U^*(s) &= \mathcal{L}[u^*(t)] \\ &= u(0) + u(1)e^{-sT} + u(2)e^{-2sT} + \ldots + u(k)e^{-ksT} + \ldots \\ &= \sum_{n=0}^{\infty} u(n)(e^{-sT})^n \end{aligned} \qquad (34)$$

- If we define $z = e^{sT}$ and substitute it into the previous expression, we obtain the Z transform of the sequence.
- Both expressions represent the same concept, but each has its own advantages and disadvantages. The first definition avoids the use of impulses and Laplace transforms, greatly simplifying analysis and design. The second allows us to use the Laplace transform to study discrete-time signals.

- The response of a discrete-time system to a unit impulse is called the **impulse response sequence**. This impulse response sequence can be used to represent the response of a linear discrete-time system to an arbitrary input sequence $\{u(k)\} = \{u(0), u(1), \ldots, u(k), \ldots\}$.
- This sequence can be expressed as

$$\begin{aligned} u(k) &= u(0)\delta(k) + u(1)\delta(k-1) + \ldots + u(i)\delta(k-i) + \ldots \\ &= \sum_{i=0}^{\infty} u(i)\delta(k-i) \end{aligned} \tag{35}$$

- For a linear system, by applying the superposition principle, the output of the system for a given input is the sum of the impulse responses corresponding to each impulse that forms the input signal:

$$\{y(k)\} = \{h(k)\}u(0) + \{h(k-1)\}u(1) + \ldots + \{h(k-i)\}u(i) + \ldots \quad (36)$$

- Writing this expression in summation form gives the convolution of the system impulse response with the input signal:

$$y(k) = h(k) * u(k) = \sum_{i=0}^{\infty} h(k-i)u(i) \quad (37)$$

- Given a system whose discrete-time impulse response is described by the sequence $h(k)$, the relationship between the system input and output in the discrete-time domain is expressed by the convolution sum:

$$y(k) = h(k) * x(k) = \sum_{n=-\infty}^{\infty} h(n)x(k-n) \tag{38}$$

- This relationship between $x(k)$ and $y(k)$ can be expressed in the complex domain using the Z transform as:

$$Y(z) = H(z)X(z) \tag{39}$$

- where $H(z)$ is the Z transform of $h(k)$.

- The relationship between the system input $x(k)$ and output $y(k)$ can be expressed in the complex domain using the Z transform as:

$$Y(z) = H(z)X(z) \tag{40}$$

- where $H(z)$ is the Z transform of $h(k)$ and is called the **discrete-time transfer function of the system**.

  The transfer function of a discrete-time system is the Z transform of its impulse response:

$$H(z) = \sum_{n=-\infty}^{\infty} h(n)z^{-n} \tag{41}$$

- From this expression, it is straightforward to obtain the frequency response by evaluating $H(z)$ on the unit circle:

$$H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}} \tag{42}$$