Control Engineering II

Topic 6:
Controller Design by Direct Synthesis

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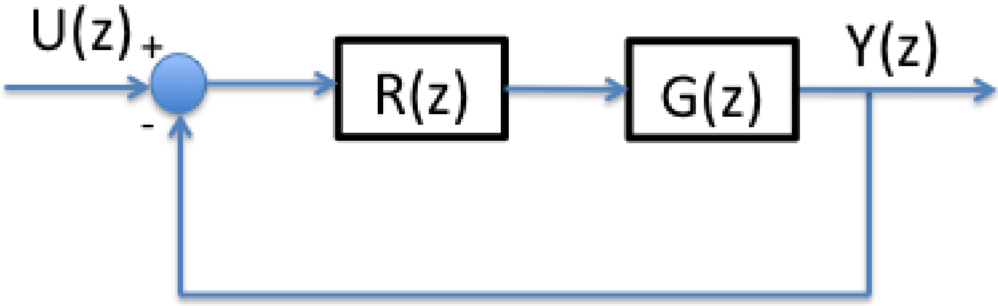
Design a regulator by direct synthesis for the system of the following figure that meets the following specifications: $M_p \leq 10$, $n_s \leq 10$ samples. Consider two situations:

a) Do not impose any condition for the system gain.

b) Impose zero position error



$$G(z) = \frac{z - 0.8}{(z + 0.5)(z - 0.5)}$$

Solution

First, the position of the poles must be calculated according to the given specifications:

$$n_s = \frac{\pi}{\sigma} = 10 \rightarrow \quad \sigma \approx 0.314$$

$$M_p = e^{\frac{-\pi\sigma}{\theta}} = 0.1 \rightarrow \quad \theta = 24.55^o$$

$$p_{1,2} = e^{-\sigma} cos\theta \pm je^{-\sigma} sen\theta$$

$$p_{1,2} = 0.664 \pm j0.304 \rightarrow \quad P(z) = z^2 - 1.328z + 0.533$$

a) The closed-loop transfer function will be:

$$F(z) = \frac{1}{z^2 - 1.328z + 0.533}$$

It must meet the stability and causality conditions:
Causality: The system to be controlled has a difference between the degree of the denominator and that of the numerator equal to 1, and the closed-loop model equal to 2, so this condition is met.

Stability: There are no poles or zeros outside the unit circle, so there are no stability problems.
We calculate the regulator:

$$R(z) = \frac{1}{G(z)} \frac{F(z)}{1 - F(z)} = \frac{(z + 0.5)(z - 0.5)}{(z - 0.8)(z^2 - 1.328z - 0.467)}$$

b) For the system to have zero position error, since it is unit feedback, the closed-loop system gain must be equal to unity. Thus the model will be:

$$F(z) = \frac{K}{z^2 - 1.328z + 0.533}$$
$$F(1) = \frac{K}{1 - 1.328 + 0.533} = 1 \rightarrow K = 0.205$$

The regulator in this case will be:

$$R(z) = \frac{1}{G(z)} \frac{F(z)}{1 - F(z)} = \frac{0.205(z + 0.5)(z - 0.5)}{(z - 0,8)(z^2 - 1.328z + 0.328)}$$

In this section we reach the same result if we make the reasoning described in the transparencies, assuming a pole at -1 in the open-loop function and applying the stability restrictions that say we should not cancel unstable or critically stable poles or zeros.

$$F(z) = \frac{Nf(z)}{z^m(z^2 - 1.328z + 0.533)}$$

As zero position error is requested, the open-loop system must have at least one pole at z=1, and we must apply the stability condition not to cancel unstable or critically stable poles or zeros:

$$R(z)G(z) = \frac{F(z)}{1 - F(z)} = \frac{Nf(z)}{Df(z) - Nf(z)} = \frac{Nf(z)}{(z - 1)[Df(z) - Nf(z)]^*}$$

Equating both expressions:

$$z^m(z^2 - 1.328z + 0.533) - Nf(z) = (z - 1)[Df(z) - Nf(z)]^*$$

For the regulator to be physically realizable:

$$gr[Dg(z)] - gr[Ng(z)] \leq gr[Df(z)] - gr[Nf(z)]$$
$$2 - 1 \leq m + 2 - gr[Nf(z)]$$

Problem 1 (3/3)

Taking, for simplicity: $m = 0 \rightarrow gr[Nf(z)] = 1$

$$Nf(z) = p_0 + p_1 z$$

So it remains: $gr[Df(z) - Nf(z)]^* = 1$

$$[Df(z) - Nf(z)]^* = n_0 + n_1 z$$

Returning to the equating:

$$z^2 - 1.328z + 0.533 - p_0 - p_1 z = (z-1)(n_0 + n_1 z) = n_1 z^2 + (n_0 - n_1)z - n_0$$

Since there is a degree of freedom, we take $p_1 = 0$

$$n_1 = 1 \quad -1.328 = n_0 - n_1 \quad 0.533 - p_0 = -n_0$$

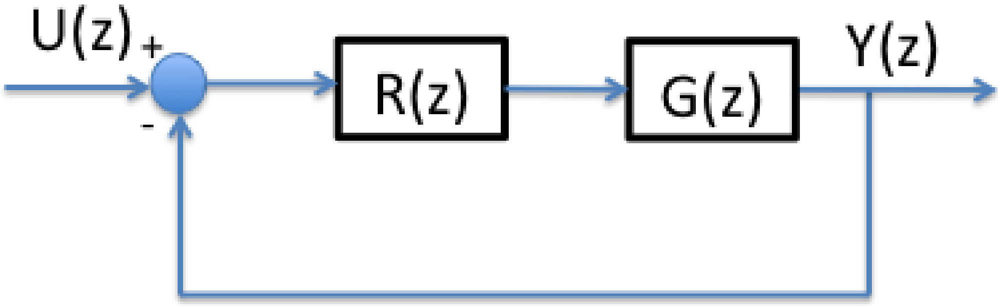
$$n_1 = 1 \quad n_0 = -0.328 \quad p_0 = 0.205$$

$$F(z) = \frac{0.205}{z^2 - 1.328z + 0.533}$$

$$R(z) = \frac{1}{G(z)} \frac{F(z)}{1 - F(z)} = \frac{0.205(z+0.5)(z-0.5)}{(z-0,8)(z^2 - 1.328z + 0.328)}$$

The same solution is reached.

Design a regulator by direct synthesis for the system of the following figure that meets the following specifications: $e_p = 0$**,** $n_s \approx 5$ **and** $n_p \approx 3$ **samples.**



$$G(z) = \frac{(z-0.5)(z-0.1)}{(z-1)(z-1.2)(z-0.6)}$$

Solution

First, the position of the closed-loop poles is calculated from the specifications:

$$n_p = \tfrac{\pi}{\theta} \approx 3 \rightarrow \theta \approx \tfrac{\pi}{3}$$
$$n_s = \tfrac{\pi}{\sigma} \approx 5 \rightarrow \sigma \approx \tfrac{\pi}{5}$$

The dominant poles and the characteristic equation are:

$$p_{1,2} = 0.2667 \pm j0.4620 \quad P(z) = z^2 - 0.53z + 0.28$$

We proceed to design a regulator that meets the specifications.

$$F(z) = \frac{Nf(z)}{z^m(z^2 - 0.53z + 0.28)}$$

As zero position error is requested, the open-loop system must have at least one pole at z=1, and we must apply the stability condition not to cancel unstable or critically stable poles or zeros. Additionally, the open-loop function has an unstable pole at z=1.2:

$$R(z)G(z) = \frac{F(z)}{1-F(z)} = \frac{Nf(z)}{Df(z) - Nf(z)} = \frac{Nf(z)}{(z-1)(z-1.2)[Df(z) - Nf(z)]^*}$$

Equating both expressions:

$$z^m(z^2 - 0.53z + 0.28) - Nf(z) = (z-1)(z-1.2)[Df(z) - Nf(z)]^*$$

For the regulator to be physically realizable:

$$gr[Dg(z)] - gr[Ng(z)] \leq gr[Df(z)] - gr[Nf(z)]$$
$$3 - 2 \leq m + 2 - gr[Nf(z)]$$

Taking, for simplicity: $m = 0 \rightarrow gr[Nf(z)] = 1$

$$Nf(z) = p_0 + p_1 z$$

So it remains: $gr[Df(z) - Nf(z)]^* = 0$

$$[Df(z) - Nf(z)]^* = n_0$$

Returning to the equating:

$$z^2 - 0.53z + 0.28 - p_0 - p_1 z = (z-1)(z-1.2)n_0 = n_0 z^2 - 2.2n_0 z + 1.2n_0$$

It is a system of equations with three equations and three unknowns:

$$n_0 = 1 \quad -(0.53 + p_1) = -2.2n_0 \quad 0.28 - p_0 = 1.2n_0$$
$$n_0 = 1 \quad p_1 = 1.67 \quad p_0 = -0.92$$
$$F(z) = \frac{1.67z - 0.92}{z^2 - 0.53z + 0.28}$$

The sought-after regulator is:

$$R(z) = \frac{1}{G(z)} \frac{F(z)}{1 - F(z)} = \frac{(z-0.6)(1.67z - 0.92)}{(z-0.5)(z-0.1)}$$

We verify that in the open loop unstable poles or zeros are not cancelled and there is a pole at z=1:

$$R(z)G(z) = \frac{1.67z - 0.92}{(z-1)(z-1.2)}$$

Design a regulator by direct synthesis for the unit feedback system and given transfer function to meet the following specifications: $e_p = 0$**,** $M_p = 15$ **and** $n_s \approx 10$ **samples.**

$$G(z) = \frac{(z+0.5)(z-1.1)}{(z+1.2)(z-0.2)(z+0.4)}$$

Solution

First, the position of the closed-loop poles is calculated from the specifications:

$$n_s = \frac{\pi}{\sigma} = 10 \rightarrow \ \sigma \approx 0.314$$
$$M_p = e^{\frac{-\pi\sigma}{\theta}} = 0.15 \rightarrow \ \theta = 29.8076^{o}$$

The dominant poles and the characteristic equation are:

$$p_{1,2} = 0.6339 \pm j0.3631 \quad P(z) = z^2 - 1.268z + 0.5337$$

We proceed to design a regulator that meets the specifications.

$$F(z) = \frac{Nf(z)}{z^m(z^2 - 1.268z + 0.5337)}$$

As zero position error is requested, the open-loop system must have at least one pole at z=1, and we must apply the stability condition not to cancel unstable or critically stable poles or zeros. Additionally, the open-loop function has an unstable pole at z=-1.2 and a zero at z=1.1:

$$R(z)G(z) = \frac{F(z)}{1-F(z)} = \frac{Nf(z)}{Df(z) - Nf(z)} = \frac{(z-1.1)Nf(z)^*}{(z-1)(z+1.2)[Df(z) - Nf(z)]^*?}$$

Equating both expressions:

$$z^m(z^2 - 1.268z + 0.5337) - (z-1.1)Nf(z)^* = (z-1)(z+1.2)[Df(z) - Nf(z)]^*$$

Problem 3 (2/3)

For the regulator to be physically realizable:

$$gr[Dg(z)] - gr[Ng(z)] \leq gr[Df(z)] - gr[Nf(z)]$$
$$2 - 1 \leq m + 2 - gr[Nf(z)]$$

Taking, for simplicity: $m = 0 \rightarrow gr[Nf(z)] = 1, gr[Nf(z)^*] = 0$

$$Nf(z)^* = p_0$$

So it remains: $gr[Df(z) - Nf(z)]^* = 0$

$$[Df(z) - Nf(z)]^* = n_0$$

Returning to the equating:

$$z^2 - 1.268z + 0.5337 - (z - 1.1)\, p_0 = (z - 1)\,(z + 1.2)\, n_0 = n_0 z^2 + 0.2 n_0 z - 1.2 n_0$$

It is a system of equations with three equations and two unknowns that will never be satisfied, so m=0 is not enough for me.
I take m=1:

$$m = 1 \rightarrow gr[Nf(z)] = 2, gr[Nf(z)^*] = 1$$
$$Nf(z)^* = p_0 + p_1 z$$

So it remains: $gr[Df(z) - Nf(z)]^* = 1$

$$[Df(z) - Nf(z)]^* = n_0 + n_1 z$$

Returning to the equating:

$$z(z^2 - 1.268z + 0.5337) - (z - 1.1)(p_0 + p_1 z) = (z - 1)(z + 1.2)(n_0 + n_1 z)$$

Problem 3 (3/3)

It is a system of equations with four equations and four unknowns, so now it can be solved:

$$z^3 - (p_1 + 1.268)z^2 + (0.5337 - p_0 + 1.1p_1)z + 1.1p_0 =$$
$$= n_1 z^3 + (n_0 + 0.2n_1)z^2 + (0.2n_0 - 1.2n_1)z - 1.2n_0$$

$$\left.\begin{aligned} n_1 &= 1 \\ -p_1 - 1.268 &= n_0 + 0.2 \\ 0.5337 - p_0 + 1.1p_1 &= 0.2n_0 - 1.2 \\ 1.1p_0 &= -1.2n_0 \end{aligned}\right\} \rightarrow \begin{aligned} n_1 &= 1 \\ n_0 &= 0.5686 \\ p_1 &= -2.0366 \\ p_0 &= -0.6203 \end{aligned}$$

$$F(z) = \frac{(z-1.1)(-2.0366z - 0.6203)}{z(z^2 - 1.268z + 0.5337)}$$

The sought-after regulator is:

$$R(z) = \frac{1}{G(z)}\frac{F(z)}{1-F(z)} = \frac{(z-0.2)(z+0.4)(-2.0366z-0.6203)}{(z+0.5)(z-1)(z+0.5715)}$$

We verify that in the open loop unstable poles or zeros are not cancelled and there is a pole at z=1:

$$R(z)G(z) = \frac{(-2.0366z - 0.6203)(z-1.1)}{(z-1)(z+1.2)(z+0.5715)}$$

Design a regulator by direct synthesis for the unit feedback system and given transfer function to have zero position error and closed-loop poles at 0.9 and 0.8:

$$G(z) = \frac{(z+1.2)(z+0.6)}{(z-1)(z+0.5)(z-0.3)}$$

Solution

First, the closed-loop characteristic polynomial is calculated. The dominant poles and the characteristic equation are:

$$p_{1,2} = 0.9, 0.8 \quad P(z) = z^2 - 1.7z + 0.72$$

We proceed to design a regulator that meets the specifications.

$$F(z) = \frac{Nf(z)}{z^m(z^2 - 1.7z + 0.72)}$$

As zero position error is requested, the open-loop system must have at least one pole at z=1, and we must apply the stability condition not to cancel unstable or critically stable poles or zeros. Additionally, the open-loop function has an unstable zero at z=1.2:

$$R(z)G(z) = \frac{F(z)}{1-F(z)} = \frac{Nf(z)}{Df(z) - Nf(z)} = \frac{(z+1.2)Nf(z)^*}{(z-1)[Df(z) - Nf(z)]^*}$$

Equating both expressions:

$$z^m(z^2 - 1.7z + 0.72) - (z+1.2)Nf(z)^* = (z-1)[Df(z) - Nf(z)]^*$$

For the regulator to be physically realizable:

$$gr[Dg(z)] - gr[Ng(z)] \leq gr[Df(z)] - gr[Nf(z)]$$
$$2 - 1 \leq m + 2 - gr[Nf(z)]$$

Taking, for simplicity: $m = 0 \rightarrow gr[Nf(z)] = 1, gr[Nf(z)^*] = 0$

$$Nf(z)^* = p_0$$

So it remains: $gr[Df(z) - Nf(z)]^* = 1$

$$[Df(z) - Nf(z)]^* = n_0 + n_1 z$$

Problem 4 (2/3)

Returning to the equating:

$$z^2 - 1.7z + 0.72 - (z+1.2)p_0 = (z-1)(n_0 + n_1 z) = n_1 z^2 + (n_0 - n_1)z - n_0$$

It is a system of equations with three equations and three unknowns:

$$n_1 = 1 \quad -(1.7 + p_0) = n_0 - 1 \quad 0.72 - 1.2p_0 = -n_0$$

$$n_1 = 1 \quad p_0 = 0.0091 \quad n_0 = -0.7091$$

$$F(z) = \frac{0.0091(z+1.2)}{z^2 - 1.7z + 0.72}$$

The sought-after regulator is:

$$R(z) = \frac{1}{G(z)} \frac{F(z)}{1 - F(z)} = 0.0091 \frac{(z+0.5)(z-0.3)}{(z+0.6)(z-0.7091)}$$

We verify that in the open loop unstable poles or zeros are not cancelled and there is a pole at z=1:

$$R(z)G(z) = 0.0091 \frac{(z+1.2)}{(z-1)(z+0.7091)}$$

What would happen if G(z) changes?

$$G(z) = \frac{(z-1.2)(z+0.6)}{(z-1)(z+0.5)(z-0.3)}$$

The reasoning to solve it is the same, but we have the following equation:

$$z^2 - 1.7z + 0.72 - (z-1.2)p_0 = (z-1)(n_0 + n_1 z) = n_1 z^2 + (n_0 - n_1)z - n_0$$

It is a system of equations with three equations and three unknowns:

$$n_1 = 1 \quad -(1.7 + p_0) = n_0 - 1 \quad 0.72 + 1.2p_0 = -n_0$$
$$n_1 = 1 \quad p_0 = -0.1 \quad n_0 = -0.6$$
$$F(z) = \frac{(-0.1)(z-1.2)}{z^2 - 1.7z + 0.72}$$

The sought-after regulator is:

$$R(z) = \frac{1}{G(z)}\frac{F(z)}{1-F(z)} = (-0.1)\frac{(z+0.5)(z-0.3)}{(z+0.6)(z-0.6)}$$

We verify that in the open loop unstable poles or zeros are not cancelled and there is a pole at z=1:

$$R(z)G(z) = (-0.1)\frac{(z-1.2)}{(z-1)(z-0.6)}$$

Careful, in this case we obtain a regulator with negative K, and this must be taken into account if we want to draw the root locus of the system with the regulator to check that it meets the specifications.