Control Engineering II

Topic 8:
Solution of the State Equation

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Problem 1

Obtain the solution of the state equation before step input and zero initial conditions if the system is given by the following equation:

$$\frac{d}{dx}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

Solution

The solution of the state equation using the Laplace transform method is as follows:

$$x(t) = L^{-1}\left[(sI - A)^{-1}\right] x(0) + L^{-1}\left[(sI - A)^{-1} BU(s)\right]$$

The first term is zero for zero initial conditions:

$$x(t) = L^{-1}\left[(sI - A)^{-1} BU(s)\right]$$

The sought solution will be:

$$X(s) = \begin{pmatrix} s+1 & 0 \\ -1 & s+3 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{1}{s} = \begin{pmatrix} \frac{1}{s+1} & 0 \\ \frac{1}{(s+1)(s+3)} & \frac{1}{s+3} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{1}{s}$$

$$X(s) = \begin{pmatrix} \frac{1}{s(s+1)} \\ \frac{s+2}{s(s+3)(s+1)} \end{pmatrix} = \begin{pmatrix} \frac{1}{s} - \frac{1}{(s+1)} \\ \frac{2}{3}\frac{1}{s} - \frac{1}{6}\frac{1}{(s+3)} - \frac{1}{2}\frac{1}{(s+1)} \end{pmatrix}$$

Solving the inverse transform:

$$x(t) = L^{-1}\left[X(s)\right] = \begin{pmatrix} 1 - e^{-t} \\ \frac{2}{3} - \frac{e^{-3t}}{6} - \frac{e^{-t}}{2} \end{pmatrix}$$

Problem 2

Obtain the discrete equivalent of the following system:

$$\frac{d}{dx}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

Solution

First we calculate the fundamental matrix:

$$\varphi(t) = L^{-1}\left[(sI - A)^{-1}\right] = L^{-1}\left[\begin{pmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{1}{s+4} \end{pmatrix}\right] = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-4t} \end{pmatrix}$$

From this matrix we discretize the system:

$$G(T) = e^{AT} = \varphi(T) = \begin{pmatrix} e^{-T} & 0 \\ 0 & e^{-4T} \end{pmatrix}$$

$$H(T) = A^{-1}(e^{AT} - I)B = \begin{pmatrix} 1 - e^{-T} \\ 0 \end{pmatrix}$$

Problem 3

Obtain the discrete equivalent of the following system:

$$\frac{d}{dx}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

Solution

First we calculate the fundamental matrix:

$$\varphi(t) = L^{-1}\left[(sI - A)^{-1}\right] = L^{-1}\left[\begin{pmatrix} \frac{1}{s+1} & 0 \\ \frac{1}{(s+1)(s+3)} & \frac{1}{s+3} \end{pmatrix}\right]$$

$$\varphi(t) = \begin{pmatrix} e^{-t} & 0 \\ \frac{e^{-t}}{2} - \frac{e^{-3t}}{2} & e^{-3t} \end{pmatrix}$$

From this matrix we discretize the system:

$$G(T) = e^{AT} = \varphi(T) = \begin{pmatrix} e^{-T} & 0 \\ \frac{e^{-T}}{2} - \frac{e^{-3T}}{2} & e^{-3T} \end{pmatrix}$$

$$H(T) = A^{-1}(e^{AT} - I)B = \begin{pmatrix} 1 - e^{-T} \\ \frac{2}{3} - \frac{e^{-3T}}{6} - \frac{e^{-T}}{2} \end{pmatrix}$$

Problem 4

Obtain the solution of the following system when the input is a unit step:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0.4 & 0 \\ 0.3 & 0.7 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$$

Solution

The solution of the state equation for a discrete system is given by the following expression:

$$x(k) = Z^{-1}\left[(zI - G)^{-1}z\right]x(0) + Z^{-1}\left[(zI - G)^{-1}HU(z)\right]$$

Each term must be calculated to arrive at the solution:

$$G^k = Z^{-1}\left[(zI - G)^{-1}z\right] = Z^{-1}\begin{bmatrix} z - 0.4 & 0 \\ -0.3 & z - 0.7 \end{bmatrix}^{-1} z$$

$$= Z^{-1}\begin{bmatrix} \frac{z}{(z-0.4)} & 0 \\ \frac{0.3z}{(z-0.4)(z-0.7)} & \frac{z}{(z-0.7)} \end{bmatrix} = \begin{bmatrix} 0.4^k & 0 \\ 0.7^k - 0.4^k & 0.7^k \end{bmatrix}$$

$$Z^{-1}\left[(zI - G)^{-1}HU(z)\right] = Z^{-1}\left[\begin{pmatrix} \frac{1}{z-0.4} & 0 \\ \frac{0.3}{(z-0.4)(z-0.7)} & \frac{1}{z-0.7} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\frac{z}{z-1}\right]$$

$$= Z^{-1}\left[\begin{bmatrix} \frac{1}{z-0.4}\frac{z}{z-1} \\ \left[\frac{0.3}{(z-0.4)(z-0.7)} + \frac{1}{z-0.7}\right]\frac{z}{z-1} \end{bmatrix}\right] = \begin{pmatrix} \frac{5}{3} - \frac{5}{3}0.4^k \\ 5 + \frac{5}{3}0.4^k - \frac{20}{3}0.7^k \end{pmatrix}$$

Finally, the solution will be:

$$\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} 0.4^k & 0 \\ 0.7^k - 0.4^k & 0.7^k \end{bmatrix}\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} \frac{5}{3} - \frac{5}{3}0.4^k \\ 5 + \frac{5}{3}0.4^k - \frac{20}{3}0.7^k \end{bmatrix}$$

Obtain the solution of the following system when the input is a unit step:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 0.3 & 0 & 0 \\ 0 & -0.7 & 0 \\ 0.2 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} u(k)$$

Solution

The solution of the state equation for a discrete system is given by the following expression:

$$x(k) = Z^{-1}\left[(zI-G)^{-1}z\right]x(0) + Z^{-1}\left[(zI-G)^{-1}HU(z)\right]$$

Each term must be calculated to arrive at the solution:

$$G^k = Z^{-1}\left[(zI-G)^{-1}z\right] = Z^{-1}\left[\begin{pmatrix} z-0.3 & 0 & 0 \\ 0 & z+0.7 & 0 \\ 0.2 & 0 & z-0.5 \end{pmatrix}^{-1} z\right]$$

$$G^k = Z^{-1}\left[\begin{pmatrix} \frac{z}{z-0.3} & 0 & 0 \\ 0 & \frac{z}{z+0.7} & 0 \\ \frac{z}{5(z-0.5)(z-0.3)} & 0 & \frac{z}{z-0.5} \end{pmatrix}\right]$$

$$G^k = \begin{pmatrix} 0.3^k & 0 & 0 \\ 0 & (-0.7)^k & 0 \\ 0.5^k - 0.3^k & 0 & 0.5^k \end{pmatrix}$$

Problem 5 (2/2)

$$Z^{-1}\left[(zI-G)^{-1}HU(z)\right]=$$

$$=Z^{-1}\left[\begin{pmatrix}\frac{1}{z-0.3} & 0 & 0\\ 0 & \frac{1}{z+0.7} & 0\\ \frac{1}{5(z-0.5)(z-0.3)} & 0 & \frac{1}{z-0.5}\end{pmatrix}\begin{pmatrix}1\\1\\2\end{pmatrix}\frac{z}{z-1}\right]$$

$$=Z^{-1}\left[\left[\begin{matrix}\frac{1}{z-0.3}\frac{z}{z-1}\\ \frac{1}{z+0.7}\frac{z}{z-1}\\ \left[\frac{0.2}{(z-0.5)(z-0.3)}+\frac{2}{z-0.5}\right]\frac{z}{z-1}\end{matrix}\right]\right]=\begin{pmatrix}\frac{10}{7}-\frac{10}{7}0.3^k\\ \frac{10}{17}-\frac{10}{17}(-0.7)^k\\ \frac{32}{7}+\frac{10}{7}0.3^k-6*0.5^k\end{pmatrix}$$

Finally, the solution will be:

$$\begin{bmatrix}x_1(k)\\x_2(k)\\x_3(k)\end{bmatrix}=\begin{bmatrix}0.3^k & 0 & 0\\ 0 & (-0.7)^k & 0\\ 0.5^k-0.3^k & 0 & 0.5^k\end{bmatrix}\begin{bmatrix}x_1(0)\\x_2(0)\\x_3(0)\end{bmatrix}+$$

$$\begin{bmatrix}\frac{10}{7}-\frac{10}{7}(0.3)^k\\ \frac{10}{17}-\frac{10}{17}(-0.7)^k\\ \frac{32}{7}+\frac{10}{7}(0.3)^k-6(0.5)^k\end{bmatrix}$$