Control Engineering II

Topic 2:
Obtaining the Transfer Function of a System

C.A. Monje, D. Copaci, M. Malfaz, J. Muñoz,
L. Mena, L. Moreno, S. Garrido

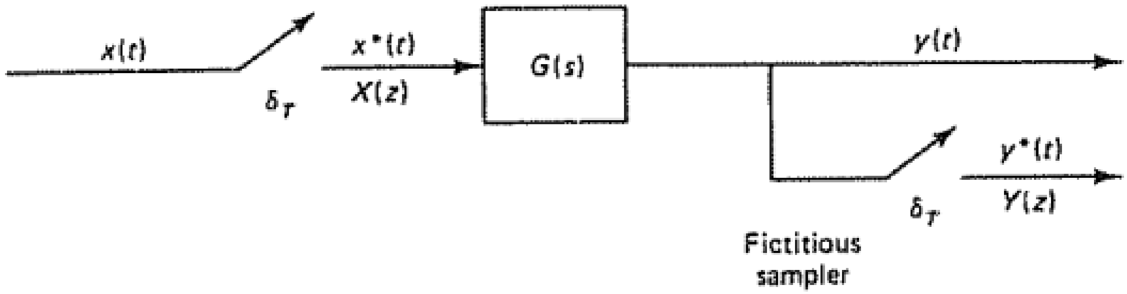
2026

uc3m | Universidad Carlos III de Madrid
OpenCourseWare

Problem 1 (1/2)

Obtain the equivalent z transfer function of the system shown in the figure, where:

$$G(s) = \frac{1-e^{-Ts}}{s}\frac{1}{s(s+1)}$$



Solution 1:

The transfer function can be rewritten as follows:

$$G(s) = (1-e^{-Ts})\left(\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}\right)$$

Taking the inverse transform we obtain the following function:

$$g(t) = (t-1+e^{-t})u(t) - [t-T-1+e^{-(t-T)}]u(t-T)$$

If we sample the signal:

$$g(kT) = (kT-1+e^{-kT})u(kT) - [kT-T-1+e^{-(kT-T)}]u(kT-T)$$

$$g(kT) = \begin{cases} 0, & k=0 \\ e^{-kT}+T-e^{-(kT-T)}, & k=1,2,3,\ldots \end{cases}$$

Finally, the z transform is calculated:

$$G(z) = \sum_{k=0}^{\infty} g(kT)z^{-k} =? \sum_{k=0}^{\infty}[e^{-kT}+T-e^{-(kT-T)}]z^{-k} - 1 - T + e^{T} =?$$

$$= (1-e^{T})\sum_{k=0}^{\infty} e^{-kT}z^{-k} + T\sum_{k=0}^{\infty} z^{-k} - 1 - T + e^{T} =?$$

$$= (1-e^{T})\frac{1}{1-e^{-T}z^{-1}} + \frac{T}{1-z^{-1}} - 1 - T + e^{T}$$

$$G(z) = \frac{(T-1+e^{-T})z^{-1} + (1-e^{-T}-Te^{-T})z^{-2}}{(1-z^{-1})(1-e^{-T}z^{-1})}$$

Solution 2:

We see that in the transfer function there is a term that is the transfer function of a zero-order hold:

$$G(s) = \frac{1-e^{-Ts}}{s}X(s)$$

In these cases, the z transform can be calculated using the following property:

$$G(z) = Z[G(s)] = (1-z^{-1})Z\left[\frac{X(s)}{s}\right] = (1-z^{-1})Z\left[\frac{1}{s^2(s+1)}\right] =$$

$$= (1-z^{-1})Z\left[\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}\right]$$

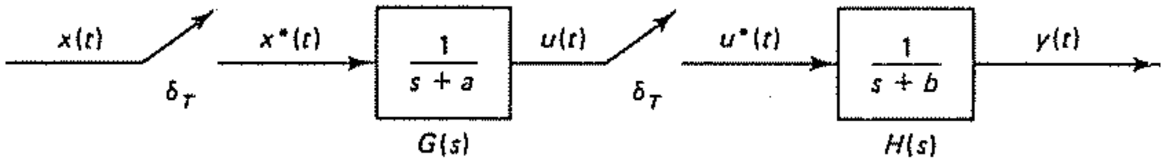
Using the tables that relate Laplace transforms of known functions with their z transforms, we obtain the solution:

$$G(z) = (1-z^{-1})\left[\frac{Tz^{-1}}{(1-z^{-1})^2} - \frac{1}{1-z^{-1}} + \frac{1}{1-e^{-T}z^{-1}}\right]$$

$$G(z) = \frac{(T-1+e^{-T})z^{-1} + (1-e^{-T}-Te^{-T})z^{-2}}{(1-z^{-1})(1-e^{-T}z^{-1})}$$

Problem 2

Obtain the equivalent transfer function of the following systems:
Case a: The transfer functions are separated by a sampler. We assume that the two samplers are synchronized and have the same sampling period.



In this case the equivalent transfer function is:

$$\frac{Y(z)}{X(z)} = \frac{Y(z)}{U(z)}\frac{U(z)}{X(z)} = H(z)G(z)$$

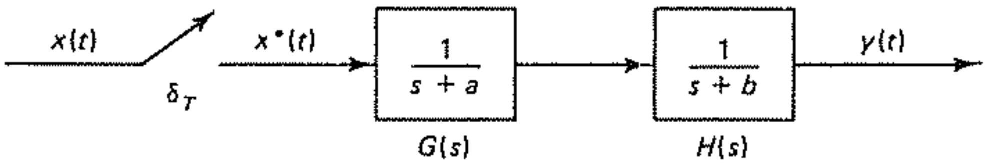
Using the tables that relate Laplace transforms to z transforms we obtain the result:

$$\frac{Y(z)}{X(z)} = H(z)G(z) = G(z)H(z) = Z\left[\frac{1}{s+a}\right] Z\left[\frac{1}{s+b}\right] =$$

$$= \frac{1}{1-e^{-aT}z^{-1}}\frac{1}{1-e^{-bT}z^{-1}}$$

$$\frac{Y(z)}{X(z)} = \frac{1}{1-e^{-aT}z^{-1}}\frac{1}{1-e^{-bT}z^{-1}}$$

Case b: There is only one sampler at the beginning.



In this case it is obtained as follows:

$$\frac{Y(z)}{X(z)} = Z[G(s)H(s)] = Z\left[\frac{1}{s+a}\frac{1}{s+b}\right] = Z\left[\frac{1}{b-a}\left(\frac{1}{s+a} - \frac{1}{s+b}\right)\right]$$

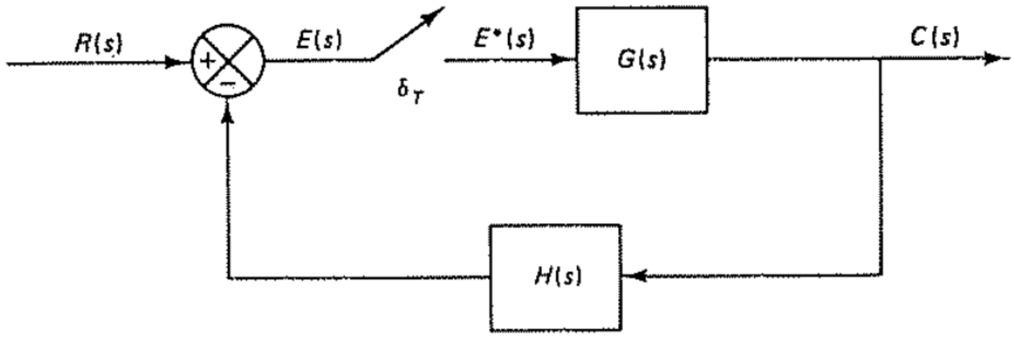
$$\frac{Y(z)}{X(z)} = \frac{1}{b-a}\left(\frac{1}{1-e^{-aT}z^{-1}} - \frac{1}{1-e^{-bT}z^{-1}}\right)$$

$$\frac{Y(z)}{X(z)} = GH(z) = \frac{1}{b-a}\left[\frac{(e^{-aT}-e^{-bT})z^{-1}}{(1-e^{-aT}z^{-1})(1-e^{-bT}z^{-1})}\right]$$

It can be observed that the equivalent transfer functions of the two systems are different.

Problem 3

Obtain the equivalent transfer function of the following system:



Solution:

In a closed-loop system the presence or absence of samplers determines its behavior. In this case we obtain the following equations from the block diagram:

$$E(s) = R(s) - H(s)C(s)$$
$$C(s) = G(s)E^*(s)$$

Substituting:

$$E(s) = R(s) - H(s)G(s)E^*(s)$$

Taking the sampled variables:

$$E^*(s) = R^*(s) - GH^*(s)E^*(s)$$
$$E^*(s) = \frac{R^*(s)}{1 + GH^*(s)}$$

As: $C^*(s) = G^*(s)E^*(s)$

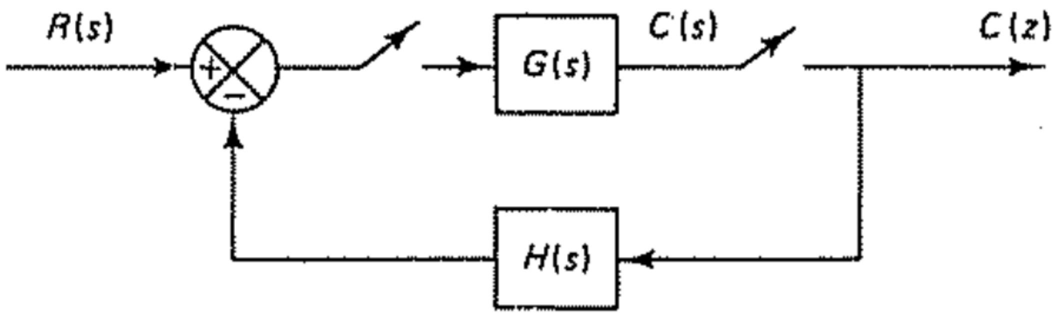
$$C^*(s) = \frac{G^*(s)R^*(s)}{1 + GH^*(s)}$$
$$C(z) = \frac{G(z)R(z)}{1 + GH(z)}$$

The solution will be:

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1 + GH(z)}$$

Problem 4

Obtain the equivalent transfer function of the following system:



Solution:

In this case we obtain the following equations from the block diagram:

$$E(s) = R(s) - H(s)C^*(s)$$
$$C(s) = G(s)E^*(s)$$
$$C^*(s) = G^*(s)E^*(s)$$

Substituting:

$$E(s) = R(s) - H(s)G^*(s)E^*(s)$$

Taking the sampled variables:

$$E^*(s) = R^*(s) - G^*(s)H^*(s)E^*(s)$$
$$E^*(s) = \frac{R^*(s)}{1 + G^*(s)H^*(s)}$$

As: $C^*(s) = G^*(s)E^*(s)$

$$C^*(s) = \frac{G^*(s)R^*(s)}{1 + G^*(s)H^*(s)}$$
$$C(z) = \frac{G(z)R(z)}{1 + G(z)H(z)}$$

The solution will be:

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1 + G(z)H(z)}$$

Problem 5

Obtain the position error for a system with unit feedback whose equivalent transfer function of the system (with hold and sampler) is:

$$G_{BSM}(z) = \frac{K(z+a)}{(z-1)^2(z-b)}$$

The transfer function of the controller is:

$$G_C(z) = \frac{K_C(z-b)}{(z-c)}$$

The value of the constants: $0 < a, b, c < 1$

Solution:

$$G_{BSM}(z)G_C(z) = \frac{KK_C(z+a)}{(z-1)^2(z-c)}$$

The system is type 2 so it will have zero error before a step input in steady state and also for a ramp input. We check it:

Step input:

$$K_p = \lim_{z\to 1} G_C(z)G_{BSM}(z) = \frac{\lim\limits_{z\to 1} KK_C(z+a)}{(z-1)^2(z-c)} = \infty$$

$$e(\infty) = \frac{\lim\limits_{z\to 1} 1}{(1+G_{BSM}(z)G_C(z))} = \frac{1}{1+K_p} = 0$$

Ramp input:

$$K_v = \frac{\lim\limits_{z\to 1} 1}{T}(z-1)G_C(z)G_{BSM}(z) = \frac{\lim\limits_{z\to 1}(z-1)KK_C(z+a)}{T(z-1)^2(z-c)} = \infty$$

$$e(\infty) = \frac{\lim\limits_{z\to 1} T}{(z-1)(1+G_{BSM}(z)G_C(z))} = \frac{1}{K_v} = 0$$

Problem 6

Obtain the position error for a system with unit feedback whose equivalent transfer function of the system (with hold and sampler) is:

$$G_{BSM}(z) = \frac{K(z+a)}{(z-b)}$$

The transfer function of the controller is:

$$G_C(z) = \frac{K_C(z-b)}{(z+a)}$$

The value of the constants: $0 < a, b < 1$

Solution:

$$G_{BSM}(z)G_C(z) = KK_C$$

The system is type 0 so it will have a constant non-zero error before a step input in steady state and infinite for a ramp input. We check it:

Step input:

$$K_p = \lim_{z\to 1} G_C(z)G_{BSM}(z) = \lim_{z\to 1} KK_C = KK_C$$

$$e(\infty) = \frac{\lim_{z\to 1} 1}{(1 + G_{BSM}(z)G_C(z))} = \frac{1}{1+K_p} = \frac{1}{1+KK_C}$$

Ramp input:

$$K_v = \frac{\lim_{z\to 1} 1}{T}(z-1)G_C(z)G_{BSM}(z) = \frac{\lim_{z\to 1}(z-1)KK_C}{T} = 0$$

$$e(\infty) = \frac{\lim_{z\to 1} T}{(z-1)(1+G_{BSM}(z)G_C(z))} = \frac{1}{K_v} = \infty$$

Problem 7

Study the behavior of the following functions when sampled with a sampling period $T_S = 1s$:

$$u_1(t) = \cos(2\pi t/8)$$
$$u_2(t) = \cos(2\pi 7t/8)$$

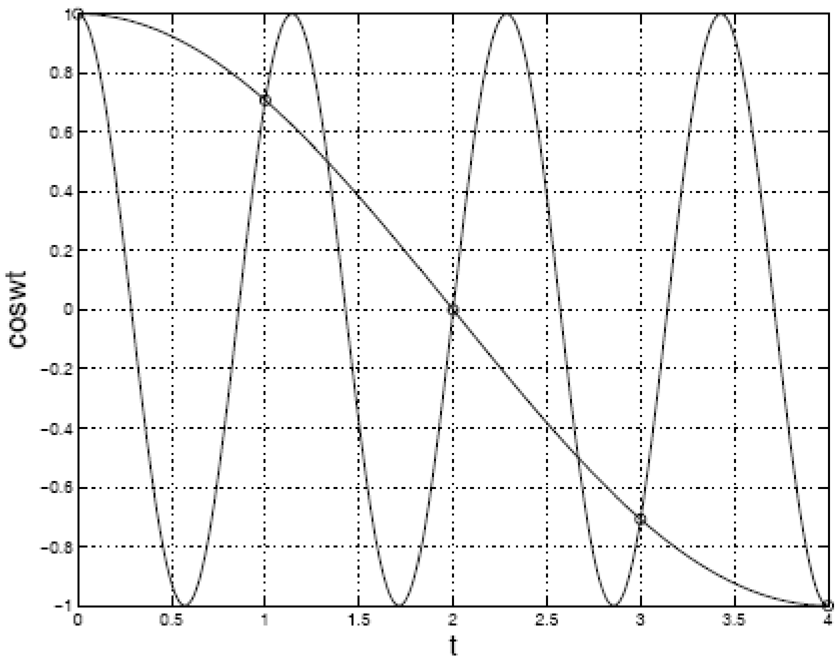
Calculate the sampling frequency for both signals according to the sampling theorem ($w_s > 2w_m$).

Solution:

If we discretize both signals with the sampling period indicated, we obtain:

$$\begin{aligned} u_1(k) &= \cos(2\pi k/8), & k = 0, 1, 2, \ldots \\ u_2(k) = \cos(2\pi 7k/8) = \cos(2\pi k - 2\pi k/8) &= \cos(2\pi k/8), & k = 0, 1, 2, \ldots \end{aligned}$$

It is observed that sampling at that frequency the sampling of both signals produces the same result, an effect known as "*aliasing*" (one signal is the alias of the other). This effect can be seen in the following figure:



Now we calculate the sampling frequencies according to the theorem:

$$u_1(t), \quad w_m = 2\pi/8, \quad w_s > 2w_m = \pi/2, \quad T_s < 4$$
$$u_2(t), \quad w_m = 2\pi 7/8, \quad w_s > 2w_m = \pi 7/2, \quad T_s < 4/7$$

It is observed that the sampling period must be smaller (or the sampling frequency higher) for the second signal.