Control Engineering II

Topic 4: Discretization of Continuous-Time Systems

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Problem 1

Obtain G(z) as the sum of residues for the following transfer function:

$$G(s) = \frac{1}{s^2(s+1)}$$

Solution

It must be taken into account that the expression has a double pole at 0 and another simple at -1, so both residues must be calculated:

$$G(z) = R_0 + R_{-1}$$

$$R_0 = \frac{1}{(2-1)!}\frac{\lim\limits_{p\to 0} d}{dp}\left[\frac{1}{(p+1)(1-e^{-T(s-p)})}\right] = \lim_{p\to 0}.$$

$$= \frac{(-1)(1-e^{-Ts}-e^{-Ts}T)}{(1-e^{-Ts})^2} \underset{z=e^{sT}}{=} \frac{(-1)(1-z^{-1}-Tz^{-1})}{(1-z^{-1})^2} = \frac{(-z)(z-1-T)}{(z-1)^2}$$

$$R_{-1} = \lim_{p\to -1}\left[\frac{1}{p^2[1-e^{-T(s-p)}]}\right] = \frac{1}{1-e^{-Ts}e^{-T}} \underset{z=e^{sT}}{=} \frac{1}{1-e^{-T}z^{-1}} = \frac{z}{z-e^{-T}}$$

$$G(z) = \frac{-z(z-T-1)}{(z-1)^2} + \frac{z}{z-e^{-T}} = \frac{(T-1+e^{-T})z^2 + (1-e^{-T}-Te^{-T})z}{(z-1)^2(z-e^{-T})}$$

$$G(z) = \frac{(T-1+e^{-T})z^{-1} + (1-e^{-T}-Te^{-T})z^{-2}}{(1-z^{-1})^2(1-e^{-T}z^{-1})}$$

Problem 2

Obtain the Z-transform of the following function by partial fraction expansion and using the residue method:

$$X(s) = \frac{s}{(s+1)^2(s+2)}$$

Solution

a) First we expand in partial fractions and calculate the coefficients:

$$X(s) = \frac{A}{(s+1)^2} + \frac{B}{(s+1)} + \frac{C}{s+2}$$

$$A = \left[(s+1)^2 X(s)\right]\Big|_{s=-1} = \frac{s}{(s+2)}\Big|_{s=-1} = -1$$

$$B = \frac{d}{ds}\left[(s+1)^2 X(s)\right]\Big|_{s=-1} = \frac{d}{ds}\left[\frac{s}{(s+2)}\right]\Big|_{s=-1} = \frac{2}{(s+2)^2}\Big|_{s=-1} = 2$$

$$C = [(s+2)X(s)]|_{s=-2} = \frac{s}{(s+1)^2}\Big|_{s=-2} = -2$$

$$X(s) = \frac{-1}{(s+1)^2} + \frac{2}{(s+1)} - \frac{2}{s+2}$$

Since the Z-transforms of these functions are known, we write the result:

$$X(z) = \frac{-Te^{-T}z^{-1}}{(1-e^{-T}z^{-1})^2} + 2\left(\frac{1}{1-e^{-T}z^{-1}}\right) - 2\left(\frac{1}{1-e^{-2T}z^{-1}}\right)$$

$$X(z) = \frac{2-2e^{-T}z^{-1} - Te^{-T}z^{-1}}{(1-e^{-T}z^{-1})^2} - \frac{2}{1-e^{-2T}z^{-1}}$$

b) We now use the residue method:

$$X(z) = R_{-1} + R_{-2}$$

$$R_{-1} = \frac{1}{(2-1)!}\frac{\lim\limits_{p\to -1} d}{dp}\left[\frac{p}{(p+2)[1-e^{-T(s-p)}]}\right] =?$$

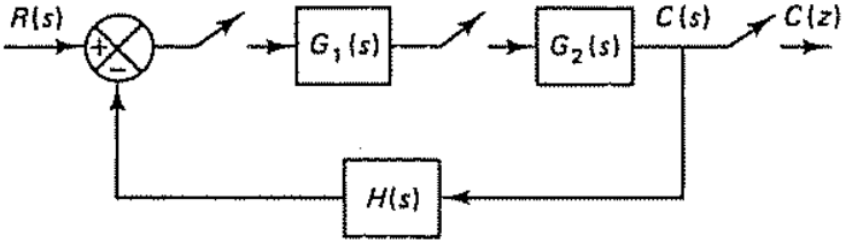
$$= \lim_{p\to -1} .$$

$$R_{-2} = \lim_{p\to -2}\left[\frac{p}{(p+1)^2(1-e^{-T(s-p)})}\right] = \frac{-2}{1-e^{-Ts}e^{-2T}} = \frac{-2z}{z-e^{-2T}}$$

$$X(z) = \frac{2-2e^{-T}z^{-1} - Te^{-T}z^{-1}}{(1-e^{-T}z^{-1})^2} - \frac{2}{1-e^{-2T}z^{-1}}$$

Problem 3

Obtain the equivalent transfer function of the following system:



Solution

In this case we obtain the following equations from the block diagram:

$$E(s) = R(s) - H(s)C(s)$$
$$C(s) = G_2(s)U^*(s)$$
$$U(s) = G_1(s)E^*(s)$$

We have called U to the signal that leaves the block G_1.
Substituting:

$$E(s) = R(s) - H(s)G_2(s)U^*(s)$$

Taking the sampled variables:

$$U^*(s) = G_1^*(s)E^*(s)$$
$$E^*(s) = R^*(s) - G_2H^*(s)U^*(s) = R^*(s) - G_2H^*(s)G_1^*(s)E^*(s)$$
$$E^*(s) = \frac{R^*(s)}{1 + G_1^*(s)G_2H^*(s)}$$

As: $C^*(s) = G_2^*(s)U^*(s) = G_2^*(s)G_1^*(s)E^*(s)$

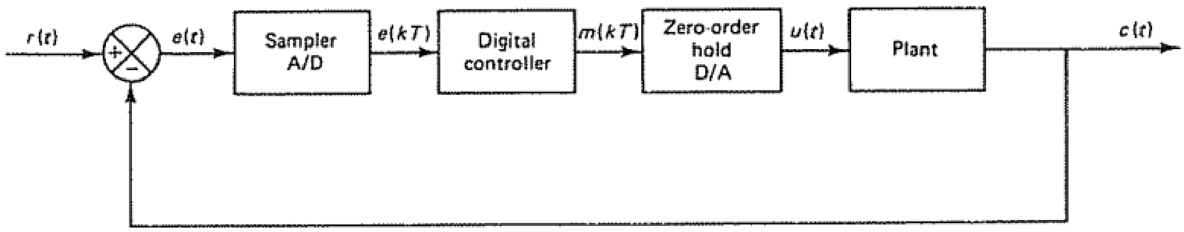
$$C^*(s) = \frac{G_1^*(s)G_2^*(s)R^*(s)}{1 + G_1^*(s)G_2H^*(s)}$$
$$C(z) = \frac{G_1(z)G_2(z)R(z)}{1 + G_1(z)G_2H(z)}$$

The solution will be:

$$\frac{C(z)}{R(z)} = \frac{G_1(z)G_2(z)}{1 + G_1(z)G_2H(z)}$$

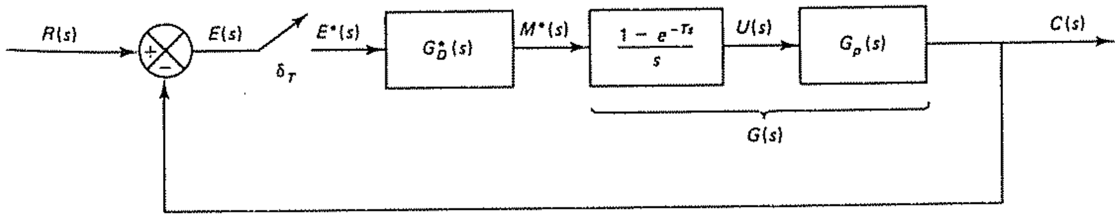
Problem 4

Calculate the equivalent transfer function of the digital control system represented in the following figure:



Solution

The system is composed of an analog-digital converter, a digital controller, a zero-order hold and the plant. If we take into account the transfer functions of the system:



The transfer function of the digital controller is $G_D^*(s)$, and the transfer function including the zero-order hold is:

$$G(s) = \frac{1 - e^{-Ts}}{s} G_p(s)$$

From the figure we get the following relationships:

$$C(s) = G(s)G_D^*(s)E^*(s)$$
$$C^*(s) = G^*(s)G_D^*(s)E^*(s)$$
$$C(z) = G(z)G_D(z)E(z)$$

As: $E(z) = R(z) - C(z)$
We have: $C(z) = G_D(z)G(z)[R(z) - C(z)]$
Finally:

$$\frac{C(z)}{R(z)} = \frac{G_D(z)G(z)}{1 + G_D(z)G(z)}$$

Problem 5

Approximate the transfer function in z for the following function using the Euler I, Euler II and Trapezoidal methods:

$$G(s) = \frac{s}{(s+1)^2}$$

Solution

$$G_e(z) = G(z)|_{s=\frac{z-1}{T}} = \frac{\frac{z-1}{T}}{\left(\frac{z-1}{T}+1\right)^2} = \frac{T(z-1)}{(z+T-1)^2}$$

Euler II:

$$G_e(z) = G(Z)|_{s=\frac{z-1}{Tz}} = \frac{\frac{z-1}{Tz}}{\left(\frac{z-1}{Tz}+1\right)^2} = \frac{Tz(z-1)}{[z(1+T)-1]^2}$$

Trapezoidal:

$$G_e(z) = G(Z)|_{s=\frac{2}{T}\left(\frac{z-1}{z+1}\right)} = \frac{\frac{2}{T}\left(\frac{z-1}{z+1}\right)}{\left(\frac{2}{T}\left(\frac{z-1}{z+1}\right)+1\right)^2} = \frac{2T(z^2-1)}{[z(T+2)-(T-2)]^2}$$