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Control Engineering II

Topic 7:
Modeling and Analysis of Systems in State Space

C.A. Monje, D. Copaci, M. Malfaz, J. Muñoz,
L. Mena, L. Moreno, S. Garrido

2026

Contents

Introduction

- **Classical** system analysis methods:
  - Are based on describing a system through the relationship between its input and output.
  - They are very useful for dynamic systems with a single input and a single output, being simple and requiring a relatively small number of calculations.
  - They are difficult to apply to nonlinear systems, except in a few very simple cases.
  - They are also difficult to use in systems with multiple inputs and outputs, or in systems whose parameters vary with time.
- System analysis and modeling methods based on the **state-space representation** are the most appropriate for dealing with multiple-input multiple-output (MIMO) systems, as well as systems with time-varying parameters and even nonlinear systems.

State-Space Modeling

- State-space modeling techniques are based on:
  - Describing the dynamic system by means of n **first-order differential equations** for continuous-time systems, or by means of n **difference equations** for discrete-time systems.
  - These resulting n equations are expressed using **matrix notation**, which greatly simplifies their mathematical representation.

- The idea of a **state** is a fundamental concept in state-space system representation.
- To understand intuitively what a state means, recall that a dynamic system is characterized by the fact that its output at a given instant depends not only on the current inputs but also on the **condition** of the system resulting from the inputs applied in the past.
- This condition of the system at a given instant can be characterized by a set of numerical values.
- This set of numerical values that characterizes the condition of the system at a given instant is what we call the **state of the system** at that instant.

Definition of State

Definition:
*The **state** of a dynamic system at a time instant t_0 can be defined as the **smallest set of numerical values** that is sufficient to determine the future evolution of the system for all $t \geq t_0$, provided that these numerical values and the system inputs for all $t \geq t_0$ are known.*

How Do We Determine the State?

- How can we describe the state of a physical system?
- Which system variables allow us to characterize its state?
- A first approach consists of identifying the variables that *store energy*.
- The energy stored in a system is usually a good indicator of its state.
- Let us consider an example.

Example 1

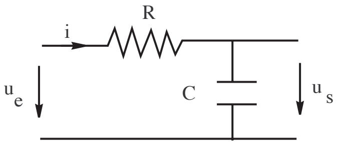


Figure 1: RC circuit.

In the system shown in the figure, the differential equation that characterizes it is

$$\frac{\mathrm{d}u_s}{\mathrm{d}t} = (u_e(t) - u_s(t))\frac{1}{RC}$$

If we take the Laplace transform, we obtain

$$sU_s(s) - u_s(0) = \frac{1}{RC}[U_e(s) - U_s(s)]$$

and, after rearranging,

$$U_s(s) = \frac{1}{\frac{1}{RC} + s}u_s(0) + \frac{\frac{1}{RC}}{\frac{1}{RC} + s}U_e(s)$$

Example 1 (Cont.)

The variable that keeps the memory of the system's energy state is the capacitor voltage u_s, and it will be the state variable of the system. For a unit-step input $u_e(t) = 1$:

$$u_s(t) = u_s(0)e^{\frac{-1}{RC}t} + U_0(1 - e^{\frac{-1}{RC}t})$$

- The formal definition due to L.A. Zadeh is one of the most satisfactory ones for expressing the properties of the concepts of state and state variable.
- Let us consider the system S, that is, the mathematical model of a real system whose inputs are the signals $u_i(t), i = 1, 2, \ldots, r$ and whose outputs are the signals $y_j(t), j = 1, 2, \ldots, m$, which are functions of time.
- The set of input signals will be denoted by the vector $\mathbf{u}(t)$ of dimension $r \times 1$, and the set of output variables by $\mathbf{y}(t)$, of dimension $m \times 1$.

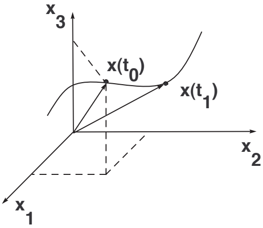


Figure 2: State concept.

The state vector $\mathbf{x}(t)$ can be considered a state variable of system S at time instant t if it satisfies the following conditions:

1. The vectors $\mathbf{x}(t_0)$ and $\mathbf{u}(t, t_0)$ uniquely determine the output $\mathbf{y}(t, t_0)$ for all initial states $\mathbf{x}(t_0) \in X$, for all values of $t \geq t_0$, and for the entire space of input functions of S. That is:

$$\mathbf{y}(t, t_0) = \mathbf{g}(\mathbf{x}(t_0), \mathbf{u}(t, t_0)) \tag{1}$$

2. If t_1 is a time instant between t_0 and t, then for each input vector $\mathbf{u}(t, t_0)$ and observed output vector $\mathbf{y}(t, t_0)$, considered from time instant t_1 so as to define the segments $\mathbf{u}(t, t_1)$ and $\mathbf{y}(t, t_1)$, there exists a non-empty subset of elements of the space X, denoted by $\mathrm{S}[\mathbf{x}(t_0); \mathbf{u}(t, t_0)]$, whose elements α satisfy the following relation:

$$\mathbf{y}(t, t_1) = \mathbf{g}(\alpha, \mathbf{u}(t, t_1)) \tag{2}$$

3. If $\mathbf{u}(t_1, t_0)$ is fixed and $\mathbf{u}(t, t_1)$ varies over the entire input-function space of S, then the intersection of the sets $\mathrm{S}[\mathbf{x}(t_0); \mathbf{u}(t, t_0)]$, considered over all values of $\mathbf{u}(t, t_1)$, is a non-empty set.

- C1. Implies that the system is **deterministic and causal**, therefore the values of the output signals at a given time instant do not depend on future inputs.
- C2. Ensures that each pair $\mathbf{u}(t, t_1)$, $\mathbf{y}(t, t_1)$ corresponds to an initial state $\mathbf{x}(t_1)$ in X.
- C3. This condition guarantees the existence of at least one state in the space X, which is associated with all possible pairs of inputs $\mathbf{u}(t, t_1)$ and outputs $\mathbf{y}(t, t_1)$, respectively.
- From C2 and C3 it follows that the state of S at time instant t is determined by the initial state $\mathbf{x}(t_0)$ and the input vector $\mathbf{u}(t, t_0)$:

$$\mathbf{x}(t) = \mathbf{f}(\mathbf{x}(t_0), \mathbf{u}(t, t_0)) \tag{3}$$

where $\mathbf{f}()$ is a unique function of its arguments.

- If a system can be described by a set of first-order differential equations or difference equations, the n state equations of an n-th order dynamic system are:

$$\begin{cases} \dot{x}_1(t) = f_1(x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_r(t)) \\ \dot{x}_2(t) = f_2(x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_r(t)) \\ \quad\vdots \\ \dot{x}_n(t) = f_n(x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_r(t)) \end{cases} \tag{4}$$

and the output equations of the system are

$$\begin{cases} y_1(t) = g_1(x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_r(t)) \\ y_2(t) = g_2(x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_r(t)) \\ \quad\vdots \\ y_m(t) = g_m(x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_r(t)) \end{cases} \tag{5}$$

Matrix Representation of State Equations

Compact Form

- If we express the previous systems of equations in matrix form, where $\mathbf{x}(t)$ is an $n \times 1$ vector, $\mathbf{u}(t)$ is an $r \times 1$ vector, and $\mathbf{y}(t)$ is an $m \times 1$ vector, we obtain

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \tag{6}$$
$$\mathbf{y}(t) = \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t)) \tag{7}$$

- If the **system is linear**, then the state equations can be written as

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \tag{8}$$
$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t) \tag{9}$$

  where the matrices $\mathbf{A}(t)$, $\mathbf{B}(t)$, $\mathbf{C}(t)$ and $\mathbf{D}(t)$ are, in the general case, time-varying matrices with dimensions $\mathbf{A}(t) : n \times n$, $\mathbf{B}(t) : n \times r$, $\mathbf{C}(t) : m \times n$, and $\mathbf{D}(t) : m \times r$.

Matrix Representation of State Equations

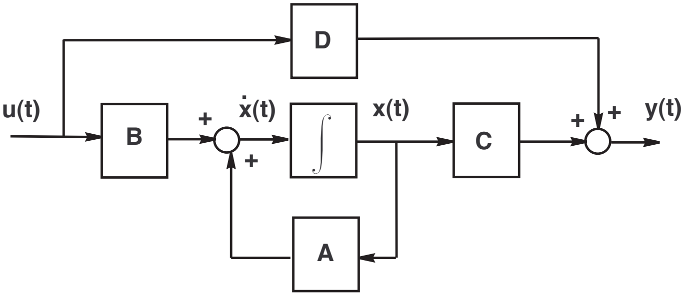


Figure 3: Block diagram of a continuous-time system represented in state-space form.

- If the system is defined by difference equations, the expressions of the state and output equations are

$$\mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k), \mathbf{u}(k)) \quad (10)$$
$$\mathbf{y}(k) = \mathbf{g}(\mathbf{x}(k), \mathbf{u}(k)) \quad (11)$$

- If the system is linear or is linearized around an equilibrium point, we have

$$\mathbf{x}(k+1) = \mathbf{G}(k)\mathbf{x}(k) + \mathbf{H}(k)\mathbf{u}(k) \quad (12)$$
$$\mathbf{y}(k) = \mathbf{C}(k)\mathbf{x}(k) + \mathbf{D}(k)\mathbf{u}(k) \quad (13)$$

Construction of the State-Space Model of a Physical System

Given a dynamic physical system, the steps to formulate a state-space model are the following:

1. Decompose the system into its basic elements so that its components, variables, inputs, outputs, and the relationships among them can be identified.
2. Selection of the system state variables, which as a first approximation can be carried out by observing the dynamic elements of the system. The number of variables of a system must be equal to the order of the differential equations that describe it.
3. Derivation of the state equations for each independent variable selected in the previous step.
4. Derivation of the system output equations from the output variables to be controlled and the selected state variables.

- Let us consider a filter built with passive elements, namely inductors and capacitors, as shown in the figure, and we want to obtain its state-space model.

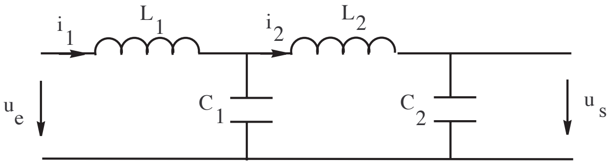


Figure 4: 4th-order Butterworth filter.

- **Solution:**
  According to Kirchhoff's laws, the system equations are:

$$
\begin{aligned}
u_e &= L_1 \frac{\mathrm{d}i_1}{\mathrm{d}t} + u_{C_1} \\
u_{C_1} &= L_2 \frac{\mathrm{d}i_2}{\mathrm{d}t} + u_{C_2} \\
C_1 \frac{\mathrm{d}u_{C_1}}{\mathrm{d}t} &= i_1 - i_2 \\
C_2 \frac{\mathrm{d}u_{C_2}}{\mathrm{d}t} &= i_2 \\
u_s &= u_{C_2}
\end{aligned}
\tag{14}
$$

- The choice of the state variables is quite clear, since the variables that store energy in the system are the capacitor voltages and the currents through the inductors.

- Taking as state variables the capacitor voltages and the currents flowing through the inductors, we obtain

$$
\begin{aligned}
\frac{\mathrm{d}i_1}{\mathrm{d}t} &= \frac{1}{L_1}u_e - \frac{1}{L_1}u_{C_1} \\
\frac{\mathrm{d}i_2}{\mathrm{d}t} &= \frac{1}{L_2}u_{C_1} - \frac{1}{L_2}u_{C_2} \\
\frac{\mathrm{d}u_{C_1}}{\mathrm{d}t} &= \frac{1}{C_1}i_1 - \frac{1}{C_1}i_2 \\
\frac{\mathrm{d}u_{C_2}}{\mathrm{d}t} &= \frac{1}{C_2}i_2
\end{aligned}
\tag{15}
$$

and the output is

$$u_s = u_{C_2}$$

In other words,

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} i_1 \\ i_2 \\ u_{C_1} \\ u_{C_2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\frac{1}{L_1} & 0 \\ 0 & 0 & \frac{1}{L_2} & -\frac{1}{L_2} \\ \frac{1}{C_1} & -\frac{1}{C_1} & 0 & 0 \\ 0 & \frac{1}{C_2} & 0 & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ u_{C_1} \\ u_{C_2} \end{bmatrix} + \begin{bmatrix} \frac{1}{L_1} \\ 0 \\ 0 \\ 0 \end{bmatrix} u_e \tag{16}$$

and the output equation is

$$u_s = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ u_{C_1} \\ u_{C_2} \end{bmatrix} \tag{17}$$

Linearization of Systems in State Space

- Assuming a given operating point of the system $(\mathbf{x} = \mathbf{x}_0, \mathbf{u} = \mathbf{u}_0)$, a linear approximation of the system in the neighborhood of that operating point is obtained from the Taylor series expansion.
- The Taylor series expansion for a multivariable system defined by the system of equations

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \tag{18}$$

$$\mathbf{y}(t) = \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t)) \tag{19}$$

at the operating point $(\mathbf{x} = \mathbf{x}_0, \mathbf{u} = \mathbf{u}_0)$, can be expressed as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathbf{x}}(t) = \mathbf{A}\tilde{\mathbf{x}}(t) + \mathbf{B}\tilde{\mathbf{u}}(t) \tag{20}$$

$$\tilde{\mathbf{y}}(t) = \mathbf{C}\tilde{\mathbf{x}}(t) + \mathbf{D}\tilde{\mathbf{u}}(t) \tag{21}$$

where $\tilde{\mathbf{x}} = \mathbf{x} - \mathbf{x}_0$, $\tilde{\mathbf{u}} = \mathbf{u} - \mathbf{u}_0$, $\tilde{\mathbf{y}} = \mathbf{y} - \mathbf{y}_0$, and the matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{D}$ are the Jacobians of $\mathbf{f}$ and $\mathbf{g}$ with respect to $\mathbf{x}$ and $\mathbf{u}$, defined by the following expressions.

- Jacobians of **f** and **g** with respect to **x** and **u**:

$$\mathbf{A} = \nabla f_x|_{\mathbf{x}_0,\mathbf{u}_0} \quad (22)$$
$$\mathbf{B} = \nabla f_u|_{\mathbf{x}_0,\mathbf{u}_0} \quad (23)$$
$$\mathbf{C} = \nabla g_x|_{\mathbf{x}_0,\mathbf{u}_0} \quad (24)$$
$$\mathbf{D} = \nabla g_u|_{\mathbf{x}_0,\mathbf{u}_0} \quad (25)$$

and recalling the expression of the Jacobian matrix

$$\nabla f_x = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \quad (26)$$

we obtain the expressions of **A**, **B**, **C** and **D** for the linearized model.

- A simplified state-space model for a drying oven is given by the following set of state equations:

$$
\begin{aligned}
\dot{x}_1 &= -a_1 x_2 \\
\dot{x}_2 &= b_1 x_1 - b_2 x_2 - b_3 u_1 \\
\dot{x}_3 &= c_1 x_2 + c_2 u_2 (T_2 - x_3) + c_3 x_2 x_3
\end{aligned}
\tag{27}
$$

  and the system is to be linearized.
- **Solution:** Substituting the equations into the Jacobian expressions, we obtain

$$
\begin{aligned}
\mathbf{A} &= \nabla f_x \big|_{\mathbf{x}_0, \mathbf{u}_0} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{bmatrix} \\
&= \begin{bmatrix} 0 & -a_1 & 0 \\ b_1 & -b_2 & 0 \\ 0 & (c_1 + c_3 x_{3,0}) & (c_3 x_{2,0} - c_2 u_{2,0}) \end{bmatrix}
\end{aligned}
\tag{28}
$$

- and for **B**

$$
\begin{aligned}
\mathbf{B} &= \nabla f_u|_{\mathbf{x}_0,\mathbf{u}_0} = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} \\ \frac{\partial f_3}{\partial u_1} & \frac{\partial f_3}{\partial u_2} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ -b_3 & 0 \\ 0 & c_2(T_2 - x_{3,0}) \end{bmatrix} \qquad (29)
\end{aligned}
$$

- Both the state-space representation and the transfer function of a system are two ways of mathematically modeling the behavior of a system.
- Both models are related. Suppose that we have obtained the state-space representation of a system, and that this system is linear and time-invariant

$$\frac{d}{dt}\mathbf{x}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \qquad (30)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \qquad (31)$$

where the matrices **A**, **B**, **C** and **D** have dimensions $\mathbf{A} : n \times n$, $\mathbf{B} : n \times r$, $\mathbf{C} : m \times n$ and $\mathbf{D} : m \times r$. If we take the Laplace transform of the state-space equations, we obtain

$$s\mathbf{X}(s) - x(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s) \qquad (32)$$

$$\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s) \qquad (33)$$

Transfer Function and State-Space Representation

- The transfer function provides the relationship between the input and the output of a system. Therefore, we must relate the system input and output. To do so,

$$\begin{aligned} s\mathbf{X}(s) - \mathbf{A}\mathbf{X}(s) &= \mathbf{B}\mathbf{U}(s) + x(0) \\ (s\mathbf{I} - \mathbf{A})\mathbf{X}(s) &= \mathbf{B}\mathbf{U}(s) + x(0) \\ \mathbf{X}(s) &= (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) + (s\mathbf{I} - \mathbf{A})^{-1}x(0) \end{aligned} \tag{34}$$

and substituting into the system output equation, we obtain

$$\begin{aligned} \mathbf{Y}(s) &= \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) + \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}x(0) + \mathbf{D}\mathbf{U}(s) \\ &= [\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}]\mathbf{U}(s) + (s\mathbf{I} - \mathbf{A})^{-1}x(0) \end{aligned} \tag{35}$$

- If we assume zero initial conditions, the transfer function of the system is

$$\mathbf{F}(s) = \frac{\mathbf{Y}(s)}{\mathbf{U}(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} \tag{36}$$

Since

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{|s\mathbf{I} - \mathbf{A}|} Adj(s\mathbf{I} - \mathbf{A})^{T}$$

it follows that the characteristic polynomial of the system is $|s\mathbf{I} - \mathbf{A}|$, and therefore the poles of the system coincide with the eigenvalues of the matrix **A**.

State-Space Representation of Systems

- In general, the system equations are not available in a form that allows the state-space model to be obtained directly.
- A system may be described by one or several differential equations, either first-order or higher-order, linear or nonlinear.
- In many physical systems, $\mathbf{D} = \mathbf{0}$, since there is no direct feedthrough from the input to the output.
- It is necessary to develop a method for converting ordinary differential equations or difference equations into first-order differential or difference equations.

Conversion of an Ordinary Differential Equation into State Equations

- The choice of state variables is not unique; therefore, it is not sufficient to present only one solution, since in some cases a particular choice of state variables may offer advantages over another.
- We will study the most common methods for selecting state variables and the types of representations they produce. These representations are known as *canonical forms*, including:
  - Controllable canonical form.
  - Observable canonical form.
  - Jordan canonical form.

Controllable Canonical Form

- In this representation method, the i-th state variable is chosen as the derivative of the $(i-1)$-th state variable.
- This choice is quite natural in many systems. For example, in mechanical systems, acceleration is the derivative of velocity, and velocity is the derivative of position (these are sometimes referred to as *phase variables*).
- Consider a time-invariant system with a single input signal u and a single output signal y, whose dynamics are described by a linear differential equation with constant coefficients

$$a_n \frac{d^n}{dt^n} y(t) + \ldots + a_1 \frac{d}{dt} y(t) + a_0 y(t) = b_m \frac{d^m}{dt^m} u(t) + \ldots + b_1 \frac{d}{dt} u(t) + b_0 u(t) \tag{37}$$

with $n \geq m$.

- If we use the differential operator $p = \frac{d}{dt}$, the previous expression becomes

$$a_n p^n y(t) + \ldots + a_1 p y(t) + a_0 y(t) = b_m p^m u(t) + \ldots + b_1 p u(t) + b_0 u(t) \quad (38)$$

Factoring out $y(t)$ and $u(t)$ on both sides, we obtain

$$(a_n p^n + \ldots + a_1 p + a_0) y(t) = (b_m p^m + \ldots + b_1 p + b_0) u(t) \quad (39)$$

and, more compactly,

$$M(p) y(t) = N(p) u(t) \quad (40)$$

where

$$\begin{aligned} M(p) &= a_n p^n + \ldots + a_1 p + a_0 & (41) \\ N(p) &= b_m p^m + \ldots + b_1 p + b_0 & (42) \end{aligned}$$

thus,

$$y(t) = \frac{N(p)}{M(p)} u(t) \quad (43)$$

Controllable Canonical Form

- If we define the first state variable as

$$x(t) = \frac{1}{M(p)}u(t) \tag{44}$$

it can be observed that equation 43 has been replaced by the following two expressions

$$\begin{aligned} x(t) &= \frac{1}{M(p)}u(t) && (45) \\ y(t) &= N(p)x(t) && (46) \end{aligned}$$

- Defining the state variables as follows

$$\begin{aligned} x(t) &= x_1(t) \\ \frac{\mathrm{d}}{\mathrm{d}t}x(t) &= x_2(t) = \frac{d}{dt}x_1(t) \\ &\vdots \\ \frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}}x(t) &= x_n(t) = \frac{d}{dt}x_{n-1}(t) && (47) \end{aligned}$$

Controllable Canonical Form

- Equation 45 can be rewritten as

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}x(t) = \frac{1}{a_n}u(t) - \frac{a_{n-1}}{a_n}x_n(t) - \ldots - \frac{a_1}{a_n}x_2(t) - \frac{a_0}{a_n}x_1(t) \qquad (48)$$

- Equations 47 and 48 can be expressed in matrix form as

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_{n-1}(t) \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ & \vdots & & & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -\frac{a_0}{a_n} & -\frac{a_1}{a_n} & -\frac{a_2}{a_n} & \ldots & -\frac{a_{n-1}}{a_n} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \frac{1}{a_n} \end{bmatrix} u(t) \qquad (49)$$

Controllable Canonical Form

- To obtain the output equation, we start from equation 46. Using the definition of the state variables 47 and assuming that $n = m$, we obtain

$$b_0 x_1(t) + b_1 x_2(t) + \ldots + b_{n-1} x_n(t) + b_n \frac{d}{dt} x_n(t) = y(t) \tag{50}$$

and substituting $\frac{d}{dt} x_n$ in this equation by 48, we obtain the following expression

$$\begin{aligned} y(t) = (b_0 - b_n \frac{a_0}{a_n}) x_1(t) + (b_1 - b_n \frac{a_1}{a_n}) x_2(t) + \ldots \\ \ldots + (b_{n-1} - b_n \frac{a_{n-1}}{a_n}) x_n(t) + \frac{b_n}{a_n} u(t) \end{aligned} \tag{51}$$

which, in matrix form, becomes

$$y(t) = \begin{bmatrix} b_0 - b_n \frac{a_0}{a_n} & b_1 - b_n \frac{a_1}{a_n} & \ldots & b_{n-1} - b_n \frac{a_{n-1}}{a_n} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \frac{b_n}{a_n} u(t) \tag{52}$$

Controllable Canonical Form

- This representation is known as the *controllable canonical form*. A common simplification is obtained when $a_n = 1$. The figure shows a block diagram of this representation.

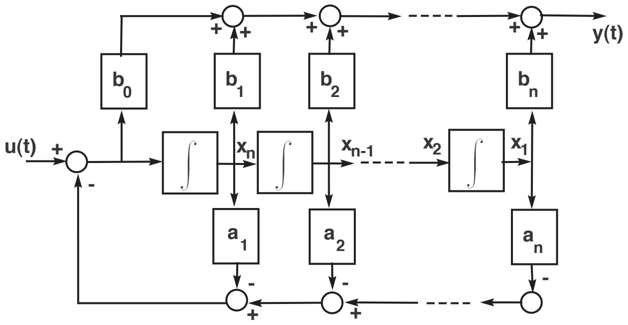


Figure 5: Graphical representation of the controllable canonical form.

Observable Canonical Form

State variables are chosen as a linear combination of the derivatives of the input and output variables.

- The state variables are selected such that x_n is a function of $y(t)$ and $u(t)$, x_{n-1} is a function of $y(t)$, $u(t)$ and their derivatives, and so on:

$$\begin{aligned}
x_n(t) &= a_n y(t) - b_n u(t) \\
x_{n-1}(t) &= a_{n-1} y(t) + a_n \frac{\mathrm{d}}{\mathrm{d}t} y(t) - b_{n-1} u(t) - b_n \frac{\mathrm{d}}{\mathrm{d}t} u(t) \\
x_{n-2}(t) &= a_{n-2} y(t) + a_{n-1} \frac{\mathrm{d}}{\mathrm{d}t} y(t) + a_n \frac{\mathrm{d}^2}{\mathrm{d}t^2} y(t) - \\
&\quad - b_{n-2} u(t) - b_{n-1} \frac{\mathrm{d}}{\mathrm{d}t} u(t) - b_n \frac{\mathrm{d}^2}{\mathrm{d}t^2} u(t) \\
&\dots \\
x_1(t) &= a_1 y(t) + a_2 \frac{\mathrm{d}}{\mathrm{d}t} y(t) + \ldots + a_n \frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}} y(t) - \\
&\quad - b_1 u(t) - b_2 \frac{\mathrm{d}}{\mathrm{d}t} u(t) - \ldots - b_n \frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}} u(t) \qquad (53)
\end{aligned}$$

- From the first equation of 53, we obtain $y(t)$ (output equation):

$$y(t) = \frac{1}{a_n} x_n(t) + \frac{b_n}{a_n} u(t) \tag{54}$$

- If we differentiate the system equations 53, we obtain:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}x_n(t) &= a_n\frac{\mathrm{d}}{dt}y(t) - b_n\frac{\mathrm{d}}{\mathrm{d}t}u(t)\\
\frac{\mathrm{d}}{\mathrm{d}t}x_{n-1}(t) &= a_{n-1}\frac{\mathrm{d}}{\mathrm{d}t}y(t) + a_n\frac{\mathrm{d}^2}{\mathrm{d}t^2}y(t) - b_{n-1}\frac{\mathrm{d}}{\mathrm{d}t}u(t) - b_n\frac{\mathrm{d}^2}{\mathrm{d}t^2}u(t)\\
&\cdots\\
\frac{\mathrm{d}}{\mathrm{d}t}x_2(t) &= a_2\frac{\mathrm{d}}{\mathrm{d}t}y(t) + a_3\frac{\mathrm{d}^2}{\mathrm{d}t^2}y(t) + \ldots + a_n\frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}}y(t) -\\
&\quad -b_2\frac{d}{dt}u(t) - b_3\frac{\mathrm{d}^2}{\mathrm{d}t^2}u(t) - \ldots - b_n\frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}}u(t)\\
\frac{\mathrm{d}}{\mathrm{d}t}x_1(t) &= a_1\frac{\mathrm{d}}{\mathrm{d}t}y(t) + a_2\frac{d^2}{dt^2}y(t) + \ldots + a_n y^{n)}(t) -\\
&\quad -b_1\frac{d}{dt}u(t) - b_2\frac{d^2}{dt^2}u(t) - \ldots - b_n\frac{d^n}{dt^n}u(t) \qquad (55)
\end{aligned}
$$

- Substituting equations 55 into the system of equations 53, we obtain:

$$
\begin{aligned}
x_n(t) &= a_n y(t) - b_n u(t) \\
x_{n-1}(t) &= a_{n-1} y(t) - b_{n-1} u(t) + \frac{d}{dt} x_n(t) \\
x_{n-2}(t) &= a_{n-2} y(t) - b_{n-2} u(t) + \frac{d}{dt} x_{n-1}(t) \\
&\cdots \\
x_1(t) &= a_1 y(t) - b_1 u(t) + \frac{d}{dt} x_2(t)
\end{aligned}
\tag{56}
$$

- If we now substitute $y(t)$ using the output equation 54, we obtain $n-1$ state equations. The only missing state equation is the one corresponding to $\dot{x}_1(t)$:

$$\begin{aligned}
x_{n-1}(t) &= \frac{a_{n-1}}{a_n}x_n(t) - (b_{n-1} - b_n\frac{a_{n-1}}{a_n})u(t) + \frac{d}{dt}x_n(t) \\
x_{n-2}(t) &= \frac{a_{n-2}}{a_n}x_n(t) - (b_{n-2} - b_n\frac{a_{n-2}}{a_n})u(t) + \frac{d}{dt}x_{n-1}(t) \\
&\vdots \\
x_1(t) &= \frac{a_1}{a_n}x_n(t) - (b_1 - b_n\frac{a_1}{a_n})u(t) + \frac{d}{dt}x_2(t)
\end{aligned} \tag{57}$$

- To obtain the missing state equation, we combine equations 57 with the original differential equation expressed in terms of the state variables that we have defined:

$$0 = \frac{a_0}{a_n}x_n(t) - (b_0 - b_n\frac{a_0}{a_n})u(t) + \frac{d}{dt}x_1(t) \tag{58}$$

- Writing equations 57 and 58 in matrix form, we obtain the set of state equations and, from equation 54, the output equation:

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 & -\frac{a_0}{a_n} \\ 1 & 0 & \dots & 0 & 0 & -\frac{a_1}{a_n} \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & -\frac{a_{n-2}}{a_n} \\ 0 & 0 & \dots & 0 & 1 & -\frac{a_{n-1}}{a_n} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix} + \begin{bmatrix} b_0 - b_n\frac{a_0}{a_n} \\ b_1 - b_n\frac{a_1}{a_n} \\ \vdots \\ b_{n-2} - b_n\frac{a_{n-2}}{a_n} \\ b_{n-1} - b_n\frac{a_{n-1}}{a_n} \end{bmatrix} u(t) \tag{59}$$

$$y(t) = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 & \frac{1}{a_n} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix} + \frac{b_n}{a_n} u(t) \tag{60}$$

- The figure shows a block diagram of this representation:

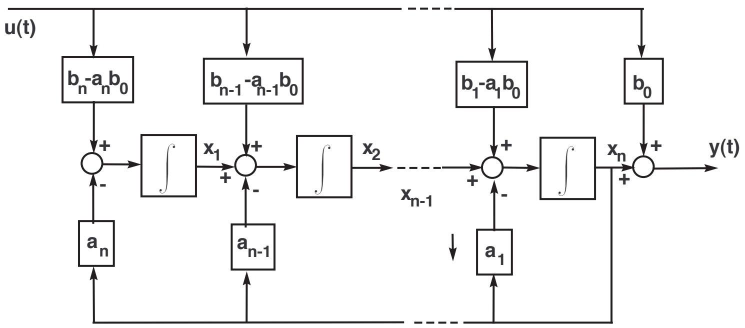


Figure 6: Graphical representation of the observable canonical form.

The choice of state variables is based on the superposition principle of linear systems; that is, the goal is to decompose the response of a linear system into its individual components.

- Consider a system whose transfer function is

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + \ldots + b_1 s + b_0}{a_n s^n + \ldots + a_1 s + a_0} \tag{61}$$

  where $m \leq n$.
- Assume that the denominator of the transfer function has no repeated roots. Then, by performing a partial-fraction decomposition, we obtain

$$\begin{aligned} G(s) &= \frac{b_m s^m + \ldots + b_1 s + b_0}{a_n(s-\lambda_1)(s-\lambda_2)\ldots(s-\lambda_n)} \\ &= \frac{b_n}{a_n} + \frac{c_1}{s-\lambda_1} + \frac{c_2}{s-\lambda_2} + \ldots + \frac{c_n}{s-\lambda_n} \end{aligned} \tag{62}$$

- Equation 62 is a linear combination of n elementary components plus a term directly related to the input, which appears when $n = m$:

$$Y(s) = \frac{c_1}{s-\lambda_1}U(s) + \frac{c_2}{s-\lambda_2}U(s) + \ldots + \frac{c_n}{s-\lambda_n}U(s) + \frac{b_n}{a_n}U(s) \quad (63)$$

- Each of these elementary components in the previous equation is the response of a system with transfer function

$$G_i(s) = \frac{1}{s-\lambda_i} \quad (64)$$

to the input signal $U(s)$. This implies that if each of these elementary components is taken as a state variable $X_i(s)$ satisfying $X_i(s) = G_i(s)U(s)$, then the state variable $x_i(t)$ defined in this way satisfies the following differential equation:

$$\frac{d}{dt}x_i(t) = \lambda_i x_i(t) + u(t) \quad (65)$$

- Therefore, we have n differential equations of the form 65, which constitute the state equations, and an output equation that in the time domain takes the form

$$y(t) = c_1 x_1(t) + c_2 x_2(t) + \ldots + c_n x_n(t) + \frac{b_n}{a_n} u(t) \tag{66}$$

and, writing the equations in matrix form, we obtain:

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u(t) \tag{67}$$

$$y(t) = [c_1 \; c_2 \; \ldots \; c_n] \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \frac{b_n}{a_n} u(t) \tag{68}$$

Jordan Canonical Form

- If all the roots of the denominator in the Jordan canonical form are simple, it is known as the *diagonal canonical form*. The figure shows a graphical representation of the diagonal canonical form of a system.

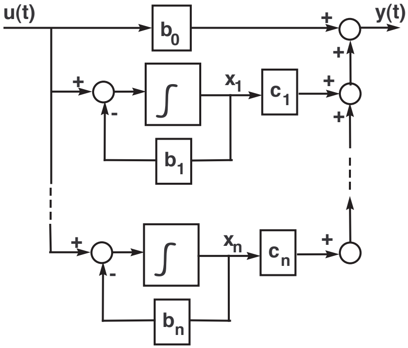


Figure 7: Graphical representation of the diagonal canonical form.

Jordan Canonical Form

- If the characteristic equation of the system has repeated roots, the partial-fraction decomposition leads to an expression of the form

$$
\begin{aligned}
G(s) &= \frac{b_m s^m + \ldots + b_1 s + b_0}{a_n (s-\lambda_1)^k (s-\lambda_2) \ldots (s-\lambda_n)} \\
&= \frac{c_1}{(s-\lambda_1)^k} + \ldots + \frac{c_k}{s-\lambda_1} + \frac{c_{k+1}}{s-\lambda_2} + \ldots + \frac{c_n}{s-\lambda_n}
\end{aligned}
\tag{69}
$$

- Assuming zero initial conditions, the response of the system to an input $U(s)$ is

$$
Y(s) = \frac{c_1}{(s-\lambda_1)^k} U(s) + \ldots + \frac{c_k}{s-\lambda_1} U(s) + \frac{c_{k+1}}{s-\lambda_2} U(s) + \ldots + \frac{c_n}{s-\lambda_n} U(s)
\tag{70}
$$

- Taking each of the fractions appearing in equation 70 as state variables, we obtain

$$
\begin{aligned}
X_1(s) &= \frac{c_1}{(s-\lambda_1)^k}U(s) = \frac{c_1}{s-\lambda_1}X_2(s) \\
X_2(s) &= \frac{c_2}{(s-\lambda_1)^{k-1}}U(s) = \frac{c_1}{s-\lambda_1}X_3(s) \\
&\vdots \\
X_k(s) &= \frac{c_k}{s-\lambda_1}U(s) \\
X_{k+1}(s) &= \frac{c_{k+1}}{s-\lambda_2}U(s) \\
&\vdots \\
X_n(s) &= \frac{c_n}{s-\lambda_n}U(s)
\end{aligned}
\tag{71}
$$

- Rewriting the previous equations 71 in the time domain, the system is expressed as a set of first-order differential equations

$$\begin{aligned}
\frac{d}{dt}x_1(t) &= \lambda_1 x_1(t) + x_2(t) \\
\frac{d}{dt}x_2(t) &= \lambda_1 x_2(t) + x_3(t) \\
&\vdots \\
\frac{d}{dt}x_k(t) &= \lambda_1 x_k(t) + u(t) \\
\frac{d}{dt}x_{k+1}(t) &= \lambda_2 x_{k+1}(t) + u(t) \\
&\vdots \\
\frac{d}{dt}x_n(t) &= \lambda_n x_n(t) + u(t)
\end{aligned} \tag{72}$$

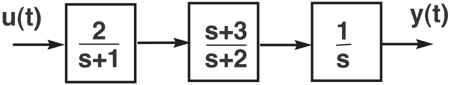
Jordan Canonical Form
Cont.

- Writing this set of equations in matrix form:

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_k(t) \\ x_{k+1}(t) \\ \vdots \\ x_n(t) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 1 & \dots & 0 & 0 & \dots & 0 \\ 0 & \lambda_1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \lambda_1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & \lambda_2 & \dots & 0 \\ \vdots & & & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & \lambda_n \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_k(t) \\ x_{k+1}(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u(t) \tag{73}$$

$$y(t) = [c_1 \ c_2 \ \dots \ c_k \ c_{k+1} \ \dots \ c_n] \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_k(t) \\ x_{k+1}(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \frac{b_n}{a_n} u(t) \tag{74}$$

Given the system shown in the figure, obtain the Jordan, controllable, and observable canonical forms.



Solution:

- *Jordan canonical form.* First, we obtain the overall transfer function of the system

$$F(s) = \frac{2(s+3)}{s(s+1)(s+2)} = \frac{-4}{s+1} + \frac{1}{s+2} + \frac{3}{s}$$

and from this, by taking the partial fractions as state variables $X_1(s) = \frac{-4}{s+1}U(s)$, $X_2(s) = \frac{1}{s+2}U(s)$ and $X_3(s) = \frac{3}{s}U(s)$, we obtain the following state-space representation

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = [-4 \ 1 \ 3] \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

- *Controllable canonical form*. Starting from the transfer function $F(s)$ of the system

$$F(s) = \frac{2(s+3)}{s(s+1)(s+2)} = \frac{2s+6}{s^3+3s^2+2s}$$

we obtain that the coefficients of the denominator polynomial are $a_3 = 1,\ a_2 = 3,\ a_1 = 2,\ a_0 = 0$ and those of the numerator are $b_1 = 2,\ b_0 = 6$. Substituting these coefficients into the controllable canonical form expression, we obtain

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = [6\ 2\ 0] \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

- *Observable canonical form*. Starting from the coefficients of the numerator and denominator polynomials of the transfer function obtained previously and substituting them into the observable canonical form expression, we arrive at

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -2 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 6 \\ 2 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

Conversion of a Difference Equation into State Equations

A process similar to that used for systems described by differential equations, but working from the *z*-transform or the difference equation itself, and using the delay operator to express the state equations in polynomial form.

- Given a time-invariant system with a single input u and a single output signal y, whose dynamics are described by a linear difference equation with constant coefficients

$$a_n y(k-n)+\ldots+a_1 y(k-1)+a_0 y(k) = b_m u(k-m)+\ldots+b_1 u(k-1)+b_0 u(k) \tag{75}$$

  with $n \geq m$.
- If we introduce the delay operator q^{-1}, the previous expression can be rewritten as

$$\begin{gathered} a_n q^{-n} y(k) + \ldots + a_1 q^{-1} y(k) + a_0 y(k) = \\ b_m q^{-m} u(k) + \ldots + b_1 q^{-1} u(k) + b_0 u(k) \end{gathered} \tag{76}$$

Conversion of a Difference Equation into State Equations

Cont.

- Factoring out $y(k)$ and $u(k)$ on both sides of the expression, we obtain:

$$(a_n q^{-n} + \ldots + a_1 q^{-1} + a_0) y(k) = (b_m q^{-m} + \ldots + b_1 q^{-1} + b_0) u(k) \tag{77}$$

- which can be expressed more compactly as a ratio of polynomials of the form

$$y(k) = \frac{N(q)}{M(q)} u(k) \tag{78}$$

- From this point, it is possible to follow a development similar to that used for continuous-time systems, leading to the same expressions.

Conversion of a Difference Equation into State Equations

Example

Given a system whose transfer function is

$$F(z) = \frac{z^2 + 0.4z + 3}{z^3 + 1.9z^2 + 1.08z + 0.18}$$

let us determine the controllable, observable, and Jordan canonical forms.

Solution:

- *Controllable canonical form*. From $F(z)$ we obtain that the coefficients of the denominator polynomial are $a_3 = 1$, $a_2 = 1.9, \quad a_1 = 1.08, \quad a_0 = 0.18$ and those of the numerator are $b_2 = 1, \; b_1 = 0.4, \; b_0 = 3$. Therefore:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.18 & -1.08 & -1.9 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = [3 \; 0.4 \; 1] \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

Conversion of a Difference Equation into State Equations

Example (Cont.)

- *Observable canonical form*. Starting from the coefficients of the numerator and denominator polynomials of the transfer function, we obtain

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 0 & -0.18 \\ 1 & 0 & -1.08 \\ 0 & 1 & -1.9 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 3 \\ 0.4 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = [0\ 0\ 1] \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

- *Jordan canonical form.* If we compute the roots of the denominator of the system transfer function and expand it, we obtain

$$F(z) = \frac{z^2 + 0.4z + 3}{(z+1)(z+0.6)(z+0.3)} = \frac{12.85}{z+1} + \frac{-26}{z+0.6} + \frac{14.14}{z+0.3}$$

and taking the partial fractions as state variables $X_1(z) = \frac{12.85}{z+1}U(z)$, $X_2(z) = \frac{26}{z+0.6}U(z)$ and $X_3(z) = \frac{14.14}{z+0.3}U(z)$, we obtain the following state-space representation

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -0.6 & 0 \\ 0 & 0 & -0.3 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = [12.85 \ -26 \ 14.14] \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

Transformations Between Representations

- Consider a system defined by

$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{Gx}(k) + \mathbf{Hu}(k) \\ \mathbf{y}(k) &= \mathbf{Cx}(k) + \mathbf{Du}(k) \end{aligned} \tag{79}$$

  and suppose we want to transform it from one state-space representation to another.
- If we define a new state vector $\mathbf{x}'(k)$ such that

$$\mathbf{x}(k) = \mathbf{Tx}'(k) \tag{80}$$

  where $\mathbf{T}$ is a *non-singular* matrix, i.e., invertible, then the new variables $\mathbf{x}'(k)$ can be expressed as a linear combination of $\mathbf{x}(k)$.
- Substituting, we obtain

$$\mathbf{Tx}'(k+1) = \mathbf{GTx}'(k) + \mathbf{Hu}(k) \tag{81}$$

  and premultiplying by the inverse of the transformation matrix, we have

$$\mathbf{x}'(k+1) = \mathbf{T}^{-1}\mathbf{GTx}'(k) + \mathbf{T}^{-1}\mathbf{Hu}(k) \tag{82}$$

Transformations Between Representations

- Therefore, if we define

$$\mathbf{G}' = \mathbf{T}^{-1}\mathbf{G}\mathbf{T} \quad (83)$$
$$\mathbf{H}' = \mathbf{T}^{-1}\mathbf{H} \quad (84)$$

the new system becomes

$$\mathbf{x}'(k+1) = \mathbf{G}'\mathbf{x}'(k) + \mathbf{H}'\mathbf{u}(k) \quad (85)$$

- Proceeding similarly for the output equation,

$$\mathbf{y}(k) = \mathbf{C}\mathbf{T}\mathbf{x}'(k) + \mathbf{D}\mathbf{u}(k) \quad (86)$$

and defining

$$\mathbf{C}' = \mathbf{C}\mathbf{T} \quad (87)$$
$$\mathbf{D}' = \mathbf{D} \quad (88)$$

we obtain a new state-space representation

$$\mathbf{x}'(k+1) = \mathbf{G}'\mathbf{x}'(k) + \mathbf{H}'\mathbf{u}(k) \quad (89)$$
$$\mathbf{y}(k) = \mathbf{C}'\mathbf{x}'(k) + \mathbf{D}'\mathbf{u}(k) \quad (90)$$

Equality of the Characteristic Equation for Different State-Space Representations

- Two state-space representations are similar if they have the same characteristic polynomial and, therefore, the same eigenvalues. Let us verify that this is true in the previous case:

$$\begin{aligned} |z\mathbf{I} - \mathbf{G}'| &= |z\mathbf{I} - \mathbf{T}^{-1}\mathbf{G}\mathbf{T}| \\ &= |z\mathbf{T}^{-1}\mathbf{T} - \mathbf{T}^{-1}\mathbf{G}\mathbf{T}| \\ &= |\mathbf{T}^{-1}(z\mathbf{I} - \mathbf{G})\mathbf{T}| \\ &= |\mathbf{T}^{-1}||z\mathbf{I} - \mathbf{G}||\mathbf{T}| \\ &= |z\mathbf{I} - \mathbf{G}| \end{aligned} \tag{91}$$

- That is, the characteristic equation of the matrix and its eigenvalues do not depend on the basis chosen to represent the system in the state space.
- The only requirement is that the transformation matrix be non-singular; therefore, infinitely many state-space representations exist.

Transformation to the Controllable Canonical Form

- Given a system of the form

$$\mathbf{x}(k+1) = \mathbf{G}\mathbf{x}(k) + \mathbf{H}u(k) \quad (92)$$

$$y(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{D}u(k) \quad (93)$$

it can be transformed into the **controllable canonical form** by means of the transformation matrix

$$\mathbf{T} = \mathbf{M}\mathbf{W} \quad (94)$$

where $\mathbf{M} = [\mathbf{H} \quad \mathbf{G}\mathbf{H} \ldots \mathbf{G}^{n-1}\mathbf{H}]$ and

$$\mathbf{W} = \begin{bmatrix} a_1 & a_2 & \ldots & a_{n-1} & 1 \\ a_2 & a_3 & \ldots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n-1} & 1 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \end{bmatrix} \quad (95)$$

where the a_i coefficients of $\mathbf{W}$ are the coefficients of the characteristic equation:

$$|z\mathbf{I} - \mathbf{G}| = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0 = 0 \quad (96)$$

Transformation to the Observable Canonical Form

- Given a system of the form

$$\mathbf{x}(k+1) = \mathbf{G}\mathbf{x}(k) + \mathbf{H}u(k) \quad (97)$$
$$y(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{D}u(k) \quad (98)$$

it can be transformed into the **observable canonical form** by means of

$$\mathbf{T} = (\mathbf{W}\mathbf{N}^*)^{-1} \quad (99)$$

where

$$\mathbf{N} = [\mathbf{C}^* \quad \mathbf{G}^* C^* \ldots (\mathbf{G}^*)^{n-1}\mathbf{C}^*] \quad (100)$$

and

$$\mathbf{W} = \begin{bmatrix} a_1 & a_2 & \ldots & a_{n-1} & 1 \\ a_2 & a_3 & \ldots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n-1} & 1 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \end{bmatrix} \quad (101)$$

where the a_i coefficients of $\mathbf{W}$ are the coefficients of the characteristic equation 96.

Transformation to Canonical Forms

Example

- Consider the system

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = [1\ 2] \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

- Obtain the transformation matrices **T** that allow us to convert this representation into the controllable and observable canonical forms.

Solution:
The characteristic equation of this system is

$$|zI - G| = z^2 + 2z + 1 = 0$$

and therefore the polynomial coefficients are $a_2 = 1, a_1 = 2, a_0 = 1$.

- **Observable canonical form**. The transformation matrix is $T = (WN^*)^{-1}$. We have

$$\mathbf{N} = \begin{bmatrix} \mathbf{C}^* & \mathbf{G}^* C^* \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$$

  and

$$\mathbf{W} = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$

- Therefore, the transformation matrix is

$$\mathbf{T} = (\mathbf{W}N^*)^{-1} = \left[\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}\right]^{-1} = \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix}$$

$$T^{-1} = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}$$

- The matrices $\mathbf{G}', \mathbf{H}', \mathbf{C}'$ of the transformed system are the following:

$$\begin{aligned}
\mathbf{G}' &= \mathbf{T}^{-1}\mathbf{G}\mathbf{T} = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} \\
\mathbf{H}' &= \mathbf{T}^{-1}\mathbf{H} = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \\
\mathbf{C}' &= \mathbf{C}\mathbf{T} = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix}
\end{aligned}$$

and the transformed system becomes

$$\begin{aligned}
\mathbf{x}'(k+1) &= \mathbf{G}'\mathbf{x}'(k) + \mathbf{H}'u(k) = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} \mathbf{x}'(k) + \begin{bmatrix} 3 \\ 2 \end{bmatrix} u(k) \\
y(k) &= \mathbf{C}'\mathbf{x}'(k) = [0\ 1]\mathbf{x}'(k)
\end{aligned}$$

- **Controllable canonical form.** The transformation matrix for converting to the controllable canonical form is $T = MW$. We have

$$\mathbf{M} = \begin{bmatrix} \mathbf{H} & \mathbf{G}H \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

and

$$\mathbf{W} = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$

- Therefore, the transformation matrix is

$$\mathbf{T} = \mathbf{MW} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\mathbf{T}^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

- The matrices $\mathbf{G}', \mathbf{H}', \mathbf{C}'$ of the transformed system are the following:

$$
\begin{aligned}
\mathbf{G}' &= \mathbf{T}^{-1}\mathbf{G}\mathbf{T} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \\
\mathbf{H}' &= \mathbf{T}^{-1}\mathbf{H} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
\mathbf{C}' &= \mathbf{C}\mathbf{T} = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \end{bmatrix}
\end{aligned}
$$

Transformation to the Jordan Canonical Form

- When the system matrix $\mathbf{G}$ has distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, the matrix $\mathbf{G}'$ of the Jordan canonical form is a diagonal matrix of the form

$$\mathbf{G}' = \mathbf{T}^{-1}\mathbf{G}\mathbf{T} = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix} = \Lambda \qquad (102)$$

- The transformation matrix $\mathbf{T}$ is obtained from the eigenvectors $\mathbf{v}_i$ corresponding to the eigenvalues λ_i, which satisfy the characteristic equation $|\lambda\mathbf{I} - \mathbf{G}| = 0$. Therefore, we have:

$$\begin{aligned} \mathbf{G}\mathbf{v}_1 &= \lambda_1\mathbf{v}_1 \\ \mathbf{G}\mathbf{v}_2 &= \lambda_2\mathbf{v}_2 \\ \vdots &= \vdots \\ \mathbf{G}\mathbf{v}_n &= \lambda_n\mathbf{v}_n \end{aligned} \qquad (103)$$

Transformation to the Jordan Canonical Form

- The transformation matrix $\mathbf{T}$ is obtained from expression 103, rewriting it in matrix form as:

$$\mathbf{G}[\mathbf{v}_1\ \mathbf{v}_2\ \ldots\ \mathbf{v}_n] = [\mathbf{v}_1\ \mathbf{v}_2\ \ldots\ \mathbf{v}_n] \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix} \tag{104}$$

and denoting by $\mathbf{T}$ the matrix whose columns are the eigenvectors,

$$\mathbf{T} = [\mathbf{v}_1\ \mathbf{v}_2\ \ldots\ \mathbf{v}_n] \tag{105}$$

the set of equalities 104 can be expressed as

$$\mathbf{GT} = \mathbf{T}\Lambda \tag{106}$$

where $\Lambda = \mathbf{G}'$ is the diagonal matrix to be obtained as a result of the transformation.

- Consider a system whose state-space representation is

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

- Obtain the transformation matrix **T** that allows us to convert this representation into the Jordan canonical form.

Solution:
The characteristic equation of this system is given by

$$|\lambda \mathbf{I} - \mathbf{G}| = (\lambda - 1)(\lambda + 1)(\lambda - 2) = 0$$

and therefore the eigenvalues are $\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = 2$. The transformation matrix for converting to the diagonal canonical form (since the roots are simple) is obtained from the eigenvectors computed from the following expression $\mathbf{G}\mathbf{v}_i = \mathbf{v}_i \lambda_i$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} v_{i1} \\ v_{i2} \\ v_{i3} \end{bmatrix} = \begin{bmatrix} v_{i1} \\ v_{i2} \\ v_{i3} \end{bmatrix} \lambda_i \tag{107}$$

where $T = [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3]$. Substituting the corresponding values of λ_i into 107, we obtain the following system of equations.

- $\lambda_1 = 1$
  - The system of equations 107 becomes

$$\begin{aligned} v_{11} &= v_{11} \\ v_{11} - v_{12} &= v_{12} \\ v_{12} + 2v_{13} &= v_{13} \end{aligned}$$

    whose solution indicates that v_{11} can be chosen arbitrarily and that $v_{12} = \frac{1}{2}v_{11}$ and $v_{13} = -v_{12}$.
  - Taking $v_{11} = 2$, we obtain $v_{12} = 1$ and $v_{13} = -1$; therefore, the eigenvector $\mathbf{v}_1$ is

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \tag{108}$$

- $\lambda_2 = -1$
  - The system of equations 107 becomes

$$\begin{aligned} v_{21} &= -v_{21} \\ v_{21} - v_{22} &= -v_{22} \\ v_{22} + 2v_{23} &= -v_{23} \end{aligned}$$

    whose solution indicates that v_{22} can be chosen arbitrarily and that $v_{23} = \frac{-1}{3} v_{22}$ and $v_{21} = 0$.
  - Taking $v_{22} = 3$, we obtain $v_{23} = -1$, and therefore the eigenvector $\mathbf{v}_2$ is

$$\mathbf{v}_2 = \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix} \tag{109}$$

- $\lambda_3 = 2$
  - The system of equations 107 becomes

$$\begin{aligned} v_{31} &= 2v_{31} \\ v_{31} - v_{32} &= 2v_{32} \\ v_{32} + 2v_{33} &= 2v_{33} \end{aligned}$$

    whose solution indicates that v_{33} can be chosen arbitrarily and that $v_{31} = 0$ and $v_{32} = 0$.
  - Taking $v_{33} = 1$, we obtain the eigenvector $\mathbf{v}_3$

$$\mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{110}$$

- The transformation matrix **T** is

$$\mathbf{T} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ -1 & -1 & 1 \end{bmatrix} \tag{111}$$

and its inverse is

$$\mathbf{T}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{-1}{6} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & 1 \end{bmatrix} \tag{112}$$

- The matrices $\mathbf{G}', \mathbf{H}', \mathbf{C}'$ of the transformed system are the following:

$$
\begin{aligned}
\mathbf{G}' &= \mathbf{T}^{-1}\mathbf{G}\mathbf{T} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{-1}{6} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ -1 & -1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \\
\mathbf{H}' &= \mathbf{T}^{-1}\mathbf{H} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{-1}{6} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{6} \\ \frac{4}{3} \end{bmatrix} \\
\mathbf{C}' &= \mathbf{C}\mathbf{T} = [1\ 2\ 1] \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ -1 & -1 & 1 \end{bmatrix} = [3\ 5\ 1]
\end{aligned} \tag{113}
$$