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Control Engineering II

Topic 2:
Obtaining the Transfer Function of a System

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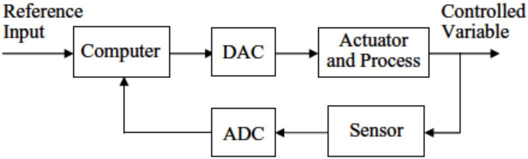


Figure 1: Digital control system.

- To control a real physical system or process using a digital controller (computer), it is necessary to measure the system, process the measurements, and apply control actions to the system.
- In most cases, the process (or plant) and the actuator are analog devices, so continuous signals must be converted into discrete signals and vice versa.

Example

Aircraft Turbine Control

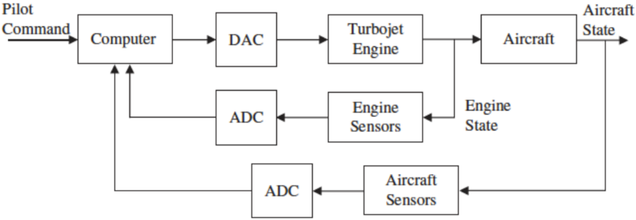


Figure 2: Aircraft turbine control system.

Difference Equations

Difference equations arise in problems where the independent variable, usually time, can only take a discrete set of values. The nonlinear difference equation

$$\begin{aligned} y(k+n) &= f[y(k+n-1), y(k+n-2), \ldots, \\ &\quad y(k+1), y(k), u(k+n), u(k+n-1), \ldots, \\ &\quad u(k+1), u(k)] \end{aligned} \tag{1}$$

is said to be of order n because n is the difference between the largest and smallest time indices appearing in the arguments $y(\cdot)$ and $u(\cdot)$.

For the moment, we will focus only on linear systems, whose expression is simpler:

$$y(k+n) + a_{n-1}y(k+n-1) + a_{n-2}y(k+n-2) + \ldots + a_1 y(k+1) + a_0 y(k) \tag{2}$$

$$= b_n u(k+n) + b_{n-1} u(k+n-1) + \ldots + b_1 u(k+1) + b_0 u(k) \tag{3}$$

Samplers and Hold Devices

- **Sampler**. A sampler is a circuit that captures the value of an analog signal at a specific instant in time.
- **Hold Device**. A hold device receives a digital signal as input and maintains the input value constant at its output for a specified period of time.
- **Analog-to-Digital Converter (A/D)**. Converts an analog signal into a digitally encoded signal.
- **Digital-to-Analog Converter (D/A)**. Converts a digitally encoded signal into an analog signal.

- In control system block diagrams, the process of sampling and converting an analog signal into a digital signal is usually represented by the following symbol, which represents an ideal sampler:

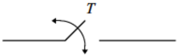


Figure 3: Sampler.

- We usually assume that:
  - The ADC outputs are exactly equal in magnitude to the inputs (i.e., rounding errors are neglected). **Unity gain**.
  - The ADC produces outputs instantaneously. **No delay**.
  - Sampling is perfectly uniform.
  - In other words, the ADC is an ideal sampler with **sampling period** T.

- The hold device converts the digital signal into a continuous-time signal. The simplest type is the **zero-order hold (ZOH)**, which keeps the signal value constant during each sampling interval, that is:

$$\{u(k)\} \to u(t) = u(k), \quad kT \leqslant t < (k+1)T, \quad k = 0, 1, 2, \ldots$$

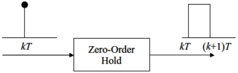


Figure 4: Zero-order hold.

- It is usually assumed that the hold device (DAC):
  - Produces outputs with exactly the same magnitude as the inputs. **Unity gain**.
  - Generates the analog output instantaneously. **No delay**.
  - Maintains the output constant throughout the entire sampling period in the case of a ZOH.
    Other hold devices include the *first-order hold*, which reconstructs the signal using straight-line segments, and the *second-order hold*, which reconstructs the signal using parabolic segments.

Hold Device: Time Response

- Time responses of zero-order and first-order hold devices.

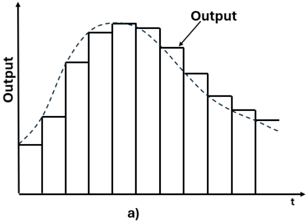


Figure 5: Hold devices: a) Zero-order hold. b) First-order hold.

Hold Device

Transfer Function

- As can be seen from the previous figure, the hold device is basically the difference between two unit-step functions shifted by one sampling period.
- Taking the Laplace transform, we obtain the transfer function of the zero-order hold:

$$G_{ZOH} = \mathcal{L}\{u(t)\} - \mathcal{L}\{u(t-T)\} = \frac{1}{s} - \frac{e^{-sT}}{s} = \frac{1-e^{-sT}}{s}$$

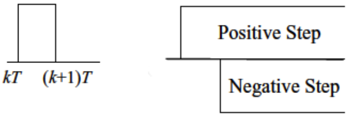


Figure 6: Hold device represented as the difference of two delayed unit-step functions.

- A discrete-time system consists of several interconnected subsystems, including one or more samplers. The location of these samplers has a significant influence on the overall transfer function.
- For two systems connected in series, $H_1(s)$ and $H_2(s)$, the Laplace transform of the output is given by

$$Y(s) = H_2(s)H_1(s)U(s)$$

- Taking the inverse Laplace transform, we obtain the time response:

$$\begin{aligned} y(t) &= \int_0^t h_2(t-\tau)\left[\int_0^\tau h_1(\tau-\lambda)u(\lambda)\,d\lambda\right]d\tau \\ &= \int_0^t u(t-\tau)\left[\int_0^\tau h_1(\tau-\lambda)h_2(\lambda)\,d\lambda\right]d\tau \\ &= \int_0^t u(t-\tau)\,h_{eq}(\tau)\,d\tau \qquad (4) \end{aligned}$$

Effect of the Sampler on the Transfer Function

Cont.

- If the output of the cascaded blocks is sampled:

$$y(kT) = \int_0^{kT} u(kT - \tau)\, h_{eq}(\tau)\, d\tau, \quad k = 1, 2, \ldots$$

- The output is the convolution of the input and the two impulse responses; therefore, **it is not possible to separate the effects of the signals that generate it**.

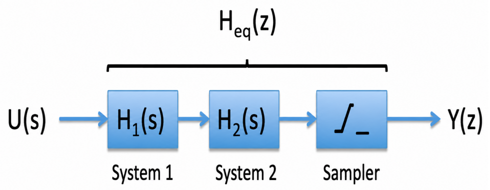


Figure 7: Transfer function–sampler configuration.

- If one of the two blocks is the sampler, the system output is the convolution of a sampled signal $u^*(t)$ with the impulse response of the continuous-time system $H(s)$.

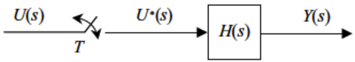


Figure 8: Sampler–transfer function configuration.

Effect of the Sampler on the Transfer Function

Cont.

- The output is a sum of shifted impulse responses of the system $h(t)$, located at the positions of the impulse train. Therefore, the resulting time-domain function is:

$$\begin{aligned} y(t) &= \int_0^t h(t-\tau)u(\tau)\,d\tau \\ &= \int_0^t h(t-\tau)\left[\sum_{k=0}^{\infty} u(kT)\delta(\tau-kT)\right] d\tau \\ &= \sum_{k=0}^{\infty} u(kT)\int_0^t h(t-\tau)\delta(\tau-kT)\,d\tau \\ &= \sum_{k=0}^{\infty} u(kT)h(t-kT) \end{aligned}$$

Effect of the Sampler on the Transfer Function

Cont.

- Sampling the output gives the convolution sum:

$$y(kT) = \sum_{i=0}^{\infty} u(iT)h(kT - iT), \quad k = 0, 1, 2, 3, \ldots$$

that is,

$$Y(z) = H(z)U(z)$$

or, in the transformed impulse-train domain,

$$Y^*(s) = H^*(s)U^*(s)$$

- For cascaded systems sampled only at the output, **their individual components cannot be separated after sampling**:

$$Y(z) = H(z)U(z) = (H_1 H_2 \dots H_n)(z)\, U(z)$$

- However, if the cascaded systems are separated by samplers, each block has both sampled inputs and outputs, and therefore its own Z-transfer function:

$$Y(z) = H(z)U(z) = H_1(z)H_2(z) \dots H_n(z)\, U(z)$$

- Examples:

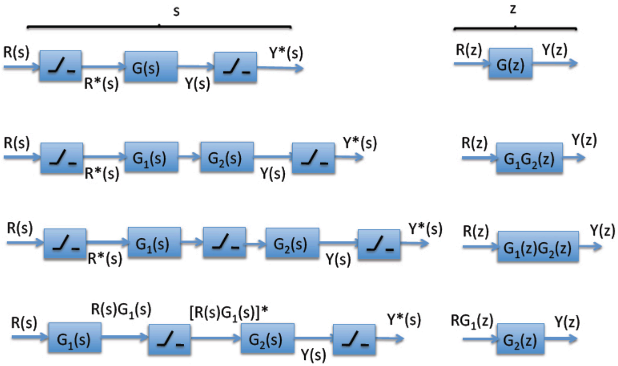


Figure 9: Examples of configurations.

- Find the equivalent sampled impulse response sequence and the equivalent Z-transfer function for a cascade connection of two analog systems with sampled input, $H_1(s) = \frac{1}{s+2}$ and $H_2(s) = \frac{2}{s+4}$, for the case where they are connected directly and for the case where they are separated by a sampler.
- **Solution:**
  - In the absence of samplers between the systems, the overall transfer function is

$$H(s) = \frac{2}{(s+2)(s+4)} = \frac{1}{s+2} - \frac{1}{s+4}$$

and the cascaded impulse response is

$$h(t) = e^{-2t} - e^{-4t}$$

- The sampled impulse response is therefore

$$h(kT) = e^{-2kT} - e^{-4kT}, \quad k = 0, 1, 2, \dots$$

- The corresponding Z-transfer function is

$$H(z) = \frac{z}{z - e^{-2T}} - \frac{z}{z - e^{-4T}} = \frac{(e^{-2T} - e^{-4T})z}{(z - e^{-2T})(z - e^{-4T})}$$

- If the systems are separated by a sampler, then each system has its own Z-transfer function:

$$H_1(z) = \frac{z}{z - e^{-2T}} \qquad H_2(z) = \frac{2z}{z - e^{-4T}}$$

- The overall Z-transfer function is

$$H(z) = H_1(z)H_2(z) = \frac{2z^2}{(z - e^{-2T})(z - e^{-4T})}$$

- Expanding into partial fractions:

$$H(z) = \frac{2}{e^{-2T} - e^{-4T}} \left[\frac{e^{-2T}z}{z - e^{-2T}} - \frac{e^{-4T}z}{z - e^{-4T}} \right]$$

- Taking the inverse transform, we obtain

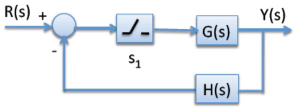
$$h(kT) = \frac{2}{e^{-2T} - e^{-4T}} \left[e^{-2T} e^{-2kT} - e^{-4T} e^{-4kT} \right]$$

$$h(kT) = \frac{2}{e^{-2T} - e^{-4T}} \left[e^{-2(k+1)T} - e^{-4(k+1)T} \right], \quad k = 0, 1, 2, \ldots$$

Feedback System with Sampler

Discrete Transfer Function

- We want to find the discrete transfer function of the feedback system with sampler shown in the figure:



- We introduce fictitious samplers at the output of any subsystem that has a sampled input.
  - This is justified because the output of a sampled system exists only at discrete time instants, and the signal is not the input to any other block.
- We introduce fictitious samplers at the inputs of a summing junction whose output is sampled.
  - This is justified because the output of a sampled summing junction is equivalent to the sum of the sampled inputs, assuming that all samplers are synchronized.

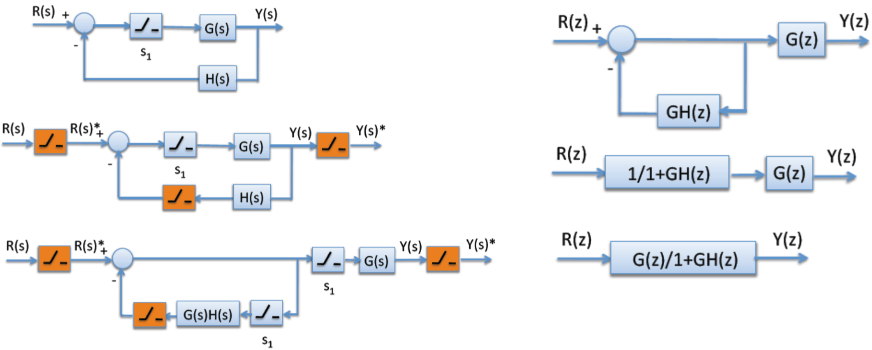


Figure 10: Equivalent transformations of a feedback system with a sampler.

- In control systems, the hold device, the system, and the sampler often appear in cascade. Since both the input and the output are sampled, it is possible to obtain the overall Z-transfer function from the Z-transfer functions of the subsystems.

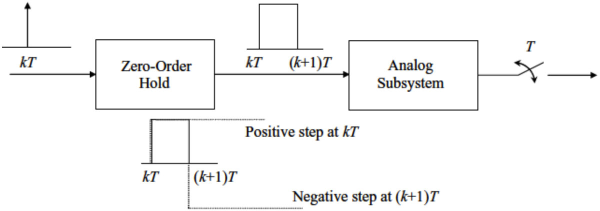


Figure 11: Combination of hold device, system, and sampler.

Combination of Hold Device, System, and Sampler

Transfer Function (cont.)

- In the Laplace domain, the overall transfer function is:

$$G_{HS}(s) = G_H(s)G_S(s) = (1 - e^{-sT})\frac{G_S(s)}{s}$$

  where $G_H(s)$ is the hold device and $G_S(s)$ is the system.

- In the time domain:

$$g_{HS}(t) = g_H(t) * g_S(t) = g_{S/s}(t) - g_{S/s}(t - T)$$

  with

$$g_{S/s}(t) = \mathcal{L}^{-1}\left\{\frac{G_S(s)}{s}\right\}$$

- Sampling this expression gives:

$$g_{HS}(kT) = g_{S/s}(kT) - g_{S/s}(kT - T)$$

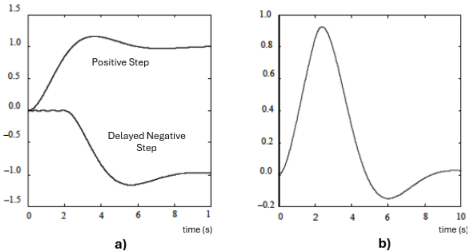


Figure 12: a) Response of an analog system to step inputs. b) Response of an analog system to a unit-pulse input.

Combination of Hold Device, System, and Sampler

Equivalent Transfer Function (cont.)

- Taking the Z transform of the previous expression, we obtain

$$\begin{aligned} G_{HSS}(z) &= (1 - z^{-1})\mathcal{Z}\{g^{*}_{S/s}(t)\} \\ &= (1 - z^{-1})\mathcal{Z}\{\mathcal{L}^{-1}[\frac{G_S(s)}{s}]^{*}\} \end{aligned} \tag{5}$$

- Written in a more compact notation:

$$G_{HSS}(z) = (1 - z^{-1})\mathcal{Z}\{\frac{G_S}{s}\} \tag{6}$$

Example 1

- Calculate the Z-transfer function of a system composed of a sampler, a hold device, and the analog system shown in the following figure:

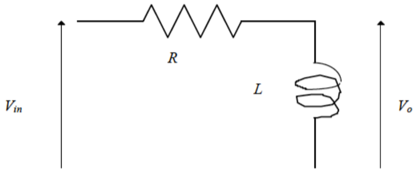


Figure 13: RL circuit.

- Relationship between the input and output voltages:

$$\frac{V_o}{V_{in}} = \frac{Ls}{Ls+R} = \frac{(L/R)s}{(L/R)s+1}$$

$$\frac{V_o}{V_{in}} = \frac{\tau s}{\tau s+1}, \quad \tau = \frac{L}{R}$$

- Using the expression for $G_{HSS}(z)$:

$$\begin{aligned} G_{HSS}(z) &= (1-z^{-1})\mathcal{Z}\{\frac{G_S}{s}\} \\ &= (1-z^{-1})\mathcal{Z}\{\frac{1}{s+1/\tau}\} \\ &= \frac{z-1}{z}\left\{\frac{z}{z-e^{-T/\tau}}\right\} \\ &= \frac{z-1}{z-e^{-T/\tau}} \end{aligned}$$

Example 2

- Calculate the Z-transfer function of a system composed of a sampler, a hold device, and the analog system formed by an amplifier and a DC motor, whose s-domain transfer function is:

$$G(s) = \frac{\theta(s)}{V_{in}(s)} = \frac{K}{s(\tau_m s + 1)}$$

- Given:

$$\frac{G(s)}{s} = \frac{K/\tau_m}{s^2(s + 1/\tau_m)}$$

- Expanding into partial fractions:

$$\frac{G(s)}{s} = K\left[\frac{A_{11}}{s^2} + \frac{A_{12}}{s} + \frac{A_2}{s + 1/\tau_m}\right]$$

$$A_{11} = \left.\frac{1/\tau_m}{s + 1/\tau_m}\right]_{s=0} = 1, A_{12} = \left.\frac{d}{ds}\left(\frac{1/\tau_m}{s + 1/\tau_m}\right)\right]_{s=0} = -1$$

$$A_2 = \left.\frac{1/\tau_m}{s^2}\right]_{s=-1/\tau_m} = \tau_m$$

Example 2
Solution

- Using the expression of the H-S-S set, $G_{HSS}(z)$:

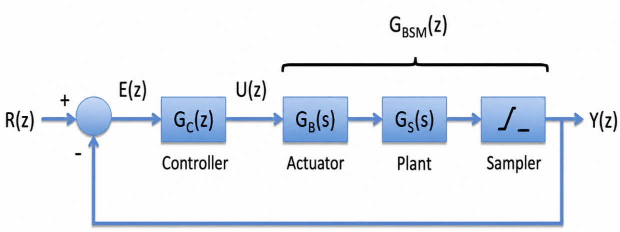
$$
\begin{aligned}
G_{HSS}(z) &= (1-z^{-1})\mathcal{Z}\{\frac{G_S}{s}\} \\
&= (1-z^{-1})\mathcal{Z}\{K[\frac{1}{s^2}-\frac{\tau_m}{s}+\frac{\tau_m}{s+1/\tau_m}]\} \\
&= K\tau_m[\frac{T/\tau_m}{z-1}-1+\frac{z-1}{z-e^{-T/\tau_m}}] \\
&= K\tau_m\left[\frac{(T/\tau_m+e^{-T/\tau_m}-1)z+[1-e^{-T/\tau_m}(T/\tau_m+1)]}{(z-1)(z-e^{-T/\tau_m})}\right]
\end{aligned}
$$

Closed-Loop Transfer Function

System with Discrete-Time Control

- The closed-loop transfer function of a system that includes a discrete-time controller $G_C(z)$ and the discrete equivalent $G_{HSS}(z)$ of a system that includes a hold device, a continuous-time system, and a sampler is given by

$$G_{cl}(z) = \frac{G_C(z)G_{HSS}(z)}{1 + G_C(z)G_{HSS}(z)} \quad (7)$$



- whose closed-loop characteristic equation is given by

$$1 + G_C(z)G_{HSS}(z) = 0 \quad (8)$$

- The Z-transfer function of a system with a hold device (ADC), sampler (DAC), and an analog system is given by:

```
>> G = tf(num, den)
>> Gd = c2d(G,T, 'method')
```

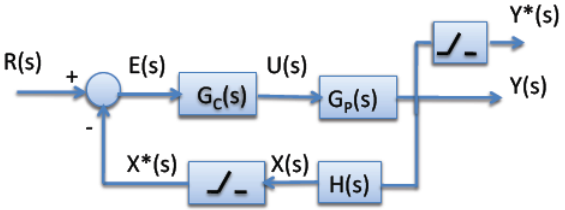
  where method is the method used to obtain the discrete transfer function, and T is the sampling period.

- Example: $G(s) = (2s^2 + 4s + 3)/(s^3 + 4s^2 + 6s + 8)$

```
>> G = tf([2 4 3], [1 4 6 8])
>> Gd = c2d(G, 0.1, 'zoh') % zero-order hold
>> Gd = c2d(G, 0.1, 'foh') % first-order hold
```

Example

- Calculate the Laplace transfer functions of the continuous and sampled systems for the system:



- The signal $x(t)$ has the Laplace transform:

$$X(s) = H(s)G_P(s)G_C(s)E(s) \tag{9}$$

This transform involves three multiplications in the s domain, which correspond to three convolutions in the time domain.

- From the block diagram, we can see that $E(s) = R(s) - X^*(s)$. Substituting this into the expression for $X(s)$ gives:

$$X(s) = H(s)G_P(s)G_C(s)\,[R(s) - X^*(s)] \tag{10}$$

- Therefore, the sampled variable has the Laplace transform:

$$X^*(s) = HG_PG_CR^*(s) - HG_PG_C^*(s)X^*(s) \tag{11}$$

- Solving for $X^*(s)$:

$$X^*(s) = \frac{HG_PG_CR^*(s)}{1 + HG_PG_C^*(s)} \tag{12}$$

- Then $E(s)$ is

$$E(s) = R(s) - \frac{HG_P G_C R^*(s)}{1 + HG_P G_C^*(s)} \tag{13}$$

- Operating:

$$Y(s) = G_C(s)G_P(s)\left[R(s) - \frac{HG_P G_C R^*(s)}{1 + HG_P G_C^*(s)}\right] \tag{14}$$

- Thus, the sampled output becomes

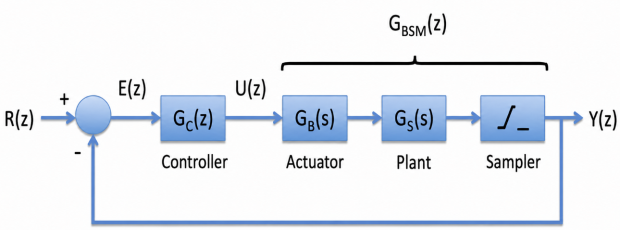
$$Y^*(s) = G_C G_P R^*(s) - G_C G_P^*(s)\frac{HG_P G_C R^*(s)}{1 + HG_P G_C^*(s)} \tag{15}$$

- Applying $z = e^{sT}$, we rewrite it in the z domain:

$$Y(z) = G_C G_P R(z) - G_C G_P(z)\frac{HG_P G_C R(z)}{1 + HG_P G_C(z)} \tag{16}$$

Steady-State Error of a System

Closed Loop



- The error $E(z)$ is given by

$$E(z) = \frac{R(z)}{1 + G_C(z)G_{BSM}(z)} \tag{17}$$

- Applying the final value theorem:

$$\begin{aligned} e(\infty) &= \lim_{z\to 1}(1 - z^{-1})E(z) \\ &= \lim_{z\to 1} \frac{(z-1)R(z)}{z(1 + G_C(z)G_{BSM}(z))} \end{aligned} \tag{18}$$

- The type of a system is the number n of unit poles, that is, poles at $(z-1)$, of the system Z-transfer function.

$$F(z) = \frac{N(z)}{(z-1)^n D(z)}, \quad n \geqslant 0$$

- The transfer function of a unit-step input is given by

$$R(z) = \frac{z}{z-1} \tag{19}$$

- Substituting it into the error expression, we obtain

$$\begin{aligned} e(\infty) &= \lim_{z\to 1} \frac{1}{(1+G_C(z)G_{BSM}(z))} \\ &= \frac{1}{1+K_p} \end{aligned} \tag{20}$$

- where $K_p = \lim_{z\to 1} G_C(z)G_{BSM}(z)$ is the so-called position error constant, which has a finite value for type-0 systems and is infinite for type-1 or higher systems.

- The transfer function of a sampled ramp input is given by

$$R(z) = \frac{Tz}{(z-1)^2} \tag{21}$$

- Substituting it into the error expression, we obtain

$$\begin{aligned} e(\infty) &= \lim_{z\to 1} \frac{T}{(z-1)(1+G_C(z)G_{BSM}(z))} \\ &= \frac{1}{K_v} \end{aligned} \tag{22}$$

- where $K_v = \lim_{z\to 1} \frac{1}{T}(z-1)G_C(z)G_{BSM}(z)$ is the so-called velocity error constant. It is zero for type-0 systems, finite for type-1 systems, and infinite for type-2 or higher systems.

Example

- Obtain the position error for a unity-feedback system whose equivalent plant transfer function, including hold device and sampler, is

$$G_{BSM}(z) = \frac{K(z+a)}{(z-1)(z-b)}$$

and whose controller is

$$G_C(z) = \frac{K_c(z-b)}{(z-c)}$$

with $0 < a, b, c < 1$.

- **Solution:**

$$G_{ol}(z) = G_C(z)G_{BSM}(z) = \frac{KK_c(z+a)}{(z-1)(z-c)}$$

This is a type-1 system, so it will have zero steady-state error for a step input. For a ramp input:

$$\begin{aligned} e(\infty) &= \lim_{z\to 1} \frac{T}{(z-1)(1+G_C(z)G_{BSM}(z))} \\ &= \frac{T(1-c)}{KK_c(1+a)} \end{aligned} \tag{23}$$

It can be observed that the steady-state error for a ramp input decreases as the controller gain K_c increases, and depends on the positions of the controller pole and zero.

Sampling Theorem

- **Theorem:**

  A band-limited signal $f(t)$, whose Fourier transform $F(j\omega)$ is nonzero in the interval $-\omega_m \leqslant \omega \leqslant \omega_m$ and zero elsewhere, can be reconstructed from the discrete-time waveform

$$f^*(t) = \sum_{k=-\infty}^{+\infty} f(t)\delta(t - kT)$$

  if and only if the sampling frequency $\omega_s = 2\pi/T$ satisfies the condition:

$$\omega_s > 2\omega_m$$

- Consider the sequence of unit impulses

$$\delta_T = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

and its Fourier transform

$$\delta_T = \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s)$$

- Impulse sampling is obtained by multiplying the waveform $f(t)$ by δ_T. By the frequency-domain convolution theorem, the spectrum of the product of the two signals is given by the convolution of their respective spectra:

$$
\begin{aligned}
\mathcal{F}\{\delta_T(t) \times f(t)\} &= \frac{1}{2\pi}\delta_T(j\omega) * F(j\omega) \\
&= \left[\frac{1}{T}\sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s)\right] * F(j\omega) \\
&= \frac{1}{T}\sum_{n=-\infty}^{\infty} F(\omega - n\omega_s)
\end{aligned}
$$

Sampling Theorem

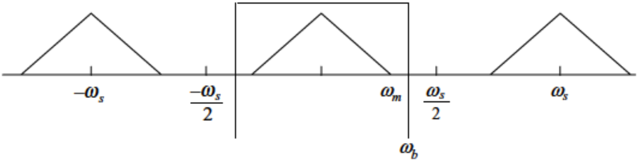


Figure 14: Spectrum of the sampled waveform.

- The spectrum of the continuous-time waveform can be recovered using an ideal low-pass filter with bandwidth ω_b in the range

$$\omega_m < \omega_b < \omega_s/2.$$

Choosing the Sampling Frequency

- Finite-bandwidth signals are an idealization associated with signals of infinite duration, since an infinitely long signal actually has infinite bandwidth.
- From a practical point of view, signals have what may be called an effective bandwidth, beyond which the effect of the spectral components becomes negligible.
- This allows us to treat signals as if they were band-limited and to choose a sampling frequency higher than the minimum specified by the sampling theorem.
- As a practical rule, the sampling frequency ω_s can be chosen as

$$\omega_s = k\omega_m, \qquad 5 \leq k \leq 10$$

Choosing the Sampling Frequency

- The choice of k depends on the application and on the cost of sampling at a given frequency: the higher the sampling frequency, the higher the cost (memory requirements).
- A closed-loop control system cannot have a sampling period shorter than the time required to measure the output. Therefore, the sampling frequency is upper-bounded by the **sensor delay**.