uc3m | Universidad Carlos III de Madrid
OpenCourseWare

Control Engineering II

Topic 4:
Discretization of Continuous-Time Systems

C.A. Monje, D. Copaci, M. Malfaz, J. Muñoz,
L. Mena, L. Moreno, S. Garrido

2026

Contents

Discretization of a Continuous-Time System

- A sampled signal can be expressed as:

$$\begin{aligned} x^*(t) &= \sum_{k=0}^{\infty} x(t)\delta(t-kT) \\ &= x(t)\sum_{k=0}^{\infty} \delta(t-kT) \qquad (1) \end{aligned}$$

- The Laplace transform of an impulse is unity. A delay of a in a time-domain signal corresponds, in the Laplace domain, to multiplication by e^{-as}. If we take the Laplace transform of the summation in Equation 1, we obtain

$$\begin{aligned} \mathcal{L}\left[\sum_{k=0}^{\infty} \delta(t-kT)\right] &= \mathcal{L}[\delta(t)] + \mathcal{L}[\delta(t-T)] + \mathcal{L}[\delta(t-2T)] + \ldots \\ &= 1 + e^{-Ts} + e^{-2Ts} + \ldots \\ &= \frac{1}{1-e^{-Ts}} \qquad (2) \end{aligned}$$

- Taking the Laplace transform of the sampled signal $x^*(t)$ yields

$$X^*(s) = \mathcal{L}[x^*(t)] = \mathcal{L}\left[x(t)\sum_{k=0}^{\infty}\delta(t-kT)\right] \tag{3}$$

- We observe that $X^*(s)$ is the Laplace transform of the product of two functions in the time domain. Recall that the Laplace transform of the product of two functions is given by

$$\mathcal{L}[f(t)g(t)] = \int_0^{\infty} f(t)g(t)e^{-st}\mathrm{d}t = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(p)G(s-p)\mathrm{d}p \tag{4}$$

Discretization of a Continuous-Time System

- Applying this expression to Equation 3, and taking into account that the Laplace transform of $x(t)$ is $X(s)$ and that the Laplace transform of the Dirac impulse train is given by Equation 2, we obtain the following convolution integral:

$$X^*(s) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} X(p) \frac{1}{1 - e^{-T(s-p)}} \mathrm{d}p \tag{5}$$

- The evaluation of this convolution integral along the contour from $c - j\infty$ to $c + j\infty$ can be carried out using the residue theorem by forming a closed contour consisting of the line itself and a circle Γ of infinite radius in the left-half plane that encloses all poles of $X(p)$.

Discretization of a Continuous-Time System

- In this way, Equation 5 can be expressed as

$$\begin{aligned} X^*(s) &= \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X(p)\frac{1}{1-e^{-T(s-p)}}\mathrm{d}p \\ &= \frac{1}{2\pi j}\oint X(p)\frac{1}{1-e^{-T(s-p)}}\mathrm{d}p \\ &\quad -\frac{1}{2\pi j}\int_{\Gamma} X(p)\frac{1}{1-e^{-T(s-p)}}\mathrm{d}p \end{aligned} \tag{6}$$

- If $X(s)$ is expressed as a rational function

$$X(s) = \frac{N(s)}{D(s)},$$

where the degree of the denominator is greater than that of the numerator, the second term in Equation 6 becomes zero. Therefore,

$$X^*(s) = \frac{1}{2\pi j}\oint X(p)\frac{1}{1-e^{-T(s-p)}}\mathrm{d}p \tag{7}$$

- This integral is solved by applying Cauchy's integral formula and is equal to the sum of the residues R inside the closed contour corresponding to the poles of $X(p)$, that is,

$$X^*(s) = \sum R\left[X(p)\frac{1}{1-e^{-T(s-p)}}\right]_{p=p_i} \tag{8}$$

- The residue expressions are the following:
  - Simple poles ($p = p_i$).
    In this case, the residue at the pole is

$$R_{p_i} = \lim_{p \to p_i}\left[(p-p_i)\frac{X(p)}{1-e^{-T(s-p)}}\right] \tag{9}$$

  - Multiple poles ($p = p_j$, with multiplicity order n). In this case, the residue at the pole of multiplicity n is

$$R_{p_j} = \frac{1}{(n-1)!}\lim_{p \to p_j}\frac{\mathrm{d}^{\,n-1}}{\mathrm{d}p^{\,n-1}}\left[(p-p_j)^n\frac{X(p)}{1-e^{-T(s-p)}}\right] \tag{10}$$

- Consider a system with transfer function

$$G(s) = \frac{2}{s+2}.$$

In this case, there is a single residue:

$$\begin{aligned} R_{p_i=-2} &= \lim_{p \to -2} \left[(p+2) \frac{\frac{2}{(p+2)}}{1 - e^{-T(s-p)}} \right] \\ &= \lim_{p \to -2} \frac{2}{1 - e^{-T(s-p)}} = \frac{2}{1 - e^{-Ts} e^{-2T}} \end{aligned}$$

Discretization of a Continuous-Time System

Example (Cont.)

- Therefore

$$G^*(s) = \frac{2}{1 - e^{-Ts}e^{-2T}} \tag{11}$$

and taking into account that $z = e^{Ts}$, we obtain the corresponding z-transform:

$$G(z) = \frac{2}{1 - z^{-1}e^{-2T}} \tag{12}$$

- If we assume a sampling period of 0.1 seconds, the z-transform of the sampled signal $G(s)$ becomes

$$G(z) = \frac{2}{1 - 0.8187z^{-1}} \tag{13}$$

Time-domain response curves of $G(s)$ and $G(z)$ without and with a zero-order hold.

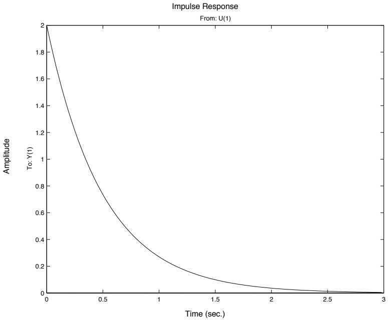


Figure 1: Response of $G(s)$.

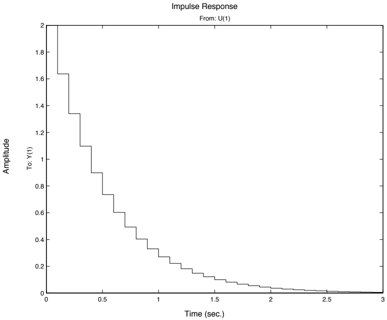


Figure 2: Response of $G(z)$.

Equivalent Discrete-Time Transfer Function

- Suppose that for a given system we know its continuous-time transfer function and we are interested in obtaining its **discrete-time equivalent**.
- This equivalent discrete-time system must have both a discrete input and a discrete output. Since the real system is continuous, a signal hold is introduced at the system input and a sampler at its output, as shown in the figure:

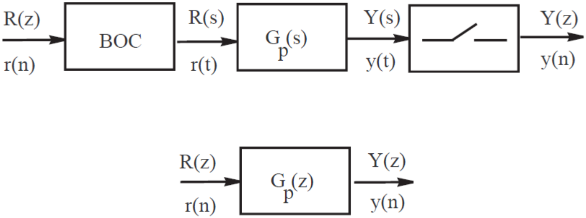


Figure 3: Equivalent discrete-time transfer function.

Equivalent Discrete-Time Transfer Function

- Let us examine the equations describing the system. First, we have previously seen that the transfer function of the zero-order hold is

$$G_B(s) = \frac{1 - e^{-Ts}}{s}$$

From this, the combined hold-process system has the following transfer function in the Laplace domain:

$$G(s) = G_B(s)G_p(s) = \frac{1 - e^{-Ts}}{s} G_p(s) \tag{14}$$

Equivalent Discrete-Time Transfer Function

- On the other hand, the input signal to the system is a sampled signal $R^*(s)$, and therefore the output of the process is

$$Y(s) = G(s)R^*(s) \tag{15}$$

If this signal $Y(s)$ is sampled in order to obtain a discrete output signal, the expression becomes

$$Y^*(s) = G^*(s)R^*(s) \tag{16}$$

which, in discrete form, can be written as

$$Y(z) = G(z)R(z) \tag{17}$$

Equivalent Discrete-Time Transfer Function

- We seek to obtain the sampled transfer function $G^*(s)$, where

$$G(s) = G_B(s)G_p(s) = \frac{1 - e^{-Ts}}{s} G_p(s)$$

Manipulating this transfer function yields

$$G(s) = \frac{G_p(s)}{s} - \frac{e^{-Ts}G_p(s)}{s}$$

that is, we have the response of $G_p(s)$ to a unit-step input,

$$\frac{G_p(s)}{s},$$

minus the same response delayed by T.

Equivalent Discrete-Time Transfer Function

- If we take the z-transform of $G(s)$, we obtain

$$\begin{aligned}
\mathcal{Z}[G(s)] &= \mathcal{Z}\left[\frac{G_p(s)}{s} - \frac{e^{-Ts}G_p(s)}{s}\right] \\
&= \mathcal{Z}\left[\frac{G_p(s)}{s}\right] - \mathcal{Z}\left[\frac{e^{-Ts}G_p(s)}{s}\right] \\
&= (1 - z^{-1})\mathcal{Z}\left[\frac{G_p(s)}{s}\right] \qquad (18)
\end{aligned}$$

and if we compute

$$\mathcal{Z}\left[\frac{G_p(s)}{s}\right]$$

using the residue method or any other suitable technique, we obtain the equivalent discrete-time transfer function of the continuous-time system.

Example 1

- Consider a system whose continuous-time transfer function is

$$G_p(s) = \frac{3}{s+1},$$

and suppose we wish to obtain its discrete-time equivalent.

- **Solution:**
  - Assuming that a zero-order hold is used, we have

$$G(z) = (1 - z^{-1})\mathcal{Z}\left[\frac{G_p(s)}{s}\right]$$

Example 1

- Let us first compute the z-transform of

$$\frac{G_p(s)}{s},$$

using the residue method. In this case, the function has two poles, and therefore there are two residues:

$$\begin{aligned} R_{p_i=-1} &= \lim_{p \to -1} \left[(p+1) \frac{\frac{3}{p(p+1)}}{1 - e^{-T(s-p)}} \right] \\ &= \lim_{p \to -1} \frac{-3}{1 - e^{-T(s-p)}} = \frac{-3}{1 - e^{-Ts} e^{-T}} \end{aligned}$$

- For the second pole:

$$\begin{aligned} R_{p_i=0} &= \lim_{p\to 0}\left[p\frac{\frac{3}{p(p+1)}}{1-e^{-T(s-p)}}\right] \\ &= \lim_{p\to 0}\frac{3}{1-e^{-T(s-p)}} = \frac{3}{1-e^{-Ts}} \end{aligned}$$

and substituting e^{Ts} by z, we obtain

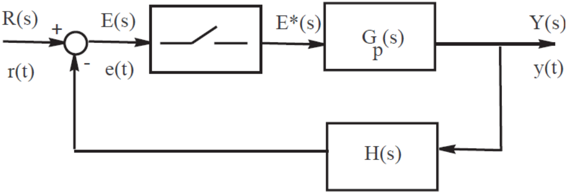
$$G(z) = (1-z^{-1})\left(\frac{3}{1-z^{-1}} - \frac{3}{1-z^{-1}e^{-T}}\right) \tag{19}$$

- If we assume a sampling period of 0.1 seconds, the z-transform of the system becomes

$$G(z) = (1-z^{-1})\left(\frac{3}{1-z^{-1}} - \frac{3}{1-0.3678z^{-1}}\right) \tag{20}$$

Sampled Transfer Function of a Typical Control Loop

- Let us now consider a typical control loop in which the error signal is sampled.



We can see that

$$
\begin{aligned}
E(s) &= R(s) - H(s)Y(s) \\
Y(s) &= G(s)E^*(s)
\end{aligned}
$$

and therefore

$$E(s) = R(s) - G(s)H(s)E^*(s)$$

Sampled Transfer Function of a Typical Control Loop

- If we sample the continuous-time error signal, we obtain

$$E^*(s) = R^*(s) - GH^*(s)E^*(s)$$

and solving for $E^*(s)$

$$E^*(s) = \frac{R^*(s)}{1 + GH^*(s)}$$

- Since

$$Y^*(s) = G^*(s)E^*(s),$$

we have

$$Y^*(s) = \frac{G^*(s)R^*(s)}{1 + GH^*(s)}$$

Sampled Transfer Function of a Typical Control Loop

- From this expression, rewriting it in terms of the z-transform, we arrive at

$$Y(z) = \frac{G(z)R(z)}{1 + GH(z)}$$

- Therefore, the sampled transfer function for a typical control loop is

$$\frac{Y(z)}{R(z)} = \frac{G(z)}{1 + GH(z)} \tag{21}$$

Transfer Function of a Computer-Based Control System

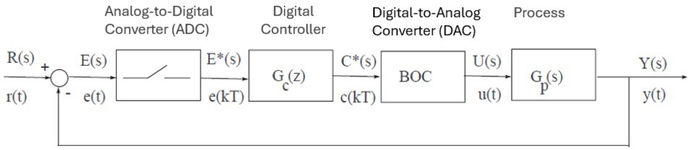


Figure 4: Computer-based control loop.

- The transfer function of the zero-order hold is

$$G_B(s) = \frac{1 - e^{-Ts}}{s}$$

and the hold-process combination has the following transfer function in the Laplace domain:

$$G(s) = G_B(s)G_p(s) = \frac{1 - e^{-Ts}}{s}G_p(s) \tag{22}$$

Transfer Function of a Computer-Based Control System

- The system output signal is

$$Y(s) = G(s)G_c^*(s)E^*(s) \tag{23}$$

or

$$Y^*(s) = G^*(s)G_c^*(s)E^*(s) \tag{24}$$

which, expressed in terms of the z-transform, becomes

$$Y(z) = G(z)G_c(z)E(z) \tag{25}$$

- In addition to sampling the error signal, we have

$$E(z) = R(z) - Y(z) \tag{26}$$

therefore

$$Y(z) = G(z)G_c(z)E(z) = G(z)G_c(z)\left[R(z) - Y(z)\right] \tag{27}$$

Transfer Function of a Computer-Based Control System

- From this, the transfer function of the control loop with sampled error signal and held controller output signal is obtained as

$$\frac{Y(z)}{R(z)} = \frac{G_c(z)G(z)}{1 + G_c(z)G(z)} \tag{28}$$

where $G(s)$ includes both the process and the zero-order hold.

Discretization of an Analog Controller

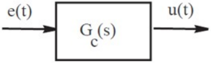
- The relationship between the Laplace transform of a sampled system and the z-transform is given by the expression

$$z = e^{Ts}.$$

- This relationship is not a rational expression, and therefore there is no simple and direct way to obtain the discrete equivalent of a given continuous-time system.
- It is necessary to obtain the discrete equivalents of both $G_p(s)$ and $G_c(s)$, which is generally a laborious process.
- It is common not to have the transfer function of the process $G_p(s)$ available, even though the transfer function of the empirically tuned analog controller $G_c(s)$ is known.
- In such situations, it is convenient to discretize only the controller, and approximate techniques are commonly used, although they can be applied to discretize any transfer function.

Discretization of an Analog Controller

- Consider a continuous-time (analog) controller that is to be replaced by a digital controller together with a sampler and a hold device implemented on a computer, such that the overall behavior is equivalent to that of the original analog controller.



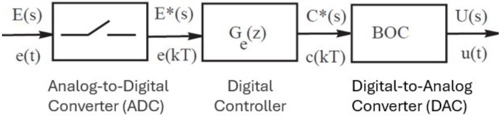


Figure 5: Discretization of an analog controller.

Numerical Approximations

- There are several approaches to perform this transformation in a fast and simple way (although not exactly).
- In general, techniques based on **numerical approximations** of the differential equation representing the continuous-time controller transfer function

$$G_c(s)$$

are commonly used. Two basic techniques are available:
  - Numerical integration (the most widely used).
  - Numerical differentiation.

- For a continuous-time controller such as

$$G_c(s) = \frac{U(s)}{E(s)} = \frac{1}{s} \qquad (29)$$

this expression corresponds to an integration of the form

$$u(t) = u(t_0) + \int_{t_0}^{t} e(t)\,\mathrm{d}t \qquad (30)$$

- Numerical approximations of this expression are based on approximating the integral of the error signal by a summation of rectangles or trapezoids, as illustrated in Figure 6.

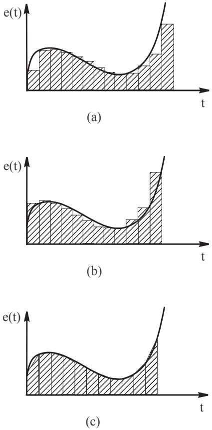


Figure 6: Numerical approximations.

- The area under the integral is approximated by a rectangle with base equal to the sampling period T and height equal to the value of the error signal at instant n.

$$u(n+1) = u(n) + Te(n) \tag{31}$$

Taking the z-transform of this equation yields

$$zU(z) = U(z) + TE(z) \tag{32}$$

and therefore

$$G_e(z) = \frac{U(z)}{E(z)} = \frac{T}{z-1} \tag{33}$$

- Comparing Equation 33 with that of the analog controller in 29, it can be observed that, to obtain the discrete equivalent using this method, it is sufficient to replace each occurrence of s in the analog controller transfer function by $(z-1)/T$

$$G_e(z) = G_c(s)\Big|_{s=\frac{z-1}{T}} \tag{34}$$

- The area under the integral is approximated by a rectangle with base equal to the sampling period T and height equal to the value of the error signal at instant n.

$$u(n+1) = u(n) + Te(n) \tag{35}$$

Taking the z-transform of this equation yields

$$zU(z) = U(z) + TE(z) \tag{36}$$

and therefore

$$G_e(z) = \frac{U(z)}{E(z)} = \frac{T}{z-1} \tag{37}$$

- Comparing Equation 37 with that of the analog controller in 29, it can be observed that, to obtain the discrete equivalent using this method, it is sufficient to replace s in $G_c(s)$ by

$$\frac{z-1}{Tz}:$$

$$G_e(z) = G_c(s)\Big|_{s=\frac{z-1}{Tz}} \tag{38}$$

- This method is also known as the **Tustin method** or the **bilinear transformation**. The area under the integral is approximated by averaging the values of the error signal at instants k and $k+1$ and multiplying by the sampling period T.

$$u(k+1) = u(k) + \frac{T}{2}[e(k) + e(k+1)] \tag{39}$$

Taking the z-transform of this equation yields

$$(z-1)U(z) = \frac{T}{2}(z+1)E(z) \tag{40}$$

and therefore

$$G_e(z) = \frac{U(z)}{E(z)} = \frac{T(z+1)}{2(z-1)} \tag{41}$$

- Comparing both expressions, it can be observed that it is sufficient to replace each occurrence of s in $G_c(s)$ by

$$\frac{2(z-1)}{T(z+1)}:$$

$$G_e(z) = G_c(s)\Big|_{s=\frac{2(z-1)}{T(z+1)}} \tag{42}$$

Example 2

- Suppose that an analog controller has been designed using classical methods for a given system, and that its transfer function in the s-domain is

$$G_c(s) = \frac{s+4}{s(s+1)}. \tag{43}$$

- **Solution:** Applying the bilinear transformation, the transfer function of the equivalent discrete controller becomes

$$G_e(z) = G_c(s)\Big|_{s=\frac{2}{T}\frac{z-1}{z+1}} = \frac{(z+1)\left[z(1+2T)+(2T-1)\right]T}{(z-1)\left[z(T+2)+(T-2)\right]} \tag{44}$$

Structure of the Discretized Controller

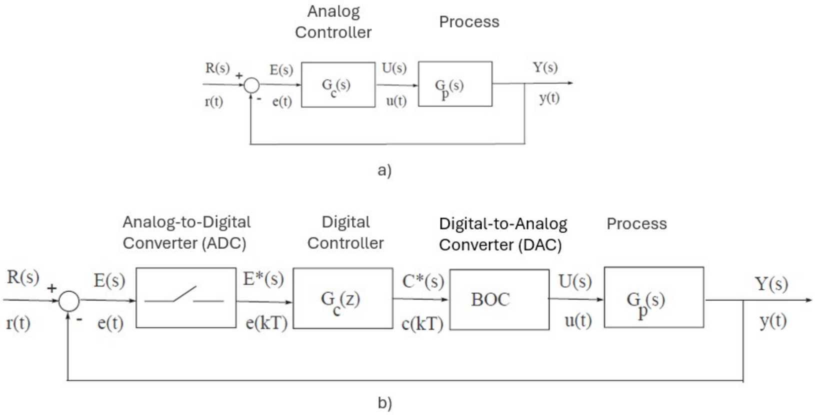


Figure 7: a) Structure of the continuous-time controller and b) equivalent discrete-time system.