Control Engineering II

Topic 1:
Z-Transform of Discrete-Time Systems

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Problem 1

Calculate the unilateral Z-transform of the unit ramp function:

$$x(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Take into account that the function can be rewritten considering the sampling period: $x(kT) = kT, \;\; k = 0, 1, 2...$

Solution:

Applying the definition of the transform and the formula that allows obtaining the summation of the resulting series, the transform would be:

$$X(z) = \sum_{k=0}^{\infty} x(kT) z^{-k} = \sum_{k=0}^{\infty} kTz^{-k} = T \sum_{k=0}^{\infty} kz^{-k}$$

$$X(z) = T(z^{-1} + 2z^{-2} + 3z^{-3} + ...)$$

$$X(z) = T\frac{z^{-1}}{(1 - z^{-1})^2} = \frac{Tz}{(z - 1)^2}$$

Problem 2

Calculate the Z-transform of the sinusoidal function:

$$x(t) = \begin{cases} \sin wt, & t \geq t \\ 0, & t < 0 \end{cases}$$

Solution:

Taking into account that $e^{jwt} = \cos wt + j \sin wt$ and $e^{-jwt} = \cos wt - j \sin wt$, we have:

- $\sin wt = \frac{1}{2j}(e^{jwt} - e^{-jwt})$
- $\cos wt = \frac{1}{2}(e^{jwt} + e^{-jwt})$

The Z-transform of the exponential function is known:

$$f(k) = e^{-\alpha k}, \quad k \geq 0 \quad \Rightarrow \quad F(z) = \frac{1}{1 - e^{-\alpha T} z^{-1}}$$

The sinusoidal function can be rewritten as the difference of exponentials, so the sought-after transform will be:

$$X(z) = \frac{1}{2j}\left(\frac{1}{1 - e^{jwT} z^{-1}} - \frac{1}{1 - e^{-jwT} z^{-1}}\right) =$$

$$= \frac{1}{2j} \frac{(e^{jwT} - e^{-jwT}) z^{-1}}{1 - (e^{jwT} + e^{-jwT}) z^{-1} + z^{-2}}$$

Then:

$$X(z) = \frac{z^{-1} \sin wT}{1 - 2z^{-1} \cos wT + z^{-2}} = \frac{z \sin wT}{z^2 - 2z \cos wT + 1}$$

Problem 3

Calculate the Z-transform of the function:

$$f(k) = \begin{cases} a^{k-1}, & k = 1, 2, 3, \ldots \\ 0, & k \leq 0 \end{cases}$$

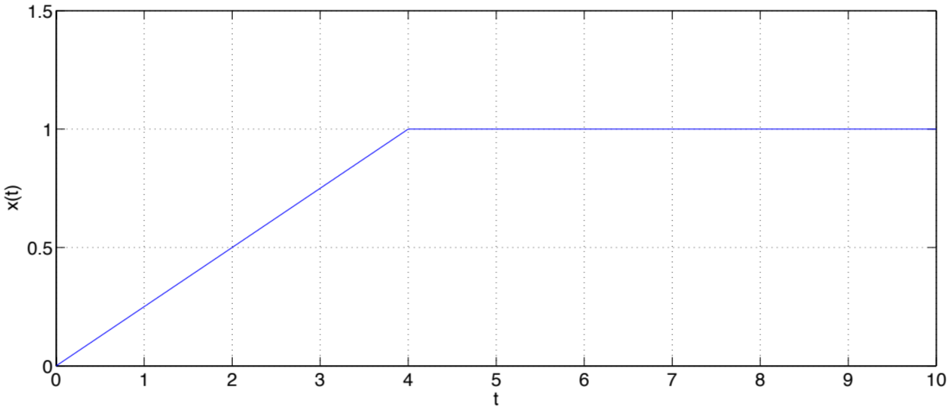
Solution:

To solve this problem one must use the real or time translation property: $Z\left[f(k-n)\right] = z^{-n}F(z)$
Since the delay in this case is one unit: $Z\left[f(k-1)\right] = z^{-1}F(z)$
The Z-transform of the function is known: $Z\left[a^k\right] = \frac{1}{1-az^{-1}}$
Therefore:

$$Z\left[f(k)\right] = Z\left[a^{k-1}\right] = z^{-1}\frac{1}{1-az^{-1}} = \frac{z^{-1}}{1-az^{-1}}$$

Problem 4

Obtain the Z-transform of the curve shown in the figure assuming that the sampling time T is 1 second.



Solution:

From the figure we obtain:

$$x\,(0) = 0, \;\; x\,(1) = 0,5 \;\; x\,(2) = 0,5$$
$$x\,(3) = 0,75 \;\; x\,(k) = 1, k = 4, 5, 6 \ldots$$

The Z-transform of $x(k)$ will be:

$$X(z) = \sum_{k=0}^{\infty} x(k)z^{-k} = 0.25z^{-1} + 0.50z^{-2} + 0.75z^{-3} + z^{-4} + z^{-5} + z^{-6} + \ldots$$

$$X(z) = 0.25(z^{-1} + 2z^{-2} + 3z^{-3}) + z^{-4}\frac{1}{1 - z^{-1}} = \frac{z^{-1} + z^{-2} + z^{-3} + z^{-4}}{4(1 - z^{-1})}$$

$$X(z) = \frac{z^{-1} + z^{-2} + z^{-3} + z^{-4}}{4(1 - z^{-1})}\frac{(1 - z^{-1})}{(1 - z^{-1})} = \frac{z^{-1}(1 - z^{-4})}{4(1 - z^{-1})^2}$$

Another way to calculate it would be from the expression of the curve:

$$x(t) = \frac{1}{4}t - \frac{1}{4}(t - 4)u(t - 4)$$

Where $u(t-4)$ is a unit step occurring when $t = 4$. Since we know the sampling time the Z-transform can be calculated as follows:

$$X(z) = Z[x(t)] = Z\left[\frac{1}{4}t\right] - Z\left[\frac{1}{4}(t - 4)u(t - 4)\right]$$

$$X(z) = \frac{1}{4}\frac{z^{-1}}{(1 - z^{-1})^2} - \frac{1}{4}\frac{z^{-4}z^{-1}}{(1 - z^{-1})^2} = \frac{z^{-1}(1 - z^{-4})}{4(1 - z^{-1})^2}$$

Problem 5

The function $y(k)$ is composed of a sum of functions $x(h)$, where $h = 1, 2, 3, ..., k$, such that:

$$y(k) = \sum_{h=0}^{k} x(h) \quad k = 0, 1, 2, ...$$

where $y(k) = 0$ for $k < 0$. Obtain the Z-transform of $y(k)$.

Solution:

If we decompose the expression:

$$y(k) = x(0) + x(1) + x(2) + ... + x(k-1) + x(k)$$
$$y(k-1) = x(0) + x(1) + x(2) + ... + x(k-1)$$

Subtracting: $y(k) - y(k-1) = x(k), \quad k = 0, 1, 2, ...$
Transforming to Z: $Z\left[y(k) - y(k-1)\right] = Z\left[x(k)\right]$
Using real or time translation, the previous expression becomes:

$$Y(z) - z^{-1}Y(z) = X(z)$$

Finally, the transform is:
$Y(z) = \frac{1}{1-z^{-1}}X(z)$, where $X(z) = Z\left[x(k)\right]$

Problem 6

Given the Z-transforms of $\sin wt$ and $\cos wt$, obtain the Z-transform of $e^{-\alpha t}\sin wt$ and $e^{-\alpha t}\cos wt$ using the complex translation theorem.

Solution:

Taking into account that $Z\left[\sin wt\right] = \frac{z^{-1}\sin wT}{1-2z^{-1}\cos wT+z^{-2}}$, we substitute z by $ze^{\alpha T}$ to obtain the Z-transform:

$$Z\left[e^{-\alpha t}\sin wt\right] = \frac{e^{-\alpha T}z^{-1}\sin wT}{1-2e^{-\alpha T}z^{-1}\cos wT + e^{-2\alpha T}z^{-2}}$$

Taking into account that $Z\left[\cos wt\right] = \frac{1-z^{-1}\cos wT}{1-2z^{-1}\cos wT+z^{-2}}$, we substitute z by $ze^{\alpha T}$ to obtain the Z-transform:

$$Z\left[e^{-\alpha t}\cos wt\right] = \frac{1-e^{-\alpha T}z^{-1}\cos wT}{1-2e^{-\alpha T}z^{-1}\cos wT + e^{-2\alpha T}z^{-2}}$$

Problem 7

Determine the final value $x(\infty)$ using the Final Value Theorem:
$X(z) = \frac{1}{1-z^{-1}} - \frac{1}{1-e^{-aT}z^{-1}}, \;\; a > 0$

Solution:

$$x(\infty) = \lim_{z \to 1} \left[(1 - z^{-1})X(z)\right] = \lim_{z \to 1} \left[(1 - z^{-1}) \left(\frac{1}{1 - z^{-1}} - \frac{1}{1 - e^{-aT}z^{-1}}\right)\right]$$

$$= \lim_{z \to 1} \left(1 - \frac{1 - z^{-1}}{1 - e^{-aT}z^{-1}}\right) = 1$$

Note: Since $X(z)$ is actually the transform of $x(t) = 1 - e^{-at}$, it can be calculated from this expression:

$$x(\infty) = \lim_{t \to \infty} \left(1 - e^{-at}\right) = 1$$

Problem 8

Find $x(k)$ **for** $k = 1, 2, 3, 4, \dots$ **when:**

$$X(z) = \frac{10z + 5}{(z-1)(z-0.2)}$$

Solution:

First we rewrite the expression to have it as a polynomial in z^{-1}:

$$X(z) = \frac{10z^{-1} + 5z^{-2}}{1 - 1.2z^{-1} + 0.2z^{-2}}$$

To solve this problem we will use the method of direct division, obtaining the inverse Z-transform by expanding $X(z)$ into a series in powers of z^{-1}. This method is useful when it is difficult to obtain the closed-form expression or only the first terms of $x(k)$ are needed.

We expand $X(z)$ to take advantage of the definition of the Z-transform:

$$X(z) = \sum_{k=0}^{\infty} x(kT)z^{-k} = x(0) + x(T)z^{-1} + x(2T)z^{-2} + \dots + x(kT)z^{-k} + \dots$$

Dividing successively, we have:

$$\begin{aligned}
\frac{10z^{-1} + 5z^{-2}}{1 - 1,2z^{-1} + 0,2z^{-2}} &= 10z^{-1} + \frac{17z^{-2} - 2z^{-3}}{1 - 1,2z^{-1} + 0,2z^{-2}} \\
&= 10z^{-1} + 17z^{-2} + \frac{18,4z^{-3} - 3,4z^{-4}}{1 - 1,2z^{-1} + 0,2z^{-2}} \\
&= 10z^{-1} + 17z^{-2} + 18,4z^{-3} + \frac{18,68z^{-4} - 3,68z^{-5}}{1 - 1,2z^{-1} + 0,2z^{-2}}
\end{aligned}$$

Therefore: $X(z) = 10z^{-1} + 17z^{-2} + 18.4z^{-3} + 18.68z^{-4} + \dots$

Comparing this result with the expansion yields the first terms of the series:

$$x(0) = 0, \ \ x(1) = 10, \ \ x(2) = 17, \ \ x(3) = 18.4, \ \ x(4) = 18.68$$

Obtain the inverse Z-transform for the following expression via partial fraction expansion:

$$X(z) = \frac{z^2 + z + 2}{(z-1)(z^2 - z + 1)}$$

Solution:

First we expand $X(z)$ into partial fractions and obtain the coefficients by equating:

$$X(z) = \frac{A}{z-1} + \frac{Bz + C}{z^2 - z + 1} = \frac{(A+B)z^2 + (-A-B+C)z + (A-C)}{(z-1)(z^2-z+1)}$$

Equating with the equation of the problem statement yields the following system of equations:

$$\left.\begin{array}{c} A + B = 1 \\ -A - B + C = 1 \\ A - C = 2 \end{array}\right\}$$

Solving:

$$A = 4, \quad B = -3, \quad C = 2$$

$$X(z) = \frac{4}{z-1} + \frac{-3z+2}{z^2 - z + 1} = \frac{4z^{-1}}{1 - z^{-1}} + \frac{-3z^{-1} + 2z^{-2}}{1 - z^{-1} + z^{-2}}$$

It should be noted that the poles of the quadratic term are complex conjugates, so the previous expression can be rewritten as:

$$X(z) = \frac{4z^{-1}}{1 - z^{-1}} - 3\left(\frac{z^{-1} - 0.5z^{-2}}{1 - z^{-1} + z^{-2}}\right) + \frac{0.5z^{-2}}{1 - z^{-1} + z^{-2}}$$

$$X(z) = 4z^{-1}\frac{1}{1 - z^{-1}} - 3z^{-1}\left(\frac{1 - 0.5z^{-1}}{1 - z^{-1} + z^{-2}}\right) + z^{-1}\frac{0.5z^{-1}}{1 - z^{-1} + z^{-2}}$$

We know that:

$$Z\left[e^{-\alpha t}\sin wt\right] = \frac{e^{-\alpha T}z^{-1}\sin wT}{1 - 2e^{-\alpha T}z^{-1}\cos wT + e^{-2\alpha T}z^{-2}}$$

$$Z\left[e^{-\alpha t}\cos wt\right] = \frac{1 - e^{-\alpha T}z^{-1}\cos wT}{1 - 2e^{-\alpha T}z^{-1}\cos wT + e^{-2\alpha T}z^{-2}}$$

We can identify $e^{-2\alpha T} = 1$ and $\cos wT = 0.5$ in this case, so we have $wT = \pi/3$ and $\sin wT = \sqrt{3}/2$. Therefore:

$$Z^{-1}\left[\frac{1-0.5z^{-1}}{1-z^{-1}+z^{-2}}\right] = 1^k \cos\frac{k\pi}{3}$$

$$Z^{-1}\left[\frac{0.5z^{-1}}{1-z^{-1}+z^{-2}}\right] = Z^{-1}\left[\frac{1}{\sqrt{3}}\frac{(\sqrt{3}/2)z^{-1}}{1-z^{-1}+z^{-2}}\right] = \frac{1}{\sqrt{3}}1^k \sin\frac{k\pi}{3}$$

Finally:

$$x(k) = 4(1^{k-1}) - 3(1^{k-1})\cos\frac{(k-1)\pi}{3} + \frac{1}{\sqrt{3}}(1^{k-1})\sin\frac{(k-1)\pi}{3}$$

Rewriting:

$$x(k) = \begin{cases} 4 - 3\cos\frac{(k-1)\pi}{3} + \frac{1}{\sqrt{3}}\sin\frac{(k-1)\pi}{3}, & k = 1,2,3,\dots \\ 0, & k \leq 0 \end{cases}$$

An alternative solution would be:

$$X(z) = 4z^{-1}\frac{4}{1-z^{-1}} - 3\left(\frac{z^{-1}}{1-z^{-1}+z^{-2}}\right) + 2z^{-1}\frac{z^{-1}}{1-z^{-1}+z^{-2}}$$

Since:

$$Z^{-1}\left[\frac{z^{-1}}{1-z^{-1}}\right] = \begin{cases} 1, & k = 1,2,3,\dots \\ 0, & k \leq 0 \end{cases}$$

$$Z^{-1}\left[\frac{z^{-1}}{1-z^{-1}+z^{-2}}\right] = \frac{2}{\sqrt{3}}(1^k)\sin\frac{k\pi}{3}$$

The solution will be:

$$x(k) = \begin{cases} 4 - 2\sqrt{3}\sin\frac{k\pi}{3} + \frac{4}{\sqrt{3}}\sin\frac{(k-1)\pi}{3}, & k = 1,2,3,\dots \\ 0, & k \leq 0 \end{cases}$$

Although this solution may seem different from the one obtained previously, both solutions are correct and have the same values for $x(k)$.

Problem 10

Obtain the inverse Z-transform for the following expression via partial fraction expansion:

$$X(z) = \frac{z+2}{(z-2)z^2}$$

Solution:

First we expand $X(z)$ into partial fractions and obtain the coefficients by equating:

$$X(z) = \frac{A}{(z-2)} + \frac{B}{z^2} + \frac{C}{z} = \frac{Az^2 + B(z-2) + Cz(z-2)}{(z-2)z^2}$$

$$\left.\begin{array}{c} A + C = 0 \\ B - 2C = 1 \\ -2B = 2 \end{array}\right\}$$

Solving:

$$A = 1, \quad B = -1, \quad C = -1$$

$$X(z) = \frac{1}{z-2} - \frac{1}{z^2} - \frac{1}{z} = \frac{z^{-1}}{1-2z^{-1}} - z^{-2} - z^{-1}$$

It should be noted that $X(z)$ has a double pole at $z = 0$, so the expansion must include the terms $1/z^2$ and $1/z$. The inverse transforms of all terms are known:

$$Z^{-1}\left[\frac{z^{-1}}{1-2z^{-1}}\right] = \begin{cases} 2^{k-1}, & k = 1, 2, 3, \ldots \\ 0, & k \leq 0 \end{cases}$$

$$Z^{-1}\left[z^{-2}\right] = \begin{cases} 1, & k = 2 \\ 0, & k \neq 2 \end{cases}$$

$$Z^{-1}\left[z^{-1}\right] = \begin{cases} 1, & k = 1 \\ 0, & k \neq 1 \end{cases}$$

Therefore, the inverse Z-transform for $X(z)$ is equal to:

$$x(k) = \begin{cases} 0 - 0 - 0 = 0, & k = 0 \\ 1 - 0 - 1 = 0, & k = 1 \\ 2 - 1 - 0 = 1, & k = 2 \\ 2^{k-1} - 0 - 0 = 2^{k-1}, & k = 3, 4, 5, \ldots \end{cases}$$

$$x(k) = \begin{cases} 0, & k = 0, 1 \\ 1, & k = 2 \\ 2^{k-1}, & k = 3, 4, 5, \ldots \end{cases}$$

Problem 11 (1/3)

Obtain the inverse Z-transform for the following expression via direct division and via partial fraction expansion:

$$X(z) = \frac{z(z+2)}{(z-1)^2}$$

Solution:

Direct division: we rewrite the expression so that the highest degree term is of degree zero:

$$X(z) = \frac{z(z+2)}{(z-1)^2} = \frac{z^2+2z}{z^2-2z+1} = \frac{1+2z^{-1}}{1-2z^{-1}+z^{-2}}$$

By successive division we obtain the values of x(k):

$$\frac{1+2z^{-1}}{1-2z^{-1}+z^{-2}} = 1 + \frac{4z^{-1}-z^{-2}}{1-2z^{-1}+z^{-2}}, \qquad X(z) = 1 + \frac{4z^{-1}-z^{-2}}{1-2z^{-1}+z^{-2}}$$

$$\frac{4z^{-1}-z^{-2}}{1-2z^{-1}+z^{-2}} = 4z^{-1} + \frac{7z^{-2}-4z^{-3}}{1-2z^{-1}+z^{-2}}, \qquad X(z) = 1 + 4z^{-1} + \frac{7z^{-2}-4z^{-3}}{1-2z^{-1}+z^{-2}}$$

$$\frac{7z^{-2}-4z^{-3}}{1-2z^{-1}+z^{-2}} = 7z^{-2} + \frac{10z^{-3}-7z^{-4}}{1-2z^{-1}+z^{-2}}, \qquad X(z) = 1 + 4z^{-1} + 7z^{-2} + \frac{10z^{-3}-7z^{-4}}{1-2z^{-1}+z^{-2}}$$

$$\frac{10z^{-3}-7z^{-4}}{1-2z^{-1}+z^{-2}} = 10z^{-3} + \frac{13z^{-4}-10z^{-5}}{1-2z^{-1}+z^{-2}}, \qquad X(z) = 1 + 4z^{-1} + 7z^{-2} + 10z^{-3} + \frac{13z^{-4}-10z^{-5}}{1-2z^{-1}+z^{-2}}$$

The result will be: $X(z) = 1 + 4z^{-1} + 7z^{-2} + 10z^{-3} + \ldots$
The first terms of $x(k)$ are: $x(0) = 1,\ \ x(1) = 4,\ \ x(2) = 7,\ \ x(3) = 10,\ \ \ldots$

Partial fraction expansion:
Since the degree of the numerator is equal to that of the denominator, we can use the first division performed above:

$$X(z) = 1 + \frac{4z^{-1}-z^{-2}}{1-2z^{-1}+z^{-2}} = 1 + \frac{4z-1}{(z-1)^2}$$

$$X(z) = 1 + \frac{4z-1}{(z-1)^2} = 1 + \frac{B}{(z-1)} + \frac{C}{(z-1)^2} = \frac{B(z-1)+C}{(z-1)^2}$$

$$\left.\begin{array}{r} B = 4 \\ C - B = -1 \end{array}\right\}$$

Problem 11 (2/3)

Solving:

$$B = 4, \;\; C = 3$$

$$X(z) = 1 + \frac{4z-1}{(z-1)^2} = 1 + \frac{4}{z-1} + \frac{3}{(z-1)^2} = 1 + \frac{4z^{-1}}{1-z^{-1}} + \frac{3z^{-2}}{(1-z^{-1})^2}$$

The inverse transforms of all fractions are known:

$$Z^{-1}[1] = \begin{cases} 1, & k = 0 \\ 0, & k = 1, 2, 3, \ldots \end{cases}$$

$$Z^{-1}\left[\tfrac{z^{-1}}{(1-z^{-1})^2}\right] = k, \;\; k = 1, 2, 3, \ldots$$

$$Z^{-1}\left[\frac{z^{-1}}{1-z^{-1}}\right] = \begin{cases} 1, & k = 1, 2, 3, \ldots \\ 0, & k \leq 0 \end{cases}$$

The solution in this case will be:

$$x(0) = 1 \;\; x(k) = 4 + 3(k-1) = 3k + 1, \;\; k = 1, 2, 3, \ldots$$

This equation can be rewritten as:

$$x(k) = 3k + 1, \;\; k = 0, 1, 2, \ldots$$

It can also be solved by expanding using the residue method:

$$X(z) = 1 + \frac{4z-1}{(z-1)^2} = 1 + H(z)$$

$$\frac{H(z)}{z} = \frac{A_0}{z} + \frac{A_1}{(z-1)^2} + \frac{A_2}{z-1}$$

$$A_0 = \left[z\frac{H(z)}{z}\right]\Bigg|_{z=0} = -1$$

$$A_1 = \left[(z-1)^2\frac{H(z)}{z}\right]\Bigg|_{z=1} = \left[\frac{4z-1}{z}\right]\Bigg|_{z=1} = 3$$

$$A_2 = \frac{d}{dz}\left[(z-1)^2\frac{H(z)}{z}\right]\Bigg|_{z=1} = \frac{d}{dz}\left[\frac{4z-1}{z}\right]\Bigg|_{z=1} = 1$$

Problem 11 (3/3)

So that $H(z)$ can be written as:

$$\frac{H(z)}{z} = \frac{-1}{z} + \frac{3}{(z-1)^2} + \frac{1}{z-1}$$

$$H(z) = -1 + \frac{3z}{(z-1)^2} + \frac{z}{z-1}$$

$$X(z) = 1 + \frac{4z-1}{(z-1)^2} = 1 - 1 + \frac{3z}{(z-1)^2} + \frac{z}{z-1} = \frac{3z}{(z-1)^2} + \frac{z}{z-1}$$

$$X(z) = \frac{3z^{-1}}{(1-z^{-1})^2} + \frac{1}{1-z^{-1}}$$

$$Z^{-1}\left[\tfrac{z^{-1}}{(1-z^{-1})^2}\right] = k, \;\; k \geq 0, \;\; Z^{-1}\left[\tfrac{1}{1-z^{-1}}\right] = 1, \;\; k \geq 0$$

The inverse transform of the first component is a ramp multiplied by three, and that of the second is a step, so the solution is as follows:

$$x(k) = 3k + 1, \;\; k = 0, 1, 2, \ldots$$

We see that this expression coincides with the result obtained with the other expansion.

Problem 12

Solve the following difference equation using the Z-transform method:

$$x(k+2) + 3x(k+1) + 2x(k) = 0, \;\; x(0) = 0, \;\; x(1) = 1$$

Solution:

First it must be taken into account that:

$$Z\left[x(k+2)\right] = z^2 X(z) - z^2 x(0) - zx(1)$$
$$Z\left[x(k+1)\right] = z^1 X(z) - zx(0)$$
$$Z\left[x(k)\right] = X(z)$$

Calculating the transform of the difference equation:

$$z^2 X(z) - z^2 x(0) - zx(1) + 3zX(z) - 3zx(0) + 2X(z) = 0$$

Substituting the given conditions and simplifying, we arrive at:

$$X(z) = \frac{z}{z^2 + 3z + 2} = \frac{z}{(z+1)(z+2)}$$

Expanding into simple fractions:

$$\frac{X(z)}{z} = \frac{A_0}{z} + \frac{A_1}{z+1} + \frac{A_2}{z+2}$$

$$A_0 = \left[z\frac{X(z)}{z}\right]\bigg|_{z=0} = 0$$

$$A_1 = \left[(z+1)\tfrac{X(z)}{z}\right]\bigg|_{z=-1} = \tfrac{1}{z+2}\big|_{z=-1} = 1$$

$$A_2 = \left[(z+2)\tfrac{X(z)}{z}\right]\bigg|_{z=-2} = \tfrac{1}{z+1}\big|_{z=-2} = -1$$

$$X(z) = \frac{z}{z+1} - \frac{z}{z+2} = \frac{1}{1+z^{-1}} - \frac{1}{1+2z^{-1}}$$

Taking into account that:

$$Z^{-1}\left[\tfrac{1}{1+z^{-1}}\right] = (-1)^k, \;\; Z^{-1}\left[\tfrac{1}{1+2z^{-1}}\right] = (-2)^k$$

The solution will be:

$$x(k) = (-1)^k - (-2)^k, \;\; k = 0, 1, 2, \ldots$$

Problem 13 (1/2)

Solve the following difference equation using the Z-transform method leaving the result as a function of $x(0)$ and $x(1)$:

$$x(k+2) + (a+b)x(k+1) + abx(k) = 0$$

where a and b are constants and $k = 0, 1, 2, \ldots$

Solution:

Calculating the transform of the difference equation:

$$[z^2X(z) - z^2x(0) - zx(1)] + (a+b)[zX(z) - zx(0)] + abX(z) = 0$$

Rearranging the expression:

$$[z^2 + (a+b)z + ab]X(z) = [z^2 + (a+b)z]x(0) + zx(1)$$
$$X(z) = \frac{[z^2+(a+b)z]x(0)+zx(1)}{z^2+(a+b)z+ab} = \frac{[z^2+(a+b)z]x(0)+zx(1)}{(z+a)(z+b)}$$

As $-a$ and $-b$ are the roots of the characteristic polynomial, two cases must be separated: a) $a \neq b$ and b) $a = b$.

a) $a \neq b$. Expanding into simple fractions:

$$\frac{X(z)}{z} = \frac{A_0}{z} + \frac{A_1}{z+a} + \frac{A_2}{z+b}$$

$$A_0 = \left[z\frac{X(z)}{z} \right]\bigg|_{z=0} = 0$$

$$A_1 = \left[(z+a)\frac{X(z)}{z} \right]\bigg|_{z=-a} = \frac{[z+a+b]x(0)+x(1)}{z+b}\bigg|_{z=-a} = \frac{bx(0)+x(1)}{b-a}$$
$$A_2 = \left[(z+b)\frac{X(z)}{z} \right]\bigg|_{z=-b} = \frac{[z+a+b]x(0)+x(1)}{z+a}\bigg|_{z=-b} = \frac{ax(0)+x(1)}{a-b}$$
$$\frac{X(z)}{z} = \frac{bx(0)+x(1)}{b-a}\frac{1}{z+a} + \frac{ax(0)+x(1)}{a-b}\frac{1}{z+b}, \quad a \neq b$$
$$X(z) = \frac{bx(0)+x(1)}{b-a}\frac{1}{1+az^{-1}} + \frac{ax(0)+x(1)}{a-b}\frac{1}{1+bz^{-1}}$$

The solution will be:

$$x(k) = \frac{bx(0)+x(1)}{b-a}(-a)^k + \frac{ax(0)+x(1)}{a-b}(-b)^k \quad a \neq b \quad k = 0, 1, 2, \ldots$$

Problem 13 (2/2)

b) $a = b$. Now there is a multiple pole, so the expansion will be done differently:

$$X(z) = \frac{(z^2 + 2az)x(0) + zx(1)}{(z+a)^2}$$

$$\frac{X(z)}{z} = \frac{A_0}{z} + \frac{A_1}{(z+a)^2} + \frac{A_2}{z+b}$$

$$A_0 = \left[z\frac{X(z)}{z} \right]\bigg|_{z=0} = 0$$

$$A_1 = \left[(z+a)^2 \frac{X(z)}{z} \right]\bigg|_{z=-a} = (z+2a)x(0) + x(1)|_{z=-a} = ax(0) + x(1)$$

$$A_2 = \frac{d}{dz}\left[(z+a)^2 \frac{X(z)}{z} \right]\bigg|_{z=-a} = \frac{d}{dz}\left[(z+2a)x(0) + x(1) \right]\Big|_{z=-a} = x(0)$$

$$X(z) = \frac{zx(0)}{z+a} + \frac{z[ax(0)+x(1)]}{(z+a)^2} = \frac{x(0)}{1+az^{-1}} + \frac{[ax(0)+x(1)]z^{-1}}{(1+az^{-1})^2}$$

The solution will be:

$$x(k) = x(0)(-a)^k + [ax(0)+x(1)]k(-a)^{k-1}, \;\; a = b \;\; k = 0, 1, 2, \ldots$$

Solve the following difference equation:

$$2x(k) - 2x(k-1) + x(k-2) = u(k)$$

$$u(k) = \begin{cases} 1, & k = 0, 1, 2, 3, \ldots \\ 0, & k < 0 \end{cases}$$

where $x(k) = 0$ for $k < 0$.

Solution:

First the Z-transform of the expression must be calculated taking into account the real or time translation property and the transform of $u(k)$ since it is known.

$$2X(z) - 2z^{-1}X(z) + z^{-2}X(z) = \frac{1}{1 - z^{-1}}$$

Solving for $X(z)$ in the previous expression:

$$X(z) = \frac{1}{1 - z^{-1}} \frac{1}{2 - 2z^{-1} + z^{-2}} = \frac{z^3}{(z-1)(2z^2 - 2z + 1)}$$

This expression can be expanded into partial fractions. The coefficients will be obtained by equating:

$$X(z) = \frac{z^3}{(z-1)(2z^2 - 2z + 1)} = \frac{Az}{(z-1)} + \frac{Bz^2 + Cz}{(2z^2 - 2z + 1)}$$

We look for expressions whose inverse transform is known. In this case it must be taken into account that the numerator is equal to z^3.

$$X(z) = \frac{(2A + B)z^3 + (-2A + C - B)z^2 + (A - C)z}{(z-1)(2z^2 - 2z + 1)}$$

$$\left.\begin{array}{c} 2A + B = 1 \\ -2A + C - B = 0 \\ A - B = 0 \end{array}\right\}$$

Problem 14 (2/2)

Solving:

$$A = 1, \;\; B = -1, \;\; C = 1$$

$$X(z) = \frac{z}{z-1} + \frac{-z^2+z}{2z^2-2z+1} = \frac{1}{1-z^{-1}} + \frac{-1+z^{-1}}{2-2z^{-1}+z^{-2}}$$

This equation can be rewritten so that known inverse transforms remain (in this case those of sine and cosine must be found):

$$X(z) = \frac{1}{1-z^{-1}} - \frac{1}{2}\frac{1-0.5z^{-1}}{(1-z^{-1}+0.5z^{-2})} + \frac{1}{2}\frac{0.5z^{-1}}{(1-z^{-1}+0.5z^{-2})}$$

Looking up in the tables the Z-transforms of sine and cosine, we can identify $e^{-2\alpha T} = 0.5$ and $\cos wT = 1/\sqrt{2}$. We have $wT = \pi/4$, $\sin wT = 1/\sqrt{2}$ and $e^{-\alpha T} = 1/\sqrt{2}$. Therefore:

$$x(k) = 1 - \frac{1}{2}e^{-\alpha kT}\cos wkT + \frac{1}{2}e^{-\alpha kT}\sin wkT$$

$$x(k) = 1 - \tfrac{1}{2}\left(\tfrac{1}{\sqrt{2}}\right)^k \cos\tfrac{k\pi}{4} + \tfrac{1}{2}\left(\tfrac{1}{\sqrt{2}}\right)^k \sin\tfrac{k\pi}{4}, \;\; k = 0, 1, 2, ...$$

$$x(0) = 0.5, \;\; x(1) = 1, \;\; x(2) = 1.25, \;\; x(3) = 1.25, \;\; \ldots$$