Control Engineering II

Topic 7:
Modeling and Analysis of Systems in State Space

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Problem 1

Linearize around an operating point the system given by the following state-space equations:

$$\begin{aligned}\dot{x}_1 &= ax_1 + bx_2x_3\\ \dot{x}_2 &= cx_3 + x_4u_1\\ \dot{x}_3 &= u_1 + x_2x_4\\ \dot{x}_4 &= x_1 + dx_3(1-u_2)\end{aligned}$$

$$\begin{aligned}y_1 &= ex_1 + x_2\\ y_2 &= x_3x_4\end{aligned}$$

Solution

To solve this problem, the Jacobians that allow the linearization of the system must be calculated:

$$\begin{aligned}\frac{d\tilde{\mathbf{x}}}{dt} &= \mathbf{A}\tilde{\mathbf{x}} + \mathbf{B}\tilde{\mathbf{u}}\\ \tilde{\mathbf{y}} &= \mathbf{C}\tilde{\mathbf{x}} + \mathbf{D}\tilde{\mathbf{u}}\end{aligned}$$

$$\mathbf{A} = \nabla_x f\big|_{x_0,u_0} = \begin{pmatrix} a & bx_{3_0} & bx_{2_0} & 0\\ 0 & 0 & c & u_{1_0}\\ 0 & x_{4_0} & 0 & x_{2_0}\\ 1 & 0 & d(1-u_{2_0}) & 0\end{pmatrix}$$

$$\mathbf{B} = \nabla_u f\big|_{x_0,u_0} = \begin{pmatrix} 0 & 0\\ x_{4_0} & 0\\ 1 & 0\\ 0 & -dx_{3_0}\end{pmatrix}$$

$$\mathbf{C} = \nabla_x g\big|_{x_0,u_0} = \begin{pmatrix} e & 1 & 0 & 0\\ 0 & 0 & x_{4_0} & x_{3_0}\end{pmatrix}$$

$$\mathbf{D} = \nabla_u g\big|_{x_0,u_0} = \begin{pmatrix} 0 & 0\\ 0 & 0\end{pmatrix}$$

Calculate the three canonical forms: Controllable, Observable and Jordan, of the system whose transfer function is the following:

$$\frac{Y(z)}{U(z)} = \frac{z+1}{z^2+1.3z+0.4}$$

Solution

The transfer function will have the following coefficients:

$$\frac{Y(z)}{U(z)} = \frac{b_1 z + b_0}{a_2 z^2 + a_1 z + a_0}$$

- *Controllable Canonical Form:*

The following expressions give us the sought canonical form:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{a_0}{a_2} & -\frac{a_1}{a_2} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{a_2} \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} b_0 - b_2\frac{a_0}{a_2} & b_1 - b_2\frac{a_1}{a_2} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \frac{b_2}{a_2} u(k)$$

If we substitute the coefficients of the transfer function we arrive at the result:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.4 & -1.3 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

- *Observable Canonical Form:*

The following expressions give us the sought canonical form:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{a_0}{a_2} & -\frac{a_1}{a_2} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} b_0 - b_2\frac{a_0}{a_2} \\ b_1 - b_2\frac{a_1}{a_2} \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 0 & \frac{1}{a_2} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \frac{b_2}{a_2} u(k)$$

If we substitute the coefficients of the transfer function we arrive at the result:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & -0.4 \\ 1 & -1.3 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

- *Jordan Canonical Form:*

First we decompose the transfer function into simple fractions:

$$\frac{Y(z)}{U(z)} = \frac{z+1}{z^2+1.3z+0.4} = \frac{5/3}{z+0.5} + \frac{-2/3}{z+0.8}$$

The following expressions give us the sought canonical form:

$$F(z) = \frac{c_1}{z-\lambda_1} + \frac{c_2}{z-\lambda_2}$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \frac{b_2}{a_2} u(k)$$

If we substitute the coefficients of the transfer function we arrive at the result:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} -0.5 & 0 \\ 0 & -0.8 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} \frac{5}{3} & -\frac{2}{3} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Problem 3 (1/4)

Calculate the three canonical forms: Controllable, Observable and Jordan, of the system whose transfer function is the following:

$$\frac{Y(z)}{U(z)} = \frac{(z+0.5)(z-0.4)}{(z+1)(z-0.5)(z-0.5)}$$

Solution

The transfer function will have the following coefficients:

$$\frac{Y(z)}{U(z)} = \frac{b_2 z^2 + b_1 z + b_0}{a_3 z^3 + a_2 z^2 + a_1 z + a_0} = \frac{z^2 + 0.1z - 0.2}{z^3 - 0.75z + 0.25}$$

- *Controllable Canonical Form:*

The following expressions give us the sought canonical form:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{a_0}{a_3} & -\frac{a_1}{a_3} & -\frac{a_2}{a_3} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{a_3} \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} b_0 - b_3\frac{a_0}{a_3} & b_1 - b_3\frac{a_1}{a_3} & b_2 - b_3\frac{a_2}{a_3} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \frac{b_3}{a_3} u(k)$$

If we substitute the coefficients of the transfer function we arrive at the result:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.25 & 0.75 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} -0.2 & 0.1 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

- *Observable Canonical Form:*

The following expressions give us the sought canonical form:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\frac{a_0}{a_3} \\ 1 & 0 & -\frac{a_1}{a_3} \\ 0 & 1 & -\frac{a_2}{a_3} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} b_0 - b_3\frac{a_0}{a_3} \\ b_1 - b_3\frac{a_1}{a_3} \\ b_2 - b_3\frac{a_2}{a_3} \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 0 & 0 & \frac{1}{a_3} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \frac{b_3}{a_3} u(k)$$

If we substitute the coefficients of the transfer function we arrive at the result:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 0 & -0.25 \\ 1 & 0 & 0.75 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} -0.2 \\ 0.1 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

Problem 3 (3/4)

- *Jordan Canonical Form:*

First we decompose the transfer function into simple fractions:

$$\frac{Y(z)}{U(z)} = \frac{z^2 + 0.1z - 0.2}{(z+1)(z-0.5)^2} = \frac{A}{(z-0.5)^2} + \frac{B}{z-0.5} + \frac{C}{z+1}$$

We calculate the value of the coefficients:

$$(B+C)z^2 + (A + 0.5B - C)z + A - 0.5B + 0.25C = z^2 + 0.1z - 0.2$$

$$\frac{Y(z)}{U(z)} = \frac{0.0667}{(z-0.5)^2} + \frac{0.6889}{z-0.5} + \frac{0.3111}{z+1}$$

The following expressions give us the sought canonical form:

$$F(z) = \frac{c_1}{(z-\lambda_1)^2} + \frac{c_2}{z-\lambda_1} + \frac{c_3}{z-\lambda_2}$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \frac{b_3}{a_3} u(k)$$

If we substitute the coefficients of the transfer function, we arrive at the result:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 0.5 & 1 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 0.0667 & 0.6889 & 0.3111 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

Problem 4 (1/4)

Calculate the three canonical forms: Controllable, Observable and Jordan, of the system whose transfer function is the following:

$$\frac{Y(z)}{U(z)} = \frac{(z+0.3)(z-0.5)(z+0.8)}{(z-1)^2(z-0.5)(z+0.2)}$$

Solution

The transfer function will have the following coefficients:

$$\frac{Y(z)}{U(z)} = \frac{b_3 z^3 + b_2 z^2 + b_1 z + b_0}{a_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a_0}$$

$$\frac{Y(z)}{U(z)} = \frac{z^3 + 0.6z^2 - 0.31z - 0.12}{z^4 - 2.3z^3 + 1.5z^2 - 0.1z - 0.1}$$

- *Controllable Canonical Form:*

The following expressions give us the sought canonical form:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \\ x_4(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{a_0}{a_4} & -\frac{a_1}{a_4} & -\frac{a_2}{a_4} & -\frac{a_3}{a_4} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{a_4} \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} b_0 - b_4\frac{a_0}{a_4} & b_1 - b_4\frac{a_1}{a_4} & b_2 - b_4\frac{a_2}{a_4} & b_3 - b_4\frac{a_3}{a_4} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{bmatrix} + \frac{b_4}{a_4} u(k)$$

If we substitute the coefficients of the transfer function we arrive at the result:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \\ x_4(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0.1 & 0.1 & -1.5 & 2.3 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} -0.12 & -0.3 & 0.6 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{bmatrix}$$

- *Observable Canonical Form*:

The following expressions give us the sought canonical form:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \\ x_4(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -\frac{a_0}{a_4} \\ 1 & 0 & 0 & -\frac{a_1}{a_4} \\ 0 & 1 & 0 & -\frac{a_2}{a_4} \\ 0 & 0 & 1 & -\frac{a_3}{a_4} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{bmatrix} + \begin{bmatrix} b_0 - b_4\frac{a_0}{a_4} \\ b_1 - b_4\frac{a_1}{a_4} \\ b_2 - b_4\frac{a_2}{a_4} \\ b_3 - b_4\frac{a_3}{a_4} \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{a_4} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{bmatrix} + \frac{b_4}{a_4} u(k)$$

Problem 4 (3/4)

If we substitute the coefficients of the transfer function we arrive at the result:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \\ x_4(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0.1 \\ 1 & 0 & 0 & 0.1 \\ 0 & 1 & 0 & -1.5 \\ 0 & 0 & 1 & 2.3 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{bmatrix} + \begin{bmatrix} -0.12 \\ -0.31 \\ 0.6 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{bmatrix}$$

- *Jordan Canonical Form:*

First we decompose the transfer function into simple fractions:

$$\frac{Y(z)}{U(z)} = \frac{z^3 + 0.6z^2 - 0.31z - 0.12}{(z-1)^2(z-0.5)(z+0.2)}$$

$$= \frac{A}{(z-1)^2} + \frac{B}{z-1} + \frac{C}{z-0.5} + \frac{D}{z+0.2}$$

We calculate the value of the coefficients:

$$\begin{aligned} &B + C + D = 1, \\ &A - 0.5B - 1.8C - 2.5D = 0.6, \\ &-0.7A - 2.5B + 0.6C + 2D = -0.31, \\ &-0.1A + B + 0.2C - 0.5D = -0.12. \end{aligned}$$

Problem 4 (4/4)

To solve this system, we use Matlab, putting the system in matrix form (Ax=B) and using the command x=A\B.

$$\frac{Y(z)}{U(z)} = \frac{2.75}{(z-1)^2} + \frac{-0.62}{z-1} + \frac{2.26}{z-0.5} + \frac{-0.64}{z+0.2}$$

The following expressions give us the sought canonical form:

$$F(z) = \frac{c_1}{(z-\lambda_1)^2} + \frac{c_2}{z-\lambda_1} + \frac{c_3}{z-\lambda_2} + \frac{c_4}{z-\lambda_3}$$

$$\begin{bmatrix} x_1\,(k+1) \\ x_2\,(k+1) \\ x_3\,(k+1) \\ x_4(k+1) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} \begin{bmatrix} x_1\,(k) \\ x_2\,(k) \\ x_3\,(k) \\ x_4(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} u(k)$$

If we substitute the coefficients of the transfer function we arrive at the result:

$$\begin{bmatrix} x_1\,(k+1) \\ x_2\,(k+1) \\ x_3\,(k+1) \\ x_4(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & -0.2 \end{bmatrix} \begin{bmatrix} x_1\,(k) \\ x_2\,(k) \\ x_3\,(k) \\ x_4(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} u(k)$$

Problem 5 (1/2)

Obtain the matrices T that allow the conversion to the observable and controllable canonical forms given the following system:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Solution

For the observable canonical form:

$$|zI - G| = (z-1)(z-3) = z^2 - 4z + 3 = 0$$

$$N = \left[C^* G^* C^* \right] = \begin{pmatrix} 3 & 5 \\ 1 & 3 \end{pmatrix}$$

$$W = \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -4 & 1 \\ 1 & 0 \end{pmatrix}$$

$$T = (WN^*)^{-1} = \begin{pmatrix} -0.25 & -0.25 \\ 0.75 & 1.75 \end{pmatrix}$$

The matrices of the transformed system are:

$$G' = T^{-1}GT = \begin{pmatrix} 0 & -3 \\ 1 & 4 \end{pmatrix} \quad H' = T^{-1}H = \begin{pmatrix} -7 \\ 3 \end{pmatrix} \quad C' = CT = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

The transformed system remains:

$$\begin{bmatrix} x_1'(k+1) \\ x_2'(k+1) \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x_1'(k) \\ x_2'(k) \end{bmatrix} + \begin{bmatrix} -7 \\ 3 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1'(k) \\ x_2'(k) \end{bmatrix}$$

Problem 5 (2/2)

For the controllable canonical form:

$$M = \begin{bmatrix} H & GH \end{bmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

$$W = \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -4 & 1 \\ 1 & 0 \end{pmatrix}$$

$$T = MW = \begin{pmatrix} -3 & 1 \\ 2 & 0 \end{pmatrix}$$

The matrices of the transformed system are:

$$G' = T^{-1}GT = \begin{pmatrix} 0 & 1 \\ -3 & 4 \end{pmatrix} \quad H' = T^{-1}H = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad C' = CT = \begin{pmatrix} -7 & 3 \end{pmatrix}$$

The transformed system remains:

$$\begin{bmatrix} x_1'(k+1) \\ x_2'(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} x_1'(k) \\ x_2'(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} -7 & 3 \end{bmatrix} \begin{bmatrix} x_1'(k) \\ x_2'(k) \end{bmatrix}$$

Obtain the transformation matrix T that allows converting to the Jordan canonical form for the following state representation:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

Solution

First, the characteristic equation is obtained:

$$|\lambda I - G| = (\lambda + 1)(\lambda - 2)(\lambda - 3) = 0$$

The eigenvalues are:

$$\lambda_1 = -1 \quad \lambda_2 = 2 \quad \lambda_3 = 3$$

We calculate the vectors for each eigenvalue:

- $\lambda_1 = -1$:

$$\begin{pmatrix} -1 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{12} \\ v_{13} \end{pmatrix} = \lambda_1 \begin{pmatrix} v_{11} \\ v_{12} \\ v_{13} \end{pmatrix}$$

We solve the following system of equations:

$$\left.\begin{matrix} -v_{11} = -v_{11} \\ v_{11} + 2v_{12} + v_{13} = -v_{12} \\ 3v_{13} = -v_{13} \end{matrix}\right\}$$

$$v_{13} = 0 \quad v_{11} = -3v_{12} \rightarrow v_{12} = 1 \quad \overrightarrow{v}_1 = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$$

- $\lambda_2 = 2$:

$$\begin{pmatrix} -1 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} v_{21} \\ v_{22} \\ v_{23} \end{pmatrix} = \lambda_2 \begin{pmatrix} v_{21} \\ v_{22} \\ v_{23} \end{pmatrix}$$

We solve the following system of equations:

$$\left.\begin{array}{c} -v_{21} = 2v_{21} \\ v_{21} + 2v_{22} + v_{23} = 2v_{22} \\ 3v_{23} = 2v_{23} \end{array}\right\}$$

$$v_{21} = 0 \quad v_{23} = 0 \rightarrow v_{22} = 1 \quad \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

- $\lambda_3 = 3$:

$$\begin{pmatrix} -1 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} v_{31} \\ v_{32} \\ v_{33} \end{pmatrix} = \lambda_3 \begin{pmatrix} v_{31} \\ v_{32} \\ v_{33} \end{pmatrix}$$

We solve the following system of equations:

$$\left.\begin{array}{c} -v_{31} = 3v_{31} \\ v_{31} + 2v_{32} + v_{33} = 3v_{32} \\ 3v_{33} = 3v_{33} \end{array}\right\}$$

$$v_{31} = 0 \quad v_{32} = v_{33} \rightarrow v_{33} = 1 \quad \vec{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

The transformation matrix T is the following:

$$T = \begin{pmatrix} -3 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Problem 6 (3/3)

The matrices of the transformed system are:

$$G' = T^{-1}GT = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \quad H' = T^{-1}H = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$C' = CT = \begin{pmatrix} -9 & 0 & 1 \end{pmatrix}$$

The transformed system remains:

$$\begin{bmatrix} x_1'(k+1) \\ x_2'(k+1) \\ x_3'(k+1) \end{bmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{bmatrix} x_1'(k) \\ x_2'(k) \\ x_3'(k) \end{bmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} u(k)$$

$$y(k) = \begin{pmatrix} -9 & 0 & 1 \end{pmatrix} \begin{bmatrix} x_1'(k) \\ x_2'(k) \\ x_3'(k) \end{bmatrix}$$