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Control Engineering II

Topic 5:
Discrete-Time PID Controllers

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Contents

- **Fixed-structure controllers.** These controllers use a transfer function defined a priori. This group includes **PID controllers**. In such controllers, the transfer function has a fixed structure.
- **Variable-structure controllers.** This second group of controllers does not have a predefined transfer-function structure. Instead, the controller structure is obtained from the desired system specifications and the transfer function of the process to be controlled. This is the case of controller design using the **direct synthesis** technique.

- Specifications express the performance characteristics desired for the system. They can be stated in different ways, using either the frequency domain or the time domain.
- Specifications related to the time-domain response of a system are usually defined with respect to the step response of a **second-order system**. For this purpose, the following theoretical transfer function is taken as the reference system:

$$G(s) = \frac{K\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \tag{1}$$

whose step response is

$$y(t) = 1 - \frac{1}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t} \sin(\omega_n t\sqrt{1-\xi^2} + \phi) \tag{2}$$

with

$$\phi = \arctan\frac{\sqrt{1-\xi^2}}{\xi} \tag{3}$$

- This system has poles located at

$$p_{1,2} = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2} \tag{4}$$

where ξ is the so-called *damping ratio* and ω_n is the *undamped natural frequency* of the system. For a system with damping ratio $0 < \xi \leq 1$ (an underdamped system), the response curve has the following form:

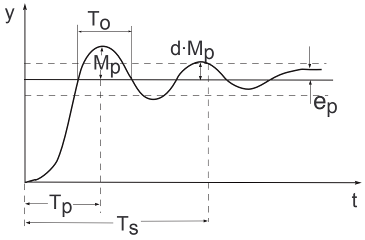


Figure 1: Response of an underdamped second-order system.

Among the parameters used to specify the response are the following:

- *Peak time*:

$$T_p = \frac{\pi}{\omega_d}$$

where ω_d is the damped frequency,

$$\omega_d = \omega_n \sqrt{1 - \xi^2}.$$

- *Decay ratio*:

$$d = e^{-2\pi \frac{\xi}{\sqrt{1-\xi^2}}}$$

- *Damped oscillation period*:

$$T_o = \frac{2\pi}{\omega_n \sqrt{1 - \xi^2}}$$

- *Overshoot*:

$$M_p = e^{-\pi \frac{\xi}{\sqrt{1-\xi^2}}} = \sqrt{d}$$

- *Settling time*:

$$T_s \simeq \frac{\pi}{\xi \omega_n}$$

- Steady-state position error for a step input, e_p.

Discretization of a PID Controller

- Nowadays, most controllers are discrete-time controllers, making it necessary to discretize the controller before implementation.
- The differential equation of a PID controller has the form

$$u(t) = K\left[e(t) + \frac{1}{T_i}\int_0^t e(\tau)\,d\tau + T_d\frac{de(t)}{dt}\right]$$

- The derivative term can be replaced by a first-order difference, and the integral term can be approximated using the first Euler method (forward rectangular approximation).

$$u(k) = K\left[e(k) + \frac{T}{T_i}\sum_{i=0}^{k-1} e(i) + \frac{T_d}{T}\left(e(k) - e(k-1)\right)\right]$$

- To obtain a recursive expression that avoids the summation term, the expression is written for $k-1$:

$$u(k-1) = K\left[e(k-1) + \frac{T}{T_i}\sum_{i=0}^{k-2} e(i) + \frac{T_d}{T}(e(k-1) - e(k-2))\right]$$

- Subtracting both expressions yields the discretized PID controller equation:

$$u(k) = u(k-1) + [q_0 e(k) + q_1 e(k-1) + q_2 e(k-2)]$$

where

$$\begin{aligned} q_0 &= K\left(1 + \frac{T_d}{T}\right) \\ q_1 &= -K\left(1 + 2\frac{T_d}{T} - \frac{T}{T_i}\right) \\ q_2 &= K\frac{T_d}{T} \end{aligned}$$

- The transfer function in the z-domain becomes

$$G(z) = \frac{U(z)}{E(z)} = \frac{q_0 + q_1 z^{-1} + q_2 z^{-2}}{1 - z^{-1}}$$

- If the integral term is approximated using the trapezoidal rule, the expression for $u(k)$ becomes

$$u(k) = K \Bigg[e(k) + \frac{T}{T_i} \Bigg(\frac{e(0)+e(k)}{2} + \sum_{i=1}^{k-1} e(i) \Bigg) + \frac{T_d}{T}(e(k) - e(k-1$$

- and the controller takes the form

$$u(k) = u(k-1) + [q_0 e(k) + q_1 e(k-1) + q_2 e(k-2)]$$

where

$$\begin{aligned} q_0 &= K\left(1 + \frac{T_d}{T} + \frac{T}{2T_i}\right) \\ q_1 &= -K\left(1 + 2\frac{T_d}{T} - \frac{T}{2T_i}\right) \\ q_2 &= K\frac{T_d}{T} \end{aligned}$$

- Suppose a continuous-time PID controller has been designed with parameters

$$K = 2, \qquad T_d = 5, \qquad T_i = 50,$$

and is to be discretized using a sampling period of

$$T = 1 \text{ s}.$$

- Using the forward rectangular integration method, the resulting controller parameters are

$$\begin{aligned} q_0 &= K\left(1 + \frac{T_d}{T}\right) = 12 \\ q_1 &= -K\left(1 + 2\frac{T_d}{T} - \frac{T}{T_i}\right) = -21.96 \\ q_2 &= K\frac{T_d}{T} = 10 \end{aligned} \tag{5}$$

Example

Discretization of a PID Controller

- If trapezoidal integration is used, the parameters become:

$$
\begin{aligned}
q_0 &= K\left(1 + \frac{T_d}{T} + \frac{T}{2T_i}\right) = 12.02 \\
q_1 &= -K\left(1 + 2\frac{T_d}{T} - \frac{T}{2T_i}\right) = -21.80 \qquad (6) \\
q_2 &= K\frac{T_d}{T} = 10
\end{aligned}
$$

Discretization Using the Residue Theorem

- Another approach to the discretization of PID controllers is to use the **residue theorem** together with a **zero-order hold** and a **sampler** with sampling period T:

$$G(z) = (1 - z^{-1}) \sum_{\text{Poles inside } C} Res\left(\frac{G(s)}{s} \frac{1}{1 - e^{pT} z^{-1}}\right)$$

- The transfer function of the zero-order hold is

$$B_0(s) = \frac{1 - e^{-sT}}{s}$$

whose equivalent numerator in the z-plane is

$$(1 - z^{-1}).$$

- Suppose we want to discretize the continuous-time controller

$$G_c(s) = 9.86\frac{s+2}{s+3.14}$$

- To discretize this controller, it is first necessary to select the sampling period and then perform the discretization. For the selection of the sampling period T, we use the empirical rule of taking $1/8$ of the damped oscillation period of the system, which is twice the peak time T_p.

$$T_o = \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{2\pi}{3.14\sqrt{1-0.5^2}} = 2.31$$

$$T = \frac{2.31}{8} = 0.28 \approx 0.2$$

- The equivalent discrete-time controller is represented in the figure.

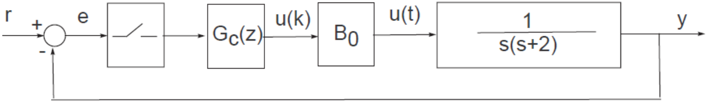


Figure 2: Structure of the control loop.

- To obtain $G_c(z)$, we use the discretization method based on the residue theorem:

$$
\begin{aligned}
G_c(z) &= (1-z^{-1})\sum R\left[G_c(s)\frac{1}{s}\frac{1}{(1-e^{pT}z^{-1})}\right] \\
&= (1-z^{-1})\sum R\left[9.86\frac{s+2}{(s+3.14)}\frac{1}{s}\frac{1}{(1-e^{pT}z^{-1})}\right] \\
&= (1-z^{-1})9.86\left[\frac{2}{3.14}\frac{1}{(1-z^{-1})}+\frac{1.14}{3.14}\frac{1}{(1-e^{-3.14\cdot 0.2}z^{-1})}\right] \\
&= (1-z^{-1})\left[6.277\frac{1}{(1-z^{-1})}+3.578\frac{1}{(1-0.533z^{-1})}\right] \\
&= 6.277+3.578\frac{(1-z^{-1})}{(1-0.533z^{-1})}=9.855\frac{z-0.702}{z-0.533}
\end{aligned}
$$

- The responses are similar, although the discrete controller response is slightly more oscillatory because the selected sampling period is the largest acceptable value, $T = 0.2$.

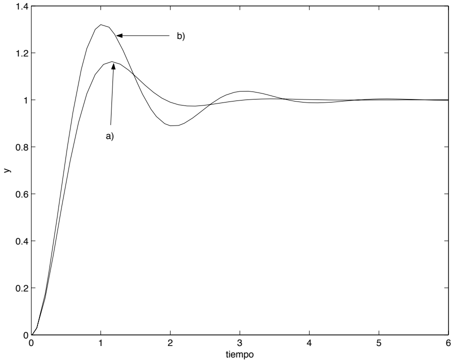


Figure 3: Response of the controlled system: a) continuous and b) discrete.

- If the sampling period is reduced, the response improves. For $T = 0.05\,s$, the discrete controller becomes

$$G_c(z) = 9.86\frac{z - 0.9073}{z - 0.8547}$$

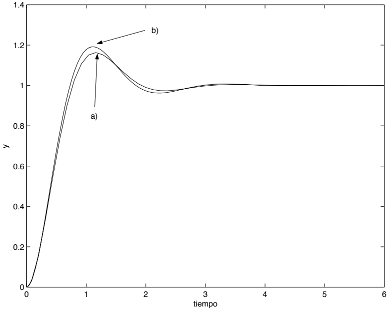
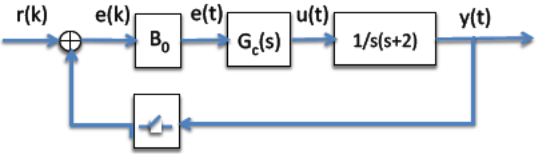


Figure 4: Response of the controlled system: a) continuous and b) discrete.

- For the system and controller of the previous example, suppose that the continuous-time process is also to be discretized using $T = 0.2\,s$. In this case, the zero-order hold changes its position.



- Applying the residue-based discretization method to both the controller and the process yields

$$
\begin{aligned}
G_cG(z) &= (1-z^{-1})\sum R\left[G_c(s)G(s)\frac{1}{s}\frac{1}{1-e^{pT}z^{-1}}\right] \\
&= (1-z^{-1})\sum R\left[9.86\frac{1}{s(s+3.14)}\frac{1}{s}\frac{1}{1-e^{pT}z^{-1}}\right] \\
&= 9.86(1-z^{-1}). \\
&\cdot\left[\frac{1}{(2-1)!}\frac{d}{ds}\left[s^2\frac{1}{s^2(s+3.14)}\frac{1}{(1-e^{pT}z^{-1})}\right]_{p=0}\right. \\
&\left. + \frac{1}{(-3.14)^2}\frac{1}{1-e^{-3.14\cdot 0.2}z^{-1}}\right]
\end{aligned}
$$

Controller and Process Discretization

Example (Cont.)

- Note that the residue associated with the multiple pole at $s = 0$ contains only the term corresponding to the highest-order factor s^2.

$$
\begin{aligned}
G_cG(z) &= 9.86(1-z^{-1}). \\
&\cdot \left[\left[\frac{-(1-e^{pT}z^{-1} + (s+3.14)(-T)e^{pT}z^{-1})}{(s+3.14)^2(1-e^{pT}z^{-1})^2} \right]_{p=0} + \right. \\
&\left. + \frac{1}{3.14^2} \frac{1}{(1-0.533z^{-1})} \right] \\
&= (1-z^{-1}) \left[\frac{(1.628z^{-1}-1)}{(1-z^{-1})^2} + \frac{1}{(1-0.533z^{-1})} \right] \\
&= \frac{(1.628z^{-1}-1)(1-0.533z^{-1}) + (1-z^{-1})^2}{(1-z^{-1})(1-0.533z^{-1})} = \\
&= 0.161 \frac{z+0.8261}{(z-1)(z-0.533)}
\end{aligned}
$$

Controller and Process Discretization

Example (Cont.)

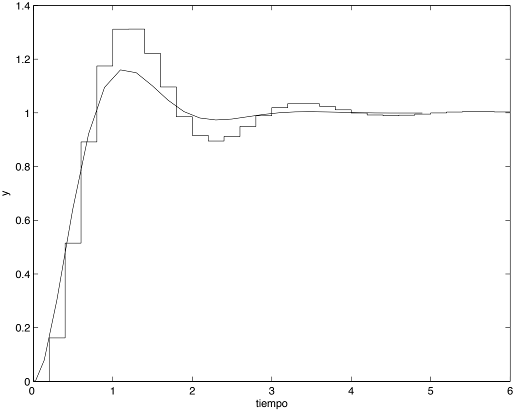


Figure 5: Results of the example closed-loop system with the continuous-time controller and its equivalent discrete-time controller.

Determination of the Sampling Frequency

- There is no fixed rule for selecting the most appropriate sampling frequency when implementing a discrete-time controller (although it must satisfy the sampling theorem).
- The choice of the sampling period depends on:
  - the desired performance specifications,
  - the process dynamics,
  - the disturbance spectrum,
  - the actuator,
  - the available measurement equipment,
  - and the computational cost, among other factors.
- Performance does not usually improve significantly below a certain sampling period, and since very small sampling periods generally increase implementation costs, a reasonable value should be selected.

- The system dynamics strongly influence the sampling period. In general, the larger the time constants, the larger the sampling period that can be used.
- Two common empirical rules are:
  - Select the sampling period so that it is at most one tenth of the smallest time constant present in the system.
  - If the system exhibits an underdamped second-order response, the sampling period can be chosen to be eight times smaller than the period of the damped oscillation.
  - If the response is overdamped second-order, the sampling period can be chosen to be eight times smaller than the system rise time.

- If we have a system whose dominant poles are located at $p_{1,2} = -1.57 \pm 2.72j$, an acceptable sampling period T according to the first rule would be

$$T = \frac{1.57}{10} = 0.157$$

- According to the second rule, the damped response of the system would have a period $t_o = 2\pi/\omega_d$, and for the case under consideration, $\omega_d = 2.72$, yielding $t_o = 2.31$:

$$T = \frac{t_o}{8} = \frac{2.31}{8} = 0.28$$

- Averaging both values, a reasonable sampling period for this system would be

$$T = 0.2 \text{ seconds.}$$

- As shown in the figure, the continuous-time and sampled responses with $T = 0.2$ seconds are very similar.

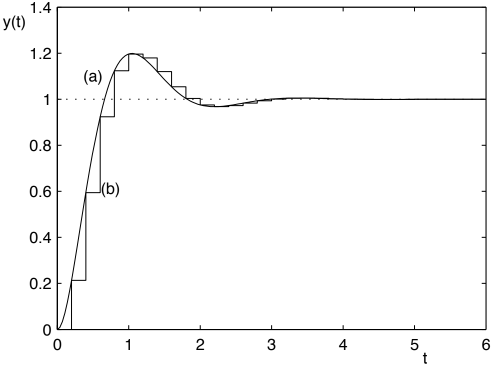


Figure 6: System response: a) continuous and b) sampled.

Design of Discrete-Time PID Controllers

- Several methods for the design of continuous-time PID controllers have been studied so far, together with their subsequent discretization.
- It is also possible to design the controller directly in discrete time. This requires:
  - Obtaining a discrete-time equivalent of the system.
  - Converting the desired time-domain specifications of the continuous-time system into specifications defined in the discrete-time z-plane.
  - Usually determining the desired closed-loop pole locations, or at least the dominant poles.

- If the desired closed-loop poles for the continuous-time system are defined in the s-plane at

$$s = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$$

- and the relationship between the Laplace transform and the z-transform is

$$z = e^{Ts},$$

  then the corresponding points in the z-plane are

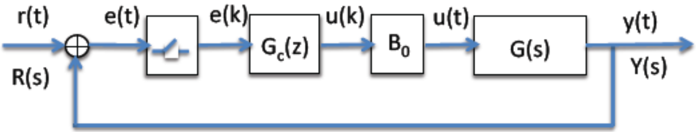
$$z = \exp\left[T\left(-\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}\right)\right]$$

- Therefore, the closed-loop pole locations in the z-plane, for a given sampling period, have the following magnitude and phase:

$$\begin{aligned} |z| &= e^{-T\xi\omega_n} \\ \angle z &= \pm T\omega_n\sqrt{1-\xi^2} = \pm T\omega_d = \pm\theta \end{aligned}$$

Design of Discrete-Time PID Controllers

- The techniques used to obtain the parameters of a discrete-time PID controller that ensures the system satisfies a given set of specifications are basically the same as those used in the continuous-time case.
- For the root-locus design method:
  - The construction rules of the root locus in the z-plane are the same.
  - The stability criterion changes, since the stable region in the z-plane is the unit circle.



- The characteristic equation of a system such as this is

$$1 + KG_c(z)G(z) = 0$$

The idea is to add poles and zeros to the root locus through the controller in order to move the roots of the closed-loop characteristic equation to the desired location.

- In the characteristic equation

$$1 + KG_c(z)G(z) = 0$$

the term K has been explicitly separated so that its variation generates the root locus, although in practice it is usually included within the controller itself.

- A point z_a lies on the root locus if it satisfies the magnitude and angle criteria of the characteristic equation, namely:

$$K = \frac{1}{|G_c(z_a)G(z_a)|}$$

$$\angle(G_c(z_a)G(z_a)) = \pm 180^\circ$$

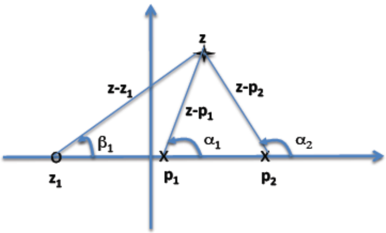
- Since K varies from zero to infinity, any point in the plane corresponds to some value of K, and therefore the magnitude criterion alone does not determine whether a point belongs to the root locus. **Membership is determined exclusively by satisfying the angle criterion.**

- Consider the simplest case where $G_c(z) = 1$ and

$$KG(z) = \frac{K(z - z_1)}{(z - p_1)(z - p_2)}$$

  where all poles and zeros are real.

- For this case, the magnitude and angle criteria are



$$\beta_1 - \alpha_1 - \alpha_2 = \pm 180^\circ$$

$$K = \frac{|z - p_1|\,|z - p_2|}{|z - z_1|}$$

Design of Discrete-Time PID Controllers

Checking Root-Locus Membership

- From a design perspective, there are two possible cases:
  - If the specifications yield a point that **belongs to the root locus**, then a proportional controller is sufficient (provided that the steady-state error specification is also satisfied). Determining the value of K corresponding to that point completely defines the controller.
  - If the desired point **does not belong to the root locus**, this indicates that an additional angle contribution is required in the angle criterion. Depending on whether the root locus must be shifted to the right or to the left, an appropriate controller will be introduced.

The **root-locus method**, introduced by Evans in 1954, is a **graphical method** for determining the closed-loop pole locations of a system from the open-loop poles and zeros.

- The root locus of a system can indicate:
  - Whether it is sufficient to vary the system gain to achieve the desired specifications (when the root locus passes through the desired closed-loop pole locations).
  - Whether it is necessary to introduce a controller that modifies the root locus so that it passes through the dominant poles required for the desired system performance.

Effect of Adding Poles

- Adding a pole to the open-loop transfer function of a system has the following effects:
  - It adds one more branch to the root locus, causing the branches to move closer together and, in some cases, even enter the positive real half-plane.
  - This tends to reduce the relative stability of the system and increases the settling time of the response.

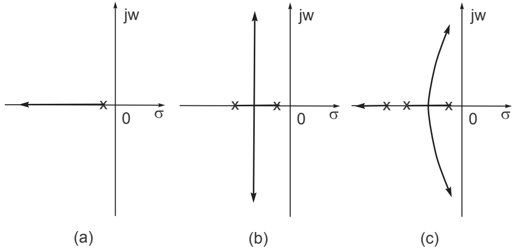


Figure 7: Effect of adding poles.

- Adding a zero to the open-loop transfer function:
  - Reduces the number of branches that go to infinity and therefore **increases the relative stability** of the system, since the branches are shifted toward the negative real part of the complex plane.
  - Makes the system **faster** by introducing a derivative effect that anticipates the system response.
  - Amplifies high-frequency noise.

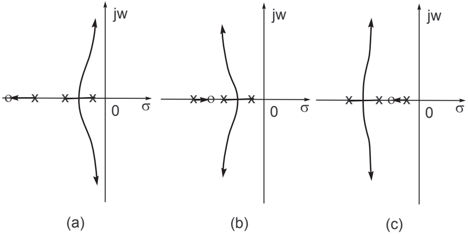


Figure 8: Effect of adding zeros.

- Proportional Controller (P)
- Proportional-Derivative Controller (PD)
- Proportional-Integral Controller (PI)
- Proportional-Integral-Derivative Controller (PID)
- Lead-Lag Controller (Phase Lead-Lag) (Real PD Controller)
- Lag-Lead Controller (Phase Lag-Lead) (Real PI Controller)

- The control action is proportional to the error, and the transfer function is the simplest possible:

$$G_c(z) = K$$

- Characteristics:
    - Introducing a P controller generates the root locus and therefore modifies the closed-loop pole locations.
    - **It modifies both the system dynamics and the steady-state error**. If the poles are adjusted to obtain a certain transient response, it is necessary to verify that the steady-state error also satisfies the specifications.
    - Since it affects both the transient and steady-state responses, it is difficult to satisfy specifications for both simultaneously.
    - If the root locus passes through the desired closed-loop pole locations, this is the first controller type that should be considered.

- The control action is proportional to the error and its derivative, and the transfer function has the form

$$G_c(z) = K_p + K_d(1 - z^{-1}) = \frac{K(z-a)}{z}$$

that is, a pole at the origin of the z-plane and a zero at a.

- Characteristics:
    - By introducing one pole and one zero, the number of branches increases by one, thus modifying the root locus.
    - The modification introduced depends on how close the pole is to the zero at the origin.
    - It tends to improve the dynamics, i.e., it reduces overshoot for the same settling time.
    - It does not usually improve steady-state performance, although this depends on the location of a.

- The control action is proportional to the error and to the integral of the error, which, when **approximated using the trapezoidal rule**, yields a transfer function of the form

$$G_c(z) = K_p + K_i \frac{(1+z^{-1})}{(1-z^{-1})} = \frac{K(z-b)}{z-1}$$

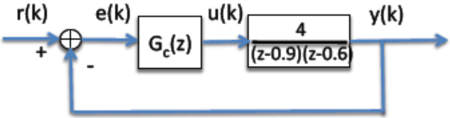
  that is, a pole at 1 and a zero at b.
- Characteristics:
  - It increases the system type by one and therefore eliminates the steady-state error for a step input (provided the original system was type 0).
  - It tends to destabilize the system, since the branches move toward the right and may leave the unit circle.

- The control action is proportional to the error, the derivative of the error, and the integral of the error, which, when **approximated using the trapezoidal method**, yields a transfer function of the form

$$G_c(z) = K_p + K_d(1 - z^{-1}) + K_i\frac{(1 + z^{-1})}{(1 - z^{-1})} = \frac{K(z - a)(z - b)}{z(z - 1)}$$

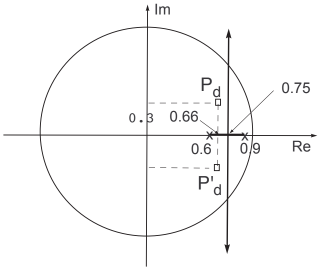
  Therefore, in general, the controller has poles at $z = 0$ and $z = 1$, and zeros at $z = a$ and $z = b$.
- Characteristics:
  - It increases the system type by one, thereby eliminating the steady-state error for a step input (provided the original system was type 0).
  - The derivative action tends to improve the transient response.

- Suppose that a discrete controller is to be designed for the following system:



- Such that the following specifications are satisfied:
  - The closed-loop poles are located at $p_{1,2} = 0.66 \pm j0.3$.
  - The steady-state error is zero.

- If we observe the root locus of the system $1 + KG(z) = 0$ in the z-plane, it does not pass through the desired pole locations, P_d. Therefore, a **P controller is not sufficient**.



- To eliminate the steady-state error, and since the system does not have any poles on the unit circle (it has no integrators), it is necessary to include an integrator in the controller.
- Since both the dynamics and the system type must be modified, a PID controller will be introduced.

- The generic expression of a discrete PID controller is

$$G_c(z) = \frac{q_0 + q_1 z^{-1} + q_2 z^{-2}}{1 - z^{-1}} = \frac{q_0 z^2 + q_1 z + q_2}{z(z-1)}$$

  - We are introducing **two poles at** $z = 1$ **and** $z = 0$ and **two zeros** whose locations and gain must be determined.
  - The specifications determine the locations of two closed-loop poles; therefore, only two PID parameters can be fixed.
  - One of the controller zeros is chosen to cancel the pole located at $z = 0.9$. Applying the magnitude and angle criteria to the expression (the pole-zero pair that cancels has been removed)

$$1 + K\frac{4(z-a)}{z(z-1)(z-0.6)} = 0$$

- Using the magnitude and angle criteria:

$$
\begin{aligned}
|G_c(z)G(z)| &= |K|\frac{\prod_{i=1}^{m}|z+z_i|}{\prod_{j=1}^{n}|z+p_j|} = 1 \\
\angle G_c(z)G(z) &= \angle K + \sum_{i=1}^{m}\angle(z+z_i) - \sum_{j=1}^{n}\angle(z+p_j) = (2q+1)\pi
\end{aligned}
$$

- We can determine the values of K and a that make the root locus pass through the poles $p_{1,2} = 0.66 \pm j\,0.3$.
- The angles associated with the poles are

$$
\begin{aligned}
\beta_1 &= 138.57^\circ \\
\beta_2 &= 78.69^\circ \\
\beta_3 &= 24.44^\circ
\end{aligned}
$$

and, in order to satisfy the angle criterion,

$$
\alpha_1 = 180^\circ - (\beta_1 + \beta_2 + \beta_3) = 61.7^\circ
$$

- From the angle α_1 we can determine the position at which the zero must be located so that it forms the required angle with the dominant pole $0.66 + j\,0.3$.
- This zero is located at

$$a = 0.265$$

- To determine the gain K, we apply the magnitude criterion:

$$4K = \frac{n_1 n_2 n_3}{m_1} = \frac{0.463 \cdot 0.306 \cdot 0.725}{0.496} = 0.302$$

and therefore

$$K = 0.075$$

- The resulting PID controller transfer function is

$$G_c(z) = \frac{0.075(z - 0.9)(z - 0.265)}{z(z - 1)}$$

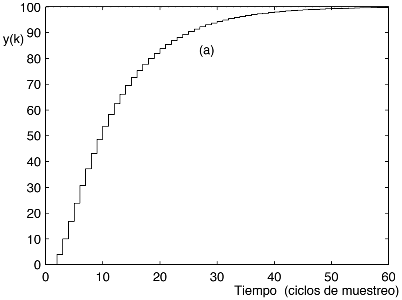


Figure 9: Open-loop system response.

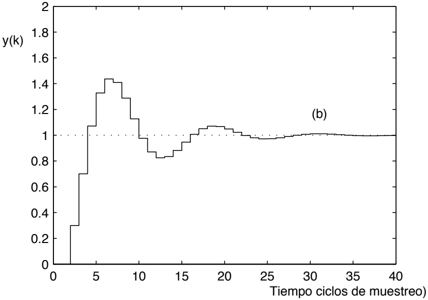


Figure 10: Closed-loop response with the designed controller.

Structure of a Real Discrete-Time PID Controller

- The structure of a discrete-time PID controller is given by the following expression:

$$u(k) = K\left[e(k) + \frac{T}{T_i}\sum_{i=0}^{k-1} e(i) + \frac{T_d}{T}\left(e(k) - e(k-1)\right)\right]$$

- Taking the z-transform, we obtain:

$$U(z) = \left[K + \frac{KT}{T_i}\frac{1}{1-z^{-1}} + \frac{KT_d}{T}(1-z^{-1})\right]E(z)$$

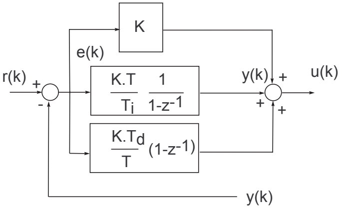


Figure 11: Theoretical structure of a discrete-time PID controller.

Several issues can be observed in the theoretical PID controller:

- The derivative of the error may take very large values. When multiplied by the derivative gain, it can **saturate the actuator** (especially in the presence of noise or when a step input is applied). *To avoid this, the derivative gain is often placed in the feedback path.*
- **Integral saturation** or **windup** occurs because, in slow systems, the integral of the error may grow to very large values. This can keep the actuator saturated for a long period of time. *Anti-windup schemes are used to disable or limit the integral action when it exceeds a specified value.*
- In some systems it is difficult to correct very small errors. *In such cases, a deadband (zero-gain region) around the desired steady-state value is introduced.*

Structure of a Real Discrete-Time Controller

Axis Controller

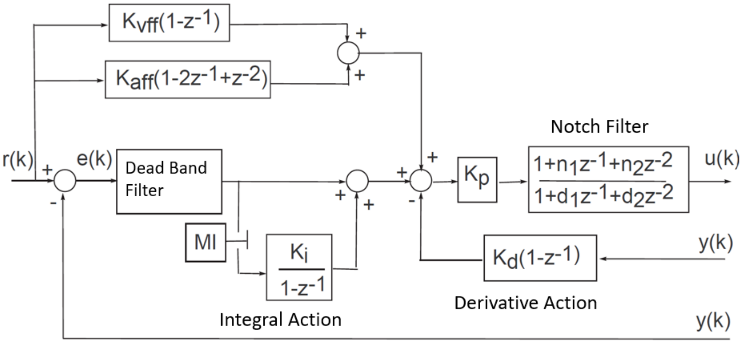


Figure 12: Structure of a discrete-time controller for motion-axis control.

Structure of a Real Discrete-Time PID Controller

- The controller also allows the introduction of two **feedforward gains**, which improve tracking performance:
  - One gain compensates for delays caused by the inertia of the system (`Kaff`).
  - The other compensates for friction in the system.
- The controller is also prepared to implement **cascade control** with two feedback loops. This approach is commonly used in motor-driven positioning systems, where:
  - The inner loop controls the motor speed.
  - The outer loop controls the position.
- The scheme also includes a **Notch filter** to eliminate resonance peaks that may appear in the system.