Control Engineering II

Topic 5: Discrete-Time PID Controllers

C.A. Monje, D. Copaci, M. Malfaz, J. Muñoz, L. Menа, L. Moreno, S. Garrido

2026

uc3m | Universidad Carlos III de Madrid
OpenCourseWare

Problem 1

Given the regulator in continuous space that presents the following transfer function, discretize it by the residue method:

$$R(s) = \frac{2}{s+3}$$

Consider for this purpose that a zero-order hold is added to the input of the regulator so that it can operate in a discrete system. The resulting transform will be that of the holder-regulator set. Use a sampling period T=0.5 s.

Solution

We know that to discretize a transfer function with a zero-order hold located before it we must use the following formula:

$$R(z) = (1 - z^{-1})Z\left[\frac{R(s)}{s}\right]$$

If we apply the residue method to calculate the transform:

$$R(z) = (1 - z^{-1})Z\left[\frac{G(s)}{s}\right] = (1 - z^{-1})\sum R_{p_i}$$

$$R\left(z\right) = \left(1 - z^{-1}\right)\left[\lim_{p\to 0}\frac{2}{(p+3)\left[1 - e^{-T(s-p)}\right]} + \lim_{p\to -3}\frac{2}{p\left[1 - e^{-T(s-p)}\right]}\right] =$$
$$= \left(1 - z^{-1}\right)\left[\frac{2}{3\left(1 - e^{-Ts}\right)} - \frac{2}{3\left(1 - e^{-3T}e^{-Ts}\right)}\right]$$

$$R(z) = (1 - z^{-1})\left[\frac{2}{3(1 - z^{-1})} - \frac{2}{3(1 - e^{-3T}z^{-1})}\right] =$$
$$= \frac{2}{3} - \frac{2}{3}\left(\frac{1 - z^{-1}}{1 - z^{-1}e^{-3T}}\right) = \frac{2}{3}\left(\frac{1 - e^{-3T}}{z - e^{-3T}}\right)$$

For T=0.5 s:

$$R(z) = \frac{2}{3}\frac{0.777}{z - 0.223}$$

Problem 2 (1/2)

Calculate the discrete PID controller if the continuous PID controller has the following parameters: $K = 2$**,** $T_d = 2.5$**,** $T_i = 40$ **and sampling period** $T = 1$ **second. Use for this purpose forward rectangular integration and trapezoidal integration.**

Solution

The differential equation of a continuous PID regulator has the following form:

$$u(t) = K[e(t) + \frac{1}{T_i}\int_0^t e(\tau)d\tau + T_d\frac{de(t)}{dt}]$$

First we calculate the discrete PID regulator that would result using forward rectangular integration or Euler's method.
If we substitute the derivative term by the first-order difference and approximate the integral term by the first of Euler's methods, the expression for the discrete domain is as follows:

$$u(k) = K[e(k) + \frac{1}{T_i}\sum_{i=0}^{k-1} e(i) + \frac{T_d}{T}(e(k) - e(k-1))]$$

We obtain the expression for k-1:

$$u(k-1) = K[e(k-1) + \frac{1}{T_i}\sum_{i=0}^{k-2} e(i) + \frac{T_d}{T}(e(k-1) - e(k-2))]$$

If we subtract this expression from the previous one and obtain the Z-transform, we arrive at the transfer function of the regulator:

$$G(z) = \frac{q_0 + q_1 z^{-1} + q_2 z^{-2}}{1 - z^{-1}}$$

$$q_0 = K\left(1 + \frac{T_d}{T}\right) = 2\left(1 + \frac{2.5}{1}\right) = 7$$

$$q_1 = -K\left(1 + 2\frac{T_d}{T} - \frac{T}{T_i}\right) = -2\left(1 + 2\frac{2.5}{1} - \frac{1}{40}\right) = -11.95$$

$$q_2 = K\frac{T_d}{T} = 2\frac{2.5}{1} = 5$$

$$G(z) = \frac{q_0 + q_1 z^{-1} + q_2 z^{-2}}{1 - z^{-1}} = \frac{7 - 11.95z^{-1} + 5z^{-2}}{1 - z^{-1}}$$

Problem 2 (2/2)

If the method used was trapezoidal integration, the integral would be approximated by this type of method:

$$u(k) = K[e(k) + \frac{T}{T_i}\left(\frac{e(0)+e(k)}{2}\right)\sum_{i=0}^{k-1} e(i) + \frac{T_d}{T}(e(k) - e(k-1))]$$

By a procedure analogous to the previous one, we arrive at the following results:

$$q_0 = K\left(1 + \frac{T_d}{T} + \frac{T}{2T_i}\right) = 2\left(1 + \frac{2.5}{1} + \frac{1}{2*40}\right) = 7.025$$
$$q_1 = -K\left(1 + 2\frac{T_d}{T} - \frac{T}{2T_i}\right) = -2\left(1 + 2\frac{2.5}{1} - \frac{1}{2*40}\right) = -11.975$$
$$q_2 = K\frac{T_d}{T} = 2\frac{2.5}{1} = 5$$

$$G(z) = \frac{q_0 + q_1 z^{-1} + q_2 z^{-2}}{1 - z^{-1}} = \frac{7.025 - 11.975z^{-1} + 5z^{-2}}{1 - z^{-1}}$$

It is observed that the transfer functions are very similar.

Calculate the simplest discrete regulator so that the following system simultaneously verifies: $e_p < 0.01$, $M_p \simeq 20$ **and** $t_s \simeq 2.1$

$$G(s) = \frac{5}{(s^2+9s+20)(s^2+3s+2)}$$

Calculate the appropriate continuous regulator and discretize it with a convenient sampling period using Euler's method.

Solution

The roots of the open-loop system are at s=-1, s=-2, s=-4 and s=-5. The root locus is as follows:

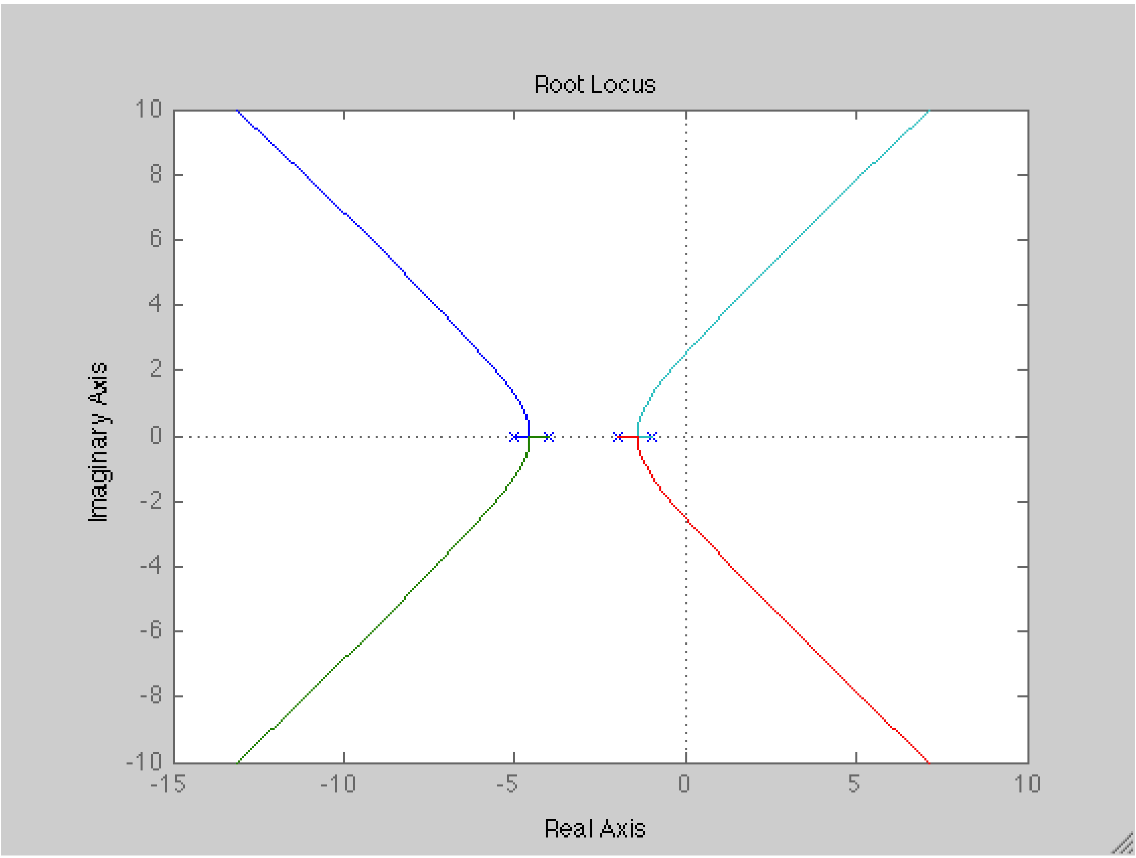
-Asymptotes: $\gamma = (1+2k)\pi/(n_p - n_z) = 45^o, 135^o, 225^o, 315^o...$

-Centroid: $z_0 = \frac{\sum \Re(p_i) - \sum \Re(z_i)}{n_p - n_z} = \frac{-1-2-4-5}{4} = -3$

-Breakaway points:

$$K = \frac{|s+4|\,|s+5|\,|s+1|\,|s+2|}{5}$$

They are obtained by differentiating this expression with respect to s and calculating the maxima and minima. On this occasion we will not calculate them since it is not necessary to solve the problem.



From the dynamic specifications we calculate the position of the dominant poles:

$$M_p = 20 \rightarrow M_p = e^{\pi/\tan\theta} \rightarrow \theta = \arctan\left(\frac{-\pi}{\ln 0.2}\right) = 1.0974rd = 62.87^{\text{o}}$$

$$t_s = 2.1 \approx \pi/\sigma \rightarrow \sigma = 1.5$$

$$p_{1,2} = -1.5 \pm j2.93$$

From the root locus it is deduced that a P-type regulator is not sufficient. Therefore, a PD regulator is tried:

$$R(s) = K(s+a)$$

Applying the argument criterion, the zero position is calculated:

$$\alpha_{-1} + \alpha_{-2} + \alpha_{-4} + \alpha_{-5} - \beta_a = 180$$

$$\alpha_{-1} = 99.68^\circ \quad \alpha_{-2} = 80.32^\circ \quad \alpha_{-4} = 49.53^\circ \quad \alpha_{-5} = 39.93^\circ \rightarrow \quad \beta_a \cong 90^\circ$$

The regulator will be as follows:

$$R(s) = K(s+1.5)$$

Applying the magnitude criterion, the value of K is calculated:

$$K = \frac{2.96 \cdot 2.96 \cdot 3.84 \cdot 4.55}{5 \cdot 2.92} = 10.6$$

$$R(s) = 10.6(s+1.5)$$

We now check the steady-state specifications:

$$K_p = \lim_{s\rightarrow 0} G(s) = \frac{10.6 \cdot 5 \cdot 1.5}{20 \cdot 2} = 1.99$$

$$e_p = \frac{1}{1+1.99} = 0.334 > 0.01$$

The designed regulator is not valid and a PID must be designed.

Problem 3 (3/3)

The integral action implies the addition of a pole at the origin. To compensate for this effect, a zero is added at a determined distance from it. As the dominant poles are located at a distance of 1.5 from the axis, a zero will be placed at one sixth of this value. Thus, the PID regulator to be included will be as follows:

$$R(s) = K\frac{(s+1.5)(s+0.25)}{s}$$

The new K is calculated with the magnitude criterion:

$$K = \frac{2.96 \cdot 2.96 \cdot 3.84 \cdot 4.55 \cdot 3.28}{5 \cdot 2.92 \cdot 3.17} = 10.96$$

$$R(s) = 10.96\frac{(s+1.5)(s+0.25)}{s} = 19.18\left(1 + \frac{1}{4.73s} + 0.57s\right)$$

It is seen that it can be written in classical form:

$$R(s) = K\left(1 + \frac{1}{T_i s} + T_d s\right)$$

To discretize this regulator by Euler's method, an adequate sampling period must be chosen. In this case, we will take a period that fulfills: $T \ll T_d \ll T_i$. For example, T=0.05. Calculating the coefficients:

$$q_0 = K\left(1 + \tfrac{T_d}{T}\right) = 19.18\left(1 + \tfrac{0.57}{0.05}\right) = 237.83$$

$$q_1 = -K\left(1 + 2\tfrac{T_d}{T} - \tfrac{T}{T_i}\right) = -19.18\left(1 + 2\tfrac{0.57}{0.05} - \tfrac{1}{4.73}\right) = -452.43$$

$$q_2 = K\tfrac{T_d}{T} = 19.18\tfrac{0.57}{0.05} = 216.65$$

Finally, the discretized controller will be:

$$G_c(z) = \frac{q_0 + q_1 z^{-1} + q_2 z^{-2}}{1 - z^{-1}} = \frac{235.83 - 452.43z^{-1} + 216.65z^{-2}}{1 - z^{-1}}$$

Given a system with step input, unit feedback and open-loop transfer function:

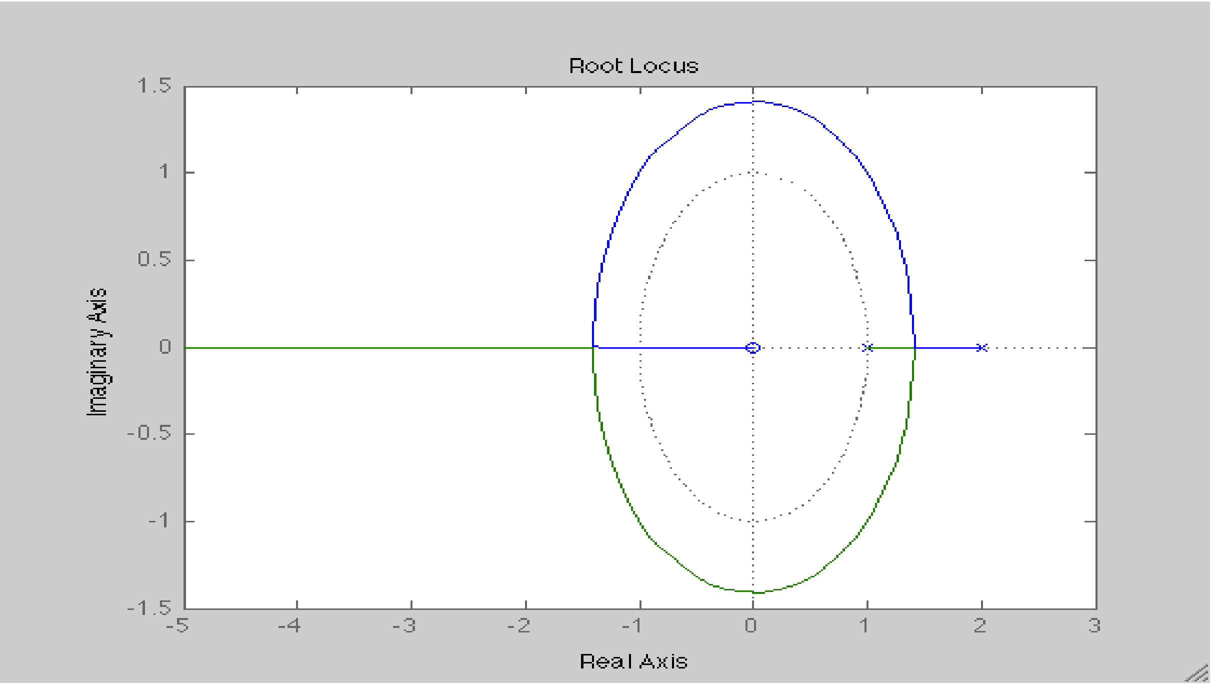
$$G(z) = \frac{z}{(z-1)(z-2)}$$

Calculate from the root locus the simplest regulator so that the system presents zero position error.

Solution

The pole at point $z = 1$ causes the position error of the system to be zero as long as the system is stable, so it is necessary to ensure the stability of the closed-loop system.

First, a proportional regulator is tried. The root locus is as follows:



The breakaway points are:

$$\frac{1}{\sigma} = \frac{1}{\sigma-1} + \frac{1}{\sigma-2}$$
$$\sigma = \pm\sqrt{2}$$

The system will always be unstable with a proportional regulator, since at least one of the poles of the closed-loop system will always be outside the unit circle. It will be necessary to modify the root locus so that the system is stable.

For this purpose we introduce a PD:

$$R(z) = K\frac{z-c}{z}$$

A zero of value c and a pole at the origin are introduced. The procedure for choosing the zero position of the regulator will be to fix an operating point and apply the argument criterion.
The new open-loop system will be:

$$R(z)G(z) = K\frac{z-c}{z}\frac{z}{(z-1)(z-2)} = K\frac{z-c}{(z-1)(z-2)}$$

We choose as an operating point $0.5 \pm 0.5j$ since it is a point for which the system is stable. And the argument criterion is applied so that the root locus passes through the operating point:

$$\alpha_2 + \alpha_1 - \beta_c = 180$$

We calculate the angles:

$$\alpha_2 = 180 - \arctan\frac{0.5}{1.5} = 161.56^o$$

$$\alpha_1 = 180 - \arctan\frac{0.5}{0.5} = 135^o$$

Isolating: $\beta_c = 116.56^o$

$$180 - 116.56 = \arctan\tfrac{0.5}{c-0.5}, \quad c = 0.7499$$

The value of K is calculated using the magnitude criterion:

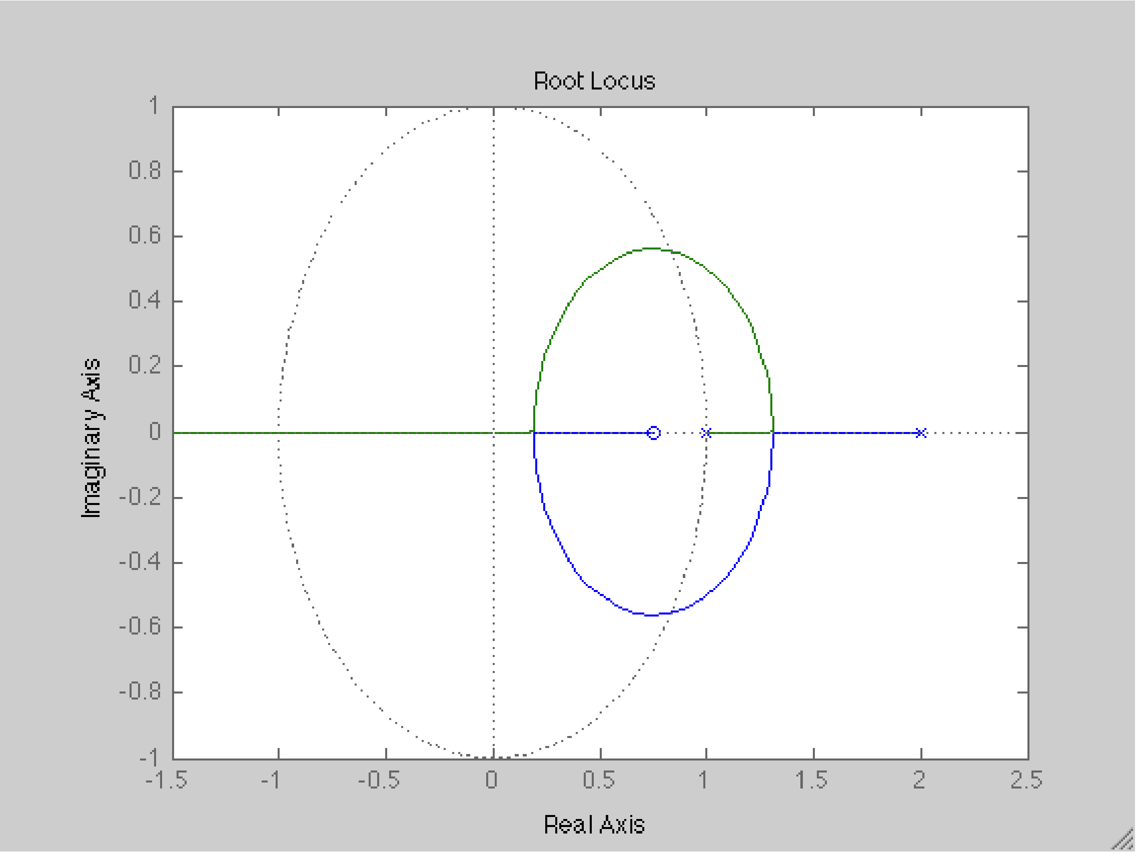
$$K = \frac{\sqrt{0.5^2 + 1.5^2}\sqrt{0.5^2 + 0.5^2}}{\sqrt{0.5^2 + (0.7499 - 0.5)^2}} = 2.0002$$

The transfer function of the regulator is:

$$R(z) = 2.0002\frac{z - 0.7499}{z}$$

Problem 4 (3/3)

The root locus of the system with the new regulator will be:



Given a system with unit feedback and open-loop transfer function:

$$G(z) = \frac{1}{(z-0.7)(z-0.9)}$$

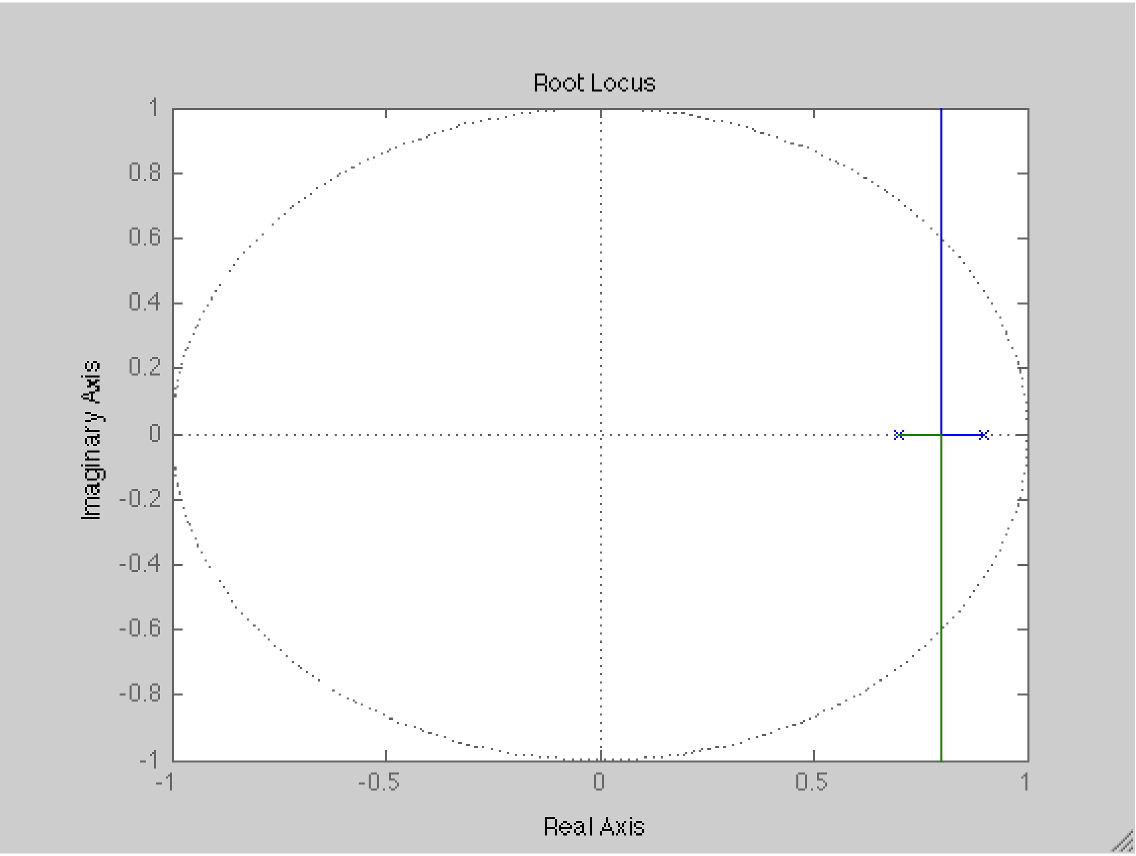
Calculate from the root locus the simplest possible discrete regulator so that the system meets the following specifications: $M_p \leq 20$**,** $n_s \leq 15$ **samples and** $e_p \leq 22$**.**

Solution

In order for the specifications to be met, the operating point must be as follows:

$$\left.\begin{array}{r} n_s = \pi/\sigma = 15 \\ M_p = e^{-\pi\sigma/\theta} = 0.2 \end{array}\right\} \rightarrow \ \sigma = 0.21 \quad \theta = 23.48^{\circ}$$

The dominant poles of the system are: $p = 0.7435 \pm 0.323j$
Let's see if we can work at the calculated operating point:



The root locus does not pass through the operating point, so it will be necessary to incorporate a PD regulator:

$$R(z) = K\frac{z-c}{z}$$

Problem 5 (2/3)

The procedure for choosing the zero position of the regulator consists of applying the argument criterion for the operating point:

$$\alpha_{0.9} + \alpha_{0.7} + \alpha_0 - \beta_c = 180$$

We calculate the angles:

$$\alpha_{0.9} = 180 - \arctan\frac{0.323}{0.9 - 0.7435} = 115.85^{\circ}$$
$$\alpha_{0.7} = \arctan\frac{0.323}{0.7435 - 0.7} = 82.33^{\circ}$$
$$\alpha_0 = \arctan\frac{0.323}{0.7435} = 23.48^{\circ}$$

Isolating: $\beta_c = 41.66^{\circ}$

$$41.66 = \arctan\tfrac{0.323}{0.7435 - c}, \quad c = 0.38$$

The value of K is calculated using the magnitude criterion:

$$K = \frac{d_{0.9}d_{0.7}d_0}{d_c} = \frac{0.3589 \cdot 0.3259 \cdot 0.8106}{0.4863} = 0.195$$

The transfer function of the regulator is:

$$R(z) = 0.195\frac{z - 0.38}{z}$$

Finally, we verify that the steady-state condition is met:

$$K_p = \lim_{z \to 1} R(z)G(z) = 4.03$$
$$e(\infty) = \frac{1}{1 + K_p} = \frac{1}{1 + 4.03} = 19.88 < 22$$

The root locus of the system with regulator is as follows: