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Control Engineering II

Topic 8:
Solution of the State Equation

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- Consider a linear time-invariant system defined by the state equations

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{aligned} \tag{1}$$

  where $\mathbf{A} : n \times n$, $\mathbf{B} : n \times r$, $\mathbf{C} : m \times n$, and $\mathbf{D} : m \times r$, and the vectors $\mathbf{x}(t) : n \times 1$, $\mathbf{u}(t) : r \times 1$, and $\mathbf{y}(t) : m \times 1$.
- Assume that $\mathbf{x}(t_0)$ is known and that the system input signal $\mathbf{u}(t)$ is known for $t \geq t_0$. The objective is to determine the vector $\mathbf{x}(t)$ for $t > t_0$.
- The homogeneous equation corresponding to 1 is

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) \tag{2}$$

  whose solution is given by:

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) \tag{3}$$

  where $\mathbf{x}(t_0)$ is the vector of initial conditions.

- The matrix $e^{\mathbf{A}t}$ can be obtained by expanding it as a power series:

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2 t^2}{2!} + \frac{\mathbf{A}^3 t^3}{3!} + \dots \tag{4}$$

- If we define a new matrix Φ as

$$\Phi(t) = e^{\mathbf{A}t} \tag{5}$$

equation 3 can be rewritten as

$$\mathbf{x}(t) = \Phi(t - t_0)\mathbf{x}(t_0) \tag{6}$$

- The solution of the non-homogeneous equation 1 has the form

$$\mathbf{x}(t) = \Phi(t - t_0)\mathbf{C}_1(t) \tag{7}$$

- Differentiating the previous equation with respect to t, we obtain

$$\begin{aligned}\frac{d}{dt}\mathbf{x}(t) &= \mathbf{A}\Phi(t-t_0)\mathbf{C}_1(t) + \Phi(t-t_0)\frac{d}{dt}\mathbf{C}_1(t) \\ &= \mathbf{A}\mathbf{x}(t) + \Phi(t-t_0)\frac{d}{dt}\mathbf{C}_1(t) \end{aligned} \tag{8}$$

and comparing equation 8 with 1, it can be observed that

$$\Phi(t-t_0)\frac{d}{dt}\mathbf{C}_1(t) = \mathbf{B}\mathbf{u}(t) \tag{9}$$

Solving for the term $\frac{d}{dt}\mathbf{C}_1(t)$ and integrating it between t_0 and t, we obtain

$$\mathbf{C}_1(t) = \int_{t_0}^{t} \Phi^{-1}(\tau - t_0)\mathbf{B}\mathbf{u}(\tau)\mathrm{d}\tau + \mathbf{C}_2 \tag{10}$$

and therefore the solution of the state equation is

Solution of the State Equation

Continuous-Time Systems

$$\begin{aligned}\mathbf{x}(t) &= \Phi(t-t_0)\mathbf{C}_1(t) = \Phi(t-t_0)\left[\int_{t_0}^{t}\Phi^{-1}(\tau-t_0)\mathbf{B}\mathbf{u}(\tau)\mathrm{d}\tau + \mathbf{C}_2\right] \\ &= \Phi(t-t_0)\mathbf{C}_2 + \Phi(t-t_0)\int_{t_0}^{t}\Phi^{-1}(\tau-t_0)\mathbf{B}\mathbf{u}(\tau)\mathrm{d}\tau \qquad (11)\end{aligned}$$

- and since

$$\Phi(t-t_0)\Phi^{-1}(\tau-t_0) = e^{\mathbf{A}(t-t_0)}e^{-\mathbf{A}(\tau-t_0)} = e^{\mathbf{A}(t-\tau)} = \Phi(t-\tau) \qquad (12)$$

equation 11 can be expressed as

$$\mathbf{x}(t) = \Phi(t-t_0)\mathbf{C}_2 + \int_{t_0}^{t}\Phi(t-\tau)\mathbf{B}\mathbf{u}(\tau)\mathrm{d}\tau \qquad (13)$$

- It remains to determine the constant vector $\mathbf{C}_2$. This vector will be obtained from the initial conditions at time t_0.
- To do so, if we particularize the solution of the state equation obtained in 13 at time t_0, and since the integral between t_0 and t is zero, we have

$$\mathbf{x}(t_0) = \Phi(0)\mathbf{C}_2 = \mathbf{C}_2 \tag{14}$$

therefore, the resulting form of the state equation solution is

$$\mathbf{x}(t) = \Phi(t - t_0)\mathbf{x}(t_0) + \int_{t_0}^{t} \Phi(t - \tau)\mathbf{B}\mathbf{u}(\tau)\mathrm{d}\tau \tag{15}$$

- The matrix $\Phi(t) = e^{\mathbf{A}t}$ is known as the *fundamental matrix* or the *state transition matrix* of the system.

- A more efficient method for solving the state equation can be obtained by applying the Laplace transform.
- Consider the system

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{aligned} \tag{16}$$

Taking Laplace transforms yields

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s) \tag{17}$$

$$\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s) \tag{18}$$

- From equation 17 it follows that

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0) + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) \tag{19}$$

- Taking the inverse Laplace transform of the previous expression, we obtain

$$\mathbf{x}(t) = \mathcal{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}]\mathbf{x}(0) + \mathcal{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s)] \tag{20}$$

- Comparing this expression with the state equation solution obtained in 15, it can be observed that:
  - The first term of equation 20 is the solution of the homogeneous equation for $\mathbf{u}(t) = 0$.
  - The second term represents the particular solution corresponding to the non-homogeneous equation with a nonzero input $\mathbf{u}(t)$.
  - Furthermore, it is straightforward to obtain the expression of the fundamental matrix $\Phi(t)$ in terms of the inverse Laplace transform, since equating the first terms of equations 20 and 15 yields

$$\mathbf{x}(t) = \Phi(t)\mathbf{x}(0) = \mathcal{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}]\mathbf{x}(0) \tag{21}$$

which implies that

$$\Phi(t) = \mathcal{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}] \tag{22}$$

- Consider the following system

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(t) \tag{23}$$

  Obtain the solution of the state equation for the system subject to a unit step input and with zero initial state.

- **Solution**:
  The solution of the state equation, using the Laplace transform, is given by the following expression

$$\mathbf{x}(t) = \mathcal{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}]\mathbf{x}(0) + \mathcal{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s)] \tag{24}$$

  in which the first term vanishes since the system has zero initial conditions. The solution is then reduced to the following expression

$$\mathbf{x}(t) = \mathcal{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s)] \tag{25}$$

- First, we obtain $[s\mathbf{I} - \mathbf{A}]$ and its inverse:

$$[s\mathbf{I} - \mathbf{A}] = \begin{bmatrix} s+2 & -1 \\ 0 & s+3 \end{bmatrix} \tag{26}$$

$$[s\mathbf{I} - \mathbf{A}]^{-1} = \begin{bmatrix} \frac{1}{s+2} & \frac{1}{(s+2)(s+3)} \\ 0 & \frac{1}{s+3} \end{bmatrix} \tag{27}$$

- and from this expression, the system response in the Laplace domain is given by

$$\mathbf{X}(s) = [(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s)] = \begin{bmatrix} \frac{1}{s+2} & \frac{1}{(s+2)(s+3)} \\ 0 & \frac{1}{s+3} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \frac{1}{s} = \begin{bmatrix} \frac{s+5}{s(s+2)(s+3)} \\ \frac{2}{s(s+3)} \end{bmatrix} \tag{28}$$

- Taking the inverse Laplace transform, we obtain the solution of the state equation.

$$\mathbf{x}(t) = \mathcal{L}^{-1}\begin{bmatrix} \frac{s+5}{s(s+2)(s+3)} \\ \frac{2}{s(s+3)} \end{bmatrix} = \begin{bmatrix} \frac{5}{6} - \frac{3}{2}e^{-2t} + \frac{2}{3}e^{-3t} \\ \frac{2}{3} - \frac{2}{3}e^{-3t} \end{bmatrix} \tag{29}$$

Discretization of Continuous-Time State Equations

- State-space controllers are commonly implemented digitally using a computer or microcontroller. Therefore, a discrete-time model is required for digital implementation.
- It is necessary to obtain discrete-time state models from the continuous-time state models of a system, usually expressed as

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{aligned} \tag{30}$$

- The objective is to obtain a discretized model of 30 of the form

$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{G}\mathbf{x}(k) + \mathbf{H}\mathbf{u}(k) \\ \mathbf{y}(k) &= \mathbf{C}\mathbf{x}(k) + \mathbf{D}\mathbf{u}(k). \end{aligned} \tag{31}$$

Discretization of State Equations

- This discretization can be carried out in two ways:
  - *Indirect Method*. This method is based on discretizing the system transfer function using conventional techniques. To do so, the system transfer function is first obtained, then a zero-order hold and a sampler are introduced, and subsequently the discrete transfer function of the system is obtained for a given sampling period T. In a second step, the discrete transfer function is converted into a discrete state-space representation using one of the methods previously presented.
  - *Direct Method*. In this method, we seek to obtain directly the expressions of $\mathbf{G}$ and $\mathbf{H}$ from the matrices $\mathbf{A}$ and $\mathbf{B}$ of the continuous-time state-space representation.

- We previously showed in 15 that the solution of the continuous-time state equation is

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\mathrm{d}\tau \tag{32}$$

- We assume that the input variables are sampled and passed through a zero-order hold, so that they remain constant until the next sampling instant, that is, $\mathbf{u}(t) = \mathbf{u}(t_0 + kT)$ for $t_0 + kT \leq t < t_0 + (k+1)T$. Then, the state equation at time $t_0 + kT$ is:

$$\mathbf{x}(t_0 + kT) = e^{\mathbf{A}kT}\mathbf{x}(t_0) + \int_{t_0}^{t_0+kT} e^{\mathbf{A}(t_0+kT-\tau)}\mathbf{B}\mathbf{u}(\tau)\mathrm{d}\tau \tag{33}$$

and at $t_0 + (k+1)T$ it is:

$$\mathbf{x}(t_0+(k+1)T) = e^{\mathbf{A}(k+1)T}\mathbf{x}(t_0)+\int_{t_0}^{t_0+(k+1)T} e^{\mathbf{A}(t_0+(k+1)T-\tau)}\mathbf{B}\mathbf{u}(\tau)\mathrm{d}\tau \tag{34}$$

- If in expression 34 we consider the state at time $t_0 + kT$ as the initial state, the expression becomes

$$\mathbf{x}(t_0+(k+1)T) = e^{\mathbf{A}T}\mathbf{x}(t_0+kT)+\int_{t_0+kT}^{t_0+(k+1)T} e^{\mathbf{A}(t_0+(k+1)T-\tau)}\mathbf{B}\mathbf{u}(\tau)\mathrm{d}\tau \quad (35)$$

- Assuming, for simplicity, that $t_0 = 0$, the previous expression can be written as

$$\mathbf{x}((k+1)T) = e^{\mathbf{A}T}\mathbf{x}(kT) + \int_{kT}^{(k+1)T} e^{\mathbf{A}((k+1)T-\tau)}\mathbf{B}\mathbf{u}(\tau)\mathrm{d}\tau \quad (36)$$

- and if we perform the change of variable $\tau = kT + t$ in the integral term of the previous equation, the resulting expression is

$$\mathbf{x}((k+1)T) = e^{\mathbf{A}T}\mathbf{x}(kT) + \int_{0}^{T} e^{\mathbf{A}(T-t)}\mathbf{B}\mathbf{u}(kT+t)\mathrm{d}t \quad (37)$$

- Since the input signals are sampled and held constant throughout the interval, $\mathbf{u}(kT+t)=\mathbf{u}(kT)$, equation 37 becomes

$$\mathbf{x}((k+1)T)=e^{\mathbf{A}T}\mathbf{x}(kT)+\int_0^T e^{\mathbf{A}(T-t)}\mathbf{B}\mathbf{u}(kT)\mathrm{d}t \tag{38}$$

and by introducing the change of variable $\lambda = T - t$, equation 38 becomes

$$\mathbf{x}((k+1)T)=e^{\mathbf{A}T}\mathbf{x}(kT)+\int_0^T e^{\mathbf{A}\lambda}\mathbf{B}\mathbf{u}(kT)\mathrm{d}\lambda \tag{39}$$

- Comparing this expression with the state equation of a discrete-time system at instant $k+1$

$$\mathbf{x}(k+1) = \mathbf{G}\mathbf{x}(k) + \mathbf{H}\mathbf{u}(k) \tag{40}$$

we obtain

$$\mathbf{G}(T) = e^{\mathbf{A}T} \tag{41}$$

$$\mathbf{H}(T) = \left(\int_0^T e^{\mathbf{A}\lambda}\mathrm{d}\lambda\right)\mathbf{B} \tag{42}$$

- In the case where matrix **A** is *non-singular*, expression 42 can be simplified. To do so, we evaluate the integral, obtaining

$$\begin{aligned}\mathbf{H}(T) &= \left(\int_0^T e^{\mathbf{A}\lambda}\mathrm{d}\lambda\right)\mathbf{B} \\ &= \mathbf{A}^{-1}[e^{\mathbf{A}T} - e^{\mathbf{A}0}]\mathbf{B} \\ &= \mathbf{A}^{-1}[e^{\mathbf{A}T} - \mathbf{I}]\mathbf{B} \qquad (43)\end{aligned}$$

- Consider the system

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(t)$$

  and suppose we want to compute its discrete-time equivalent.
- **Solution:**
  - The fundamental matrix of this system, using the Laplace transform, is given by

$$\Phi(t) = \mathcal{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}]$$

  - First, we obtain $[s\mathbf{I} - \mathbf{A}]$ and its inverse:

$$\begin{aligned} \left[s\mathbf{I} - \mathbf{A}\right] &= \begin{bmatrix} s+2 & -1 \\ 0 & s+3 \end{bmatrix} \\ \left[s\mathbf{I} - \mathbf{A}\right]^{-1} &= \begin{bmatrix} \frac{1}{s+2} & \frac{1}{(s+2)(s+3)} \\ 0 & \frac{1}{s+3} \end{bmatrix} \end{aligned}$$

- The fundamental matrix $\Phi(t)$ of the system is

$$\Phi(t) = \mathcal{L}^{-1}[(s\mathbf{I}-\mathbf{A})^{-1}] = \mathcal{L}^{-1}\begin{bmatrix} \frac{1}{s+2} & \frac{1}{(s+2)(s+3)} \\ 0 & \frac{1}{s+3} \end{bmatrix} = \begin{bmatrix} e^{-2t} & e^{-2t}-e^{-3t} \\ 0 & e^{-3t} \end{bmatrix}$$

- From the fundamental matrix, the matrices $\mathbf{G}(T)$ and $\mathbf{H}(T)$ of the discrete-time state-space model with sampling period T are obtained as

$$\mathbf{G}(T) = e^{\mathbf{A}T} = \begin{bmatrix} e^{-2T} & e^{-2T}-e^{-3T} \\ 0 & e^{-3T} \end{bmatrix}$$

$$\begin{aligned} \mathbf{H}(T) &= \left(\int_0^T e^{\mathbf{A}\lambda}\mathrm{d}\lambda\right)\mathbf{B} = \int_0^T \begin{bmatrix} e^{-2\lambda} & e^{-2\lambda}-e^{-3\lambda} \\ 0 & e^{-3\lambda} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} d\lambda \\ &= \begin{bmatrix} -\frac{3}{2}e^{-2T} + \frac{2}{3}e^{-3T} + \frac{3}{2} - \frac{2}{3} \\ -\frac{2}{3}e^{-3T} + \frac{2}{3} \end{bmatrix} \end{aligned}$$

Solution of the Discrete-Time State Equation

- Given a system represented by the discrete-time state equations

$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{G}\mathbf{x}(k) + \mathbf{H}\mathbf{u}(k) && (44) \\ \mathbf{y}(k) &= \mathbf{C}\mathbf{x}(k) + \mathbf{D}\mathbf{u}(k) && (45) \end{aligned}$$

the solution of the state equation can be obtained directly by expressing the state equation recursively:

$$\begin{aligned} \mathbf{x}(1) &= \mathbf{G}\mathbf{x}(0) + \mathbf{H}\mathbf{u}(0) \\ \mathbf{x}(2) &= \mathbf{G}\mathbf{x}(1) + \mathbf{H}\mathbf{u}(1) = \mathbf{G}^2\mathbf{x}(0) + \mathbf{G}\mathbf{H}\mathbf{u}(0) + \mathbf{H}\mathbf{u}(1) \\ \mathbf{x}(3) &= \mathbf{G}\mathbf{x}(2) + \mathbf{H}\mathbf{u}(2) = \mathbf{G}^3\mathbf{x}(0) + \mathbf{G}^2\mathbf{H}\mathbf{u}(0) + \mathbf{G}\mathbf{H}\mathbf{u}(1) + \mathbf{H}\mathbf{u}(2) \\ &\vdots \end{aligned}$$

thus obtaining the general solution for an arbitrary instant k:

$$\mathbf{x}(k) = \mathbf{G}^k\mathbf{x}(0) + \sum_{j=0}^{k-1} \mathbf{G}^{k-j-1}\mathbf{H}\mathbf{u}(j) \qquad (46)$$

- From equation 46, it can be observed that the state of the system at instant k has two basic components:
  - One depends on the initial state of the system at instant 0.
  - The other depends on the sequence of control inputs applied to the system up to instant k.
- From the previous expression, the system output can be written as

$$\mathbf{y}(k) = \mathbf{C}\mathbf{G}^{k}\mathbf{x}(0) + \mathbf{C}\sum_{j=0}^{k-1}\mathbf{G}^{k-j-1}\mathbf{H}\mathbf{u}(j) + \mathbf{D}\mathbf{u}(k) \tag{47}$$

- Equations 46 and 47 can be used to compute the values of the state variables and the output at any time instant k, but **they are not particularly convenient for direct computation**.

- A more efficient method for solving discrete-time state equations is based on the z-transform. Given the state equation

$$\mathbf{x}(k+1) = \mathbf{G}\mathbf{x}(k) + \mathbf{H}\mathbf{u}(k) \tag{48}$$

if we take the z-transform of both sides, we obtain

$$z\mathbf{X}(z) - z\mathbf{x}(0) = \mathbf{G}\mathbf{X}(z) + \mathbf{H}\mathbf{U}(z) \tag{49}$$

where $\mathbf{X}(z) = \mathcal{Z}[\mathbf{x}(k)]$ and $\mathbf{U}(z) = \mathcal{Z}[\mathbf{u}(k)]$. Grouping the terms in $\mathbf{X}(z)$ gives

$$(z\mathbf{I} - \mathbf{G})\mathbf{X}(z) = z\mathbf{x}(0) + \mathbf{H}\mathbf{U}(z) \tag{50}$$

Multiplying both sides by $(z\mathbf{I} - \mathbf{G})^{-1}$ yields

$$\mathbf{X}(z) = (z\mathbf{I} - \mathbf{G})^{-1} z\mathbf{x}(0) + (z\mathbf{I} - \mathbf{G})^{-1}\mathbf{H}\mathbf{U}(z) \tag{51}$$

Solution of the Discrete-Time State Equation

- Taking the inverse z-transform gives

$$\mathbf{x}(k) = \mathcal{Z}^{-1}[(z\mathbf{I} - \mathbf{G})^{-1}z]\mathbf{x}(0) + \mathcal{Z}^{-1}[(z\mathbf{I} - \mathbf{G})^{-1}\mathbf{H}\mathbf{U}(z)] \tag{52}$$

- Comparing expressions 46 and 52, it follows that

$$\mathbf{G}^k = \mathcal{Z}^{-1}[(z\mathbf{I} - \mathbf{G})^{-1}z] \tag{53}$$

and that

$$\sum_{j=0}^{k-1} \mathbf{G}^{k-j-1}\mathbf{H}\mathbf{u}(j) = \mathcal{Z}^{-1}[(z\mathbf{I} - \mathbf{G})^{-1}\mathbf{H}\mathbf{U}(z)] \tag{54}$$

for $k = 1, 2, 3, \ldots$

- Consider the system

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(k)$$

- **Solution:**
  - The matrix $\mathbf{G}^k$ for this system, using the z-transform, is given by

$$\mathbf{G}^k = \mathcal{Z}^{-1}[(z\mathbf{I} - \mathbf{G})^{-1} z]$$

  - First, let us obtain $[z\mathbf{I} - \mathbf{G}]$ and its inverse:

$$\begin{aligned} [z\mathbf{I} - \mathbf{G}] &= \begin{bmatrix} z - 0.8 & 0 \\ 0 & z - 0.5 \end{bmatrix} \\ [z\mathbf{I} - \mathbf{G}]^{-1} &= \begin{bmatrix} \dfrac{1}{z-0.8} & 0 \\ 0 & \dfrac{1}{z-0.5} \end{bmatrix} \end{aligned}$$

- The matrix $\mathbf{G}^k$ is therefore

$$\mathbf{G}^k = \mathcal{Z}^{-1}[(z\mathbf{I} - \mathbf{G})^{-1}z] = \mathcal{Z}^{-1}\begin{bmatrix} \dfrac{z}{z-0.8} & 0 \\ 0 & \dfrac{z}{z-0.5} \end{bmatrix} = \begin{bmatrix} 0.8^k & 0 \\ 0 & 0.5^k \end{bmatrix}$$

Solution of the Discrete-Time State Equation

Example (Cont.)

- The effect of the input on the system up to instant k is obtained using the z-transform and assuming a unit-step input:

$$
\begin{aligned}
\sum_{j=0}^{k-1} \mathbf{G}^{k-j-1}\mathbf{H}u(j) &= \mathcal{Z}^{-1}[(z\mathbf{I}-\mathbf{G})^{-1}\mathbf{H}U(z)] \\
&= \mathcal{Z}^{-1}\left(\begin{bmatrix} \frac{1}{z-0.8} & 0 \\ 0 & \frac{1}{z-0.5} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}\frac{z}{z-1}\right) \\
&= \mathcal{Z}^{-1}\begin{bmatrix} \frac{z}{(z-0.8)(z-1)} \\ \frac{2z}{(z-0.5)(z-1)} \end{bmatrix} \\
&= \begin{bmatrix} -5\cdot 0.8^k + 5 \\ -4\cdot 0.5^k + 4 \end{bmatrix}
\end{aligned}
$$

Solution of the Discrete-Time State Equation

Example (Cont.)

- Therefore, the response of the system at any time instant k can be obtained as

$$\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} 0.8^k & 0 \\ 0 & 0.5^k \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} -5 \cdot 0.8^k + 5 \\ -4 \cdot 0.5^k + 4 \end{bmatrix}$$

- It can be observed that the state vector consists of two components:
  - The first term corresponds to the effect of the initial conditions and decays with time because the eigenvalues (0.8 and 0.5) are inside the unit circle.
  - The second term corresponds to the forced response due to the unit-step input.
- In the steady state ($k \to \infty$), the powers 0.8^k and 0.5^k tend to zero, and therefore the state vector converges to

$$\lim_{k\to\infty} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$