uc3m | Universidad Carlos III de Madrid
OpenCourseWare

Control Engineering II

Topic 6:
Controller Design by Direct Synthesis

C.A. Monje, D. Copaci, M. Malfaz, J. Muñoz,
L. Mena, L. Moreno, S. Garrido

2026

Contents

- Classical PID controllers were originally based on the use of physical devices to achieve the different control actions.
  - The controller structure is **fixed**, which constrains the controller design to the control actions incorporated into it.
  - If the system is complex, **a PID controller may not be sufficient** to achieve satisfactory performance. The required controller may be more complex than what can be realized using a classical PID controller.
- The use of computers makes it possible to design and implement complex control algorithms.
  - We will study the analytical design of discrete-time controllers that are not restricted to a basic transfer-function structure (**direct synthesis method**).

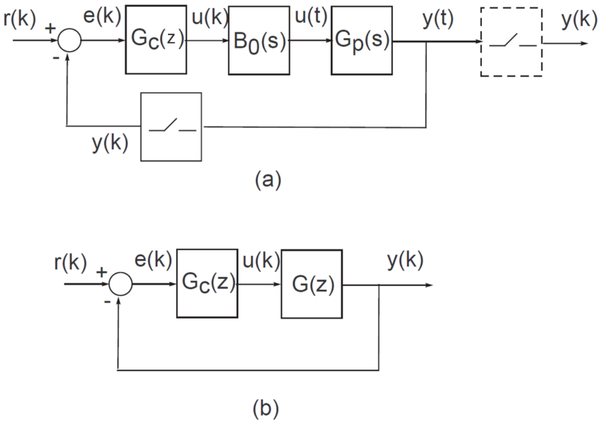

Figure 1: a) Control structure and b) discrete equivalent of the system.

- It is necessary to obtain the discrete equivalent of the system since this is an analytical design method.
- For a system such as the one shown in the previous figure (discrete equivalent), the z-transform of the zero-order hold and process combination is given by

$$
\begin{aligned}
G(z) &= \mathcal{Z}[B_0(s)G_p(s)] = \mathcal{Z}\left[\frac{1-e^{-Ts}}{s}G_p(s)\right] \\
&= (1-z^{-1}) \sum_{\text{Poles in } C} Res_{s=p_i}\left(\frac{G_p(s)}{s}\frac{1}{1-e^{sT}z^{-1}}\right)
\end{aligned}
$$

- The desired transfer function must be equal to the closed-loop transfer function of the equivalent system:

$$F(z) = \frac{Y(z)}{R(z)} = \frac{G_c(z)G(z)}{1 + G_c(z)G(z)}$$

- Solving for the controller transfer function $G_c(z)$:

$$G_c(z) = \frac{F(z)}{G(z) - F(z)G(z)} = \frac{F(z)}{G(z)\left[1 - F(z)\right]}$$

- Since the discrete transfer function of the plant $G(z)$ and the desired closed-loop transfer function $F(z)$ are known, the discrete controller transfer function $G_c(z)$ can be determined directly.

The controller design process consists of three steps:

- Determine the desired closed-loop transfer function $F(z)$ so that the design specifications are satisfied.
- Obtain $G_c(z)$ by substituting the selected $F(z)$ into the previously derived expression for the controller.
- Implement the resulting controller.

- Three fundamental aspects must be considered in the controller design:
  - The resulting controller must be physically **realizable**, which means that its transfer function must be causal. In other words, the degree of the numerator must be less than or equal to the degree of the denominator.
  - It is desirable that the controller be as **simple** as possible. Different choices of $F(z)$ may satisfy the same design specifications and lead to different controllers.
  - In practice, exact pole-zero cancellation is not possible. Therefore, it is **not advisable to cancel unstable or critically stable poles or zeros** of the plant, since this may result in an unstable system.

Physical Realizability Constraints

To guarantee physical realizability, the discrete transfer function of the controller must be causal.

- The numerator polynomial degree must be less than or equal to the denominator polynomial degree.
- Suppose the controller transfer function is given by

$$
\begin{aligned}
G_c(z) &= \frac{b_0 + b_1 z^{-1} + \ldots + b_m z^{-m}}{1 + a_1 z^{-1} + \ldots + a_n z^{-n}} \\
&= \frac{F(z)}{G(z)[1 - F(z)]} = \frac{Ng_c(z)}{Dg_c(z)} = \frac{Nf(z)Dg(z)}{Ng(z)\,[Df(z) - Nf(z)]}
\end{aligned}
$$

- The general realizability condition is therefore

$$\deg[Dg_c(z)] \geq \deg[Ng_c(z)]$$

- Expressing $F(z)$, $G(z)$ and $G_c(z)$ in terms of their numerator and denominator polynomials:

$$
\begin{aligned}
F(z) &= \frac{Nf(z)}{Df(z)} \\
G(z) &= \frac{Ng(z)}{Dg(z)} \\
G_c(z) &= \frac{Ng_c(z)}{Dg_c(z)} = \frac{Nf(z)Dg(z)}{Ng(z)\left[Df(z) - Nf(z)\right]}
\end{aligned}
$$

- The realizability condition can be expressed as:

$$\deg[Nf(z)] + \deg[Dg(z)] \leq \deg[Ng(z)] + \deg[Df(z) - Nf(z)]$$

- Besides being physically realizable, the closed-loop system must also be realizable:

$$\deg[Nf(z)] \leq \deg[Df(z)]$$

- Taking this condition into account, the last term of the realizability condition can be simplified:

$$\deg[Nf(z)] + \deg[Dg(z)] \leq \deg[Ng(z)] + \deg[Df(z)]$$

- Rearranging the inequality:

$$\deg[Dg(z)] - \deg[Ng(z)] \leq \deg[Df(z)] - \deg[Nf(z)]$$

- The difference between the denominator and numerator degrees of $F(z)$ (which in discrete-time systems represents the system delay) must be greater than or equal to the delay of the process $G(z)$. In other words, the delay of the model $F(z)$ must be equal to or greater than the delay of the process being controlled.

Simplicity Considerations

$$G_c(z) = \frac{F(z)}{G(z)[1 - F(z)]} = \frac{Nf(z)Dg(z)}{Ng(z)[Df(z) - Nf(z)]}$$

- One might think that since the general expression of the controller transfer function depends on $F(z)$, the simpler $F(z)$ is, the simpler $G_c(z)$ will be.
- This is not necessarily true, since the process transfer function $G(z)$ also appears in the expression and it may be quite complex.
- The controller should be made as simple as the plant transfer function and the design specifications allow. To achieve this, the realizability inequality a common minimal-delay choice is to impose equality:

$$\deg[Dg(z)] - \deg[Ng(z)] = \deg[Df(z)] - \deg[Nf(z)]$$

Stability Constraints

- From the closed-loop transfer function of the plant and controller:

$$F(z) = \frac{G_c(z)G(z)}{1 + G_c(z)G(z)}$$

- Solving for $G_c(z)$:

$$G_c(z) = \frac{1}{G(z)} \frac{F(z)}{1 - F(z)}$$

- The controller transfer function contains two terms:
  - The first term is the inverse of the process transfer function.
  - The second term depends only on the desired closed-loop transfer function $F(z)$.

Stability Constraints

$$G_c(z) = \frac{1}{G(z)} \frac{F(z)}{1 - F(z)}$$

- The first term of the controller may cancel poles and zeros of the process.
- From a mathematical point of view, it would be possible to design the controller so that unstable poles and zeros of the process are cancelled in the open-loop transfer function. However, although such cancellations pose no mathematical difficulty, they are highly problematic in practice because exact cancellation is unlikely. Small modelling errors may cause instability or insufficient damping.
- Therefore, it is necessary to **avoid cancelling process poles or zeros that lie on or outside the unit circle**.

- Let the process transfer function be separated into poles and zeros located inside the unit circle ($Dg_i(z)$, $Ng_i(z)$) and those located on or outside the unit circle ($Dg_e(z)$, $Ng_e(z)$):

$$
\begin{aligned}
Ng(z) &= Ng_e(z)\, Ng_i(z) \\
Dg(z) &= Dg_e(z)\, Dg_i(z)
\end{aligned}
$$

- One way to avoid undesirable cancellations involving unstable (or critically stable) poles and zeros is:
  - Include the unstable or critically stable plant zeros in $N_f(z)$, and the unstable or critically stable plant poles in $D_f(z) - N_f(z)$.
  - In this way, the plant poles and zeros themselves are not cancelled.
  - Exact cancellation with the real process is impossible, but it can be performed safely within the controller because the $G(z)$ used there is the estimated model of the plant rather than the actual plant, which is never known exactly.

Stability Constraints

- Suppose that $F(z)/(1-F(z))$ is chosen such that:

$$\begin{aligned} Nf(z) &= Ng_e(z)\, Nf^*(z) \\ Df(z) - Nf(z) &= Dg_e(z)\, [Df(z) - Nf(z)]^* \end{aligned}$$

  where the superscript $*$ denotes the remaining part of the transfer function after extracting the corresponding factor.
- Substituting these expressions into $G_c(z)$:

$$\begin{aligned} G_c(z) &= \frac{Dg(z)}{Ng(z)} \frac{Nf(z)}{Df(z) - Nf(z)} \\ &= \frac{Dg_e(z)Dg_i(z)}{Ng_e(z)Ng_i(z)} \frac{Ng_e(z)Nf^*(z)}{Dg_e(z)[Df(z) - Nf(z)]^*} \\ &= \frac{Dg_i(z)Nf^*(z)}{Ng_i(z)[Df(z) - Nf(z)]^*} \end{aligned}$$

Stability Constraints

- In the controller expression

$$G_c(z) = \frac{Dg_i(z)Nf^*(z)}{Ng_i(z)[Df(z) - Nf(z)]^*}$$

the unstable or critically stable poles and zeros have been cancelled internally.

- The open-loop transfer function of the plant and controller becomes

$$\begin{aligned} G_c(z)G(z) &= \frac{Dg_i(z)Nf^*(z)}{Ng_i(z)[Df(z) - Nf(z)]^*}\frac{Ng_e(z)Ng_i(z)}{Dg_e(z)Dg_i(z)} \\ &= \frac{Nf^*(z)Ng_e(z)}{[Df(z) - Nf(z)]^*Dg_e(z)} \end{aligned}$$

- Therefore, the unstable or critically stable poles and zeros of the process remain present in the open-loop transfer function, as desired.

- To avoid cancellations between the controller poles and zeros and the unstable or critically stable poles and zeros of the process, the following constraints must be satisfied:
    - The zeros of the numerator of the desired closed-loop transfer function, $F(z)$, must contain the unstable or critically stable zeros of the plant:

$$Nf(z) = Ng_e(z)Nf^*(z)$$

    - The polynomial expression $[Df(z) - Nf(z)]$ must contain among its zeros the unstable or critically stable poles of the process:

$$[Df(z) - Nf(z)] = Dg_e(z)[Df(z) - Nf(z)]^*$$

Example

- Consider a system whose discrete equivalent is given by:

$$G(z) = \frac{(z - 0.5)}{(z - 0.2)(z - 0.9)}$$

- We wish to design a controller such that the closed-loop poles are located at

$$p_{1,2} = 0.3 \pm j\,0.6$$

  and the steady-state error for a unit-step input is zero.
- From the desired pole locations, the closed-loop transfer function $F(z)$ must have the form:

$$F(z) = \frac{Nf(z)}{z^m(z - 0.3 + j0.6)(z - 0.3 - j0.6)} = \frac{Nf(z)}{z^m(z^2 - 0.6z + 0.45)}$$

  where the factor z^m introduces an m-sample delay and adds m poles at the origin, without changing the specified nonzero closed-loop poles.

Example

- To obtain zero steady-state error, an integrator must be introduced through a pole at $(z-1)$ in the open-loop transfer function:

$$G(z)G_c(z) = \frac{F(z)}{1-F(z)} = \frac{Nf(z)}{[Df(z)-Nf(z)]} = \frac{Nf(z)}{(z-1)[Df(z)-Nf(z)]^*}$$

- From the expressions of $F(z)$ and $G(z)G_c(z)$:

$$[Df(z)-Nf(z)] = z^m(z^2-0.6z+0.45)-Nf(z) = (z-1)[Df(z)-Nf(z)]^*$$

- We must determine the value of m and the polynomials $Nf(z)$ and $[Df(z)-Nf(z)]^*$ in order to obtain the controller satisfying the design specifications.

Example

- In this case, the **stability constraint** does not affect the design because all poles and zeros of the process are stable.
- The **physical realizability constraint** requires:

$$\deg[Dg(z)] - \deg[Ng(z)] \leq \deg[Df(z)] - \deg[Nf(z)]$$

that is,

$$2 - 1 \leq (m + 2) - \deg[Nf(z)]$$

and therefore

$$\deg[Nf(z)] \leq m + 1$$

Example

- For **simplicity**, we choose the smallest value of m that guarantees a solution.
  - Let $m = 0$. Then $\deg[Nf(z)] \leq 1$, so
$$Nf(z) = p_0 + p_1 z$$
  - For $m = 0$,
$$\deg[Df(z) - Nf(z)] = 2$$
  - Since
$$[Df(z) - Nf(z)] = (z-1)[Df(z) - Nf(z)]^*,$$
    the polynomial $[Df(z) - Nf(z)]^*$ must be of degree 1.
  - Therefore:
$$(z^2 - 0.6z + 0.45) - (p_0 + p_1 z) = (z-1)(n_0 + n_1 z)$$

Example

- Since the polynomial is of degree 2, we obtain three equations (one per coefficient) and four unknowns (p_0, p_1, n_0, n_1).
- Therefore, there is one more unknown than equations, giving one degree of freedom.
- We add an additional condition to obtain a unique solution, for example:

$$p_1 = 0$$

- Then:

$$(z^2 - 0.6z + 0.45) - p_0 = (z - 1)(n_0 + n_1 z)$$

- Equating coefficients:

$$\begin{aligned} 1 &= n_1 \\ -0.6 &= n_0 - n_1 \\ 0.45 - p_0 &= -n_0 \end{aligned}$$

- Solving yields:

$$p_0 = 0.85, \qquad n_0 = 0.4, \qquad n_1 = 1$$

Example

- Therefore, the desired closed-loop transfer function is:

$$F(z) = \frac{0.85}{z^2 - 0.6z + 0.45}$$

- The resulting controller transfer function is:

$$G_c(z) = \frac{0.85(z - 0.2)(z - 0.9)}{(z - 0.5)(z^2 - 0.6z - 0.4)}$$

- The controller transfer function is considerably more complex than that of a PID controller, even for a relatively simple example such as this one.

Example

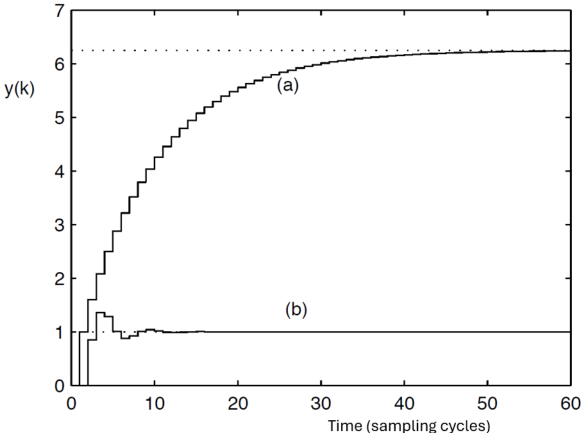

Figure 2: System response: a) open-loop and b) closed-loop with the designed controller.