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Control Engineering II

Topic 9:
State Feedback Control

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Contents

- Basic state feedback control structure:
  - Difference: the controller is placed in the feedback path.
  - The control action is determined using the state variables rather than the system output.

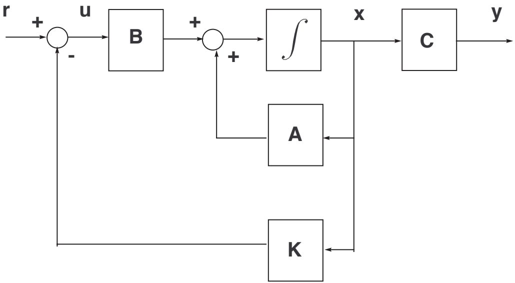


Figure 1: Block diagram of a state-feedback control system.

Observable and Controllable Modes

- Consider a continuous-time system expressed in *Jordan canonical form* with distinct eigenvalues:

$$e^{At} = \begin{pmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & \dots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & e^{\lambda_n t} \end{pmatrix}$$

- The *i*-th component of the state equation solution can be expressed as

$$x_i(t) = e^{\lambda_i (t-t_0)} x_i(t_0) + \int_{t_0}^{t} e^{\lambda_i (t-\tau)} B_i u(\tau)\, d\tau$$

- This component of the state vector solution is called a **system mode** (the mode associated with eigenvalue λ_i).

Observable and Controllable Modes

- The total system response has the form

$$\mathbf{y}(t) = C_1 x_1(t) + \ldots + C_n x_n(t)$$

where $C_1, \ldots, C_n$ are the columns of the matrix $\mathbf{C}$.

Observable and Controllable Modes

- For the response mode x_i, if the row vector $B_i = 0$, that mode is not affected by the input $\mathbf{u}(t)$.
- Furthermore, in diagonal canonical form, the state variable x_i is also independent of the other state variables. Therefore, it is impossible to influence its value either directly (through the control input) or indirectly (through another state variable). Such a mode is called a **non-controllable mode**.
- If any mode is non-controllable, some system states cannot be reached, and the system is said to be **non-controllable**.
- Similarly, a column $C_i = 0$ in matrix $\mathbf{C}$ implies that the corresponding mode cannot be observed in the system output. If $C_i \neq 0$, the mode is called an **observable mode**.

State Controllability of a System

- Consider a system

$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{G}\mathbf{x}(k) + \mathbf{H}\mathbf{u}(k) \\ \mathbf{y}(k) &= \mathbf{C}\mathbf{x}(k) \end{aligned} \qquad (1)$$

where

$$\begin{aligned} \mathbf{G} &= e^{\mathbf{A}T} \\ \mathbf{H} &= \int_0^T e^{\mathbf{A}\tau}\mathbf{B}\, d\tau \end{aligned}$$

and T is the sampling period of the system.

Definition

A control system of dimension n is said to be **completely state controllable** if there exists an input $\mathbf{u}$ defined over the interval $[0, n-1]$ such that the system can be driven from an arbitrary initial state $\mathbf{x}(0)$ to an arbitrary final state $\mathbf{x}(n)$ within a finite time interval (n sampling periods).

- Consider a system defined by the state equation

$$\mathbf{x}(k+1) = \mathbf{G}\mathbf{x}(k) + \mathbf{H}\mathbf{u}(k) \tag{2}$$

where $\mathbf{G} : n \times n$, $\mathbf{H} : n \times 1$, $\mathbf{x} : n \times 1$, and $\mathbf{u} : 1 \times 1$.

- The solution of the state equation for a linear time-invariant discrete-time system is

$$\begin{aligned} \mathbf{x}(n) &= \mathbf{G}^n\mathbf{x}(0) + \sum_{j=0}^{n-1} \mathbf{G}^{\,n-j-1}\mathbf{H}\mathbf{u}(j) \\ &= \mathbf{G}^n\mathbf{x}(0) + \mathbf{G}^{n-1}\mathbf{H}\mathbf{u}(0) + \mathbf{G}^{n-2}\mathbf{H}\mathbf{u}(1) + \ldots + \mathbf{H}\mathbf{u}(n-1) \end{aligned} \tag{3}$$

- In matrix form,

$$\mathbf{x}(n) - \mathbf{G}^n\mathbf{x}(0) = \begin{bmatrix} \mathbf{H} & \mathbf{G}\mathbf{H} & \ldots & \mathbf{G}^{n-1}\mathbf{H} \end{bmatrix} \begin{bmatrix} \mathbf{u}(n-1) \\ \mathbf{u}(n-2) \\ \vdots \\ \mathbf{u}(0) \end{bmatrix} \tag{4}$$

- Since $\mathbf{H}$ is an $n \times 1$ matrix and $\mathbf{G}$ is an $n \times n$ matrix, the matrix

$$\mathbf{M}_C = \begin{bmatrix} \mathbf{H} & \mathbf{GH} & \dots & \mathbf{G}^{n-1}\mathbf{H} \end{bmatrix} \tag{5}$$

  has dimensions $n \times n$.
- In the system of equations 4, the initial state $\mathbf{x}(0)$ and the final state $\mathbf{x}(n)$ are known, and the matrix $\mathbf{M}_C$ can be computed.
- In order to determine the control input vector, the rank of $\mathbf{M}_C$ must be equal to n. The matrix $\mathbf{M}_C$ is called the **controllability matrix**.

Definition

State Controllability Condition.
A necessary and sufficient condition for a system to be completely state controllable is that the rank of the controllability matrix be equal to n:

$$\mathrm{rank}(\mathbf{M}_C) = n$$

- Given the system

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k) \tag{6}$$

  determine the input sequence $u(0), u(1)$ that must be applied to the system so that it moves from the state $[0 \quad 0]^T$ at instant 0 to the state $[2 \quad 0]^T$ at cycle 2.

- **Solution:**
  According to equation 4, we have

$$\begin{bmatrix} x_1(2) \\ x_2(2) \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} u(1) \\ u(0) \end{bmatrix} \tag{7}$$

  and substituting $x_1(0), x_2(0), x_1(2), x_2(2)$ by the desired values of the initial and final states, we obtain

$$\begin{bmatrix} 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} u(1) \\ u(0) \end{bmatrix} \tag{8}$$

$$\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} u(1) \\ u(0) \end{bmatrix} \tag{9}$$

where

$$\mathbf{M}_C = \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix} \tag{10}$$

and since $det(\mathbf{M}_c) \neq 0$, the rank of the controllability matrix is 2, and the system is completely state controllable.

Solving the system of equations 9, we have

$$\begin{cases} 2 = u(1) - u(0) \\ 0 = u(1) - 2u(0) \end{cases} \tag{11}$$

therefore:

$$u(1) = 2u(0)$$

and the solution is $u(0) = 2, u(1) = 4$.

- Consider a second system defined by the following equations:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k) \tag{12}$$

determine the input sequence $(u(0), u(1))$ that must be applied to the system so that it moves from the state $[0 \quad 0]^T$ at instant 0 to the state $[2 \quad 0]^T$ at cycle 2.

- **Solution:**
According to equation 4, we have

$$\begin{bmatrix} x_1(2) \\ x_2(2) \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u(1) \\ u(0) \end{bmatrix} \tag{13}$$

and substituting $x_1(0), x_2(0), x_1(2), x_2(2)$ by the desired values of the initial and final states, we obtain

State Controllability
Example 2 (Cont.)

$$\begin{bmatrix} 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u(1) \\ u(0) \end{bmatrix} \tag{14}$$

$$\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u(1) \\ u(0) \end{bmatrix} \tag{15}$$

that is, $2 = u(1) - u(0)$ and, for $u(1) = 0$, we have $u(0) = -2$. Since

$$\mathbf{M}_C = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \tag{16}$$

and the rank of $\mathbf{M}_C$ is 1, the system is **not completely state controllable**.

- However, if we solve the system, we obtain that the solution gives $2 = u(1) - u(0)$; that is, if we set $u(1) = 0$, then $u(0) = -2$.
- The state is reachable from the initial point considered.

- For a system with vector input defined by

$$\mathbf{x}(k+1) = \mathbf{G}\mathbf{x}(k) + \mathbf{H}\mathbf{u}(k) \tag{17}$$

  where $\mathbf{G} : n \times n$, $\mathbf{H} : n \times r$, $\mathbf{x} : n \times 1$, $\mathbf{u} : r \times 1$.
- The controllability matrix $\mathbf{M}_C$ is an $n \times nr$ matrix, that is,

$$\mathbf{M}_C = \begin{bmatrix} \mathbf{H} & \mathbf{G}\mathbf{H} & \dots & \mathbf{G}^{n-1}\mathbf{H} \end{bmatrix} \tag{18}$$

  and the system of equations consists of n equations and nr unknowns.
- For the system to have a solution, the rank of the controllability matrix must be the smaller of n and nr, that is, n.
- Since the system has more unknowns than equations, there are $n(r-1)$ degrees of freedom for choosing the possible values of the control inputs that allow the system to move from the initial state to the final one.
- Some type of constraint or criterion is established to select, among all possible solutions, those that are of interest.

Output Controllability

- The fact that a system is completely state controllable does not guarantee that we can control the system output, except in the case where the system has a single output.
- In general, when designing a control system, the objective is to control the system output rather than its state. Therefore, it is necessary to perform an analysis similar to that used for complete state controllability, but now focused on output controllability.

Definition

A control system of dimension n is said to be **completely output controllable** if there exists an input $\mathbf{u}$ defined over the interval $[0, n-1]$ such that it is possible to drive the system from an arbitrary initial output $\mathbf{y}(0)$ to an arbitrary final output $\mathbf{y}(n)$ within a finite period of time (n sampling periods).

- Suppose the system is defined by

$$\mathbf{x}(k+1) = \mathbf{G}\mathbf{x}(k) + \mathbf{H}u(k) \quad (19)$$

$$\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) \quad (20)$$

where $\mathbf{G} : n \times n$, $\mathbf{H} : n \times 1$, $\mathbf{C} : m \times n$, $\mathbf{x} : n \times 1$, $\mathbf{u} : 1 \times 1$, $\mathbf{y} : m \times 1$, and $m \leq n$.

- Recall that the solution of the state equation for a discrete-time linear time-invariant system is

$$\begin{aligned}\mathbf{x}(n) &= \mathbf{G}^n\mathbf{x}(0) + \sum_{j=0}^{n-1} \mathbf{G}^{n-j-1}\mathbf{H}u(j) \\ &= \mathbf{G}^n\mathbf{x}(0) + \mathbf{G}^{n-1}\mathbf{H}u(0) + \mathbf{G}^{n-2}\mathbf{H}u(1) + \ldots + \mathbf{H}u(n-1)\end{aligned} \quad (21)$$

- Therefore, the system output can be expressed as

$$\mathbf{y}(n) = \mathbf{C}\mathbf{x}(n) = \mathbf{C}\mathbf{G}^n\mathbf{x}(0) + \sum_{j=0}^{n-1} \mathbf{C}\mathbf{G}^{n-j-1}\mathbf{H}u(j) \quad (22)$$

Output Controllability

Scalar Input $u(k)$

- Hence,

$$\mathbf{y}(n) - \mathbf{CG}^n\mathbf{x}(0) = \sum_{j=0}^{n-1} \mathbf{CG}^{\,n-j-1}\mathbf{Hu}(j)$$
$$= \begin{bmatrix} \mathbf{C}H & \mathbf{C}GH & \dots & \mathbf{C}G^{n-1}\mathbf{H} \end{bmatrix} \begin{bmatrix} \mathbf{u}(n-1) \\ \mathbf{u}(n-2) \\ \vdots \\ \mathbf{u}(0) \end{bmatrix} \tag{23}$$

- Since $\mathbf{H} : n \times 1$, $\mathbf{G} : n \times n$, and $\mathbf{C} : m \times n$, the output controllability matrix

$$\mathbf{M}_{CS} = \begin{bmatrix} \mathbf{C}H & \mathbf{C}GH & \dots & \mathbf{C}G^{n-1}\mathbf{H} \end{bmatrix}$$

has dimensions $m \times n$.

Output Controllability

Scalar Input $u(k)$

- In the system of equations 23, the initial and final outputs, $\mathbf{x}(0)$ and $\mathbf{y}(n)$, are known, and the matrix $\mathbf{M}_{CS}$ can be computed. In order to determine the control input vector, it is necessary to verify that the rank of the matrix $\mathbf{M}_{CS}$ is equal to m.

Definition (Output Controllability Condition)

The necessary and sufficient condition for a system to be completely output controllable is that the rank of the output controllability matrix be equal to m:

$$\mathrm{rank}(\mathbf{M}_{CS}) = m$$

Definition

Complete state controllability implies complete output controllability if and only if the m rows of the matrix $\mathbf{C}$ are linearly independent.

- Given the system

$$
\begin{aligned}
\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} &= \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k) \\
\begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} &= \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}
\end{aligned}
$$

- Determine whether the system is completely output controllable.

- **Solution:** The output controllability matrix is

$$\mathbf{M}_{CS} = \begin{bmatrix}\mathbf{C}H & \mathbf{C}GH\end{bmatrix} = \begin{bmatrix}0 & -1\\ 0 & 1\end{bmatrix} \tag{24}$$

whose rank is 1. Therefore, the system is **not** completely output controllable, although it is state controllable since

$$\mathbf{M}_C = [\mathbf{H} \quad \mathbf{G}H] = \begin{bmatrix}1 & -1\\ 1 & -2\end{bmatrix}$$

and therefore

$$\mathrm{rank}(\mathbf{M}_C) = 2.$$

- This result could have been anticipated, since the rows of the matrix **C** are not linearly independent.
- In this case, $\mathbf{d}etM_C = -1 \neq 0$, which implies that any state can be controlled.
- However, in the output controllability matrix $\mathbf{M}_{CS}$, the minor associated with y_1 is -2, whereas the minor associated with y_2 is zero. Therefore, the output y_1 is controllable, while y_2 is linearly dependent on the first output.
- This means that we can freely assign one output (y_1), but the other output (y_2) will be constrained by the value of the first.

Observability of a System

- An important aspect is the possibility of estimating or observing the values taken by state variables that cannot be measured directly.
- This estimation is performed using measurements of the system output over a certain minimum number of sampling periods, together with the values of the control signal applied to the system.

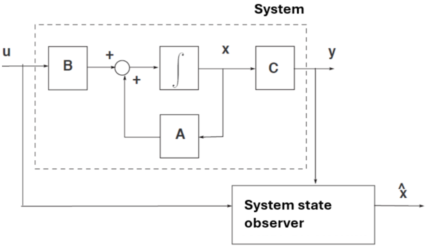


Figure 2: Concept of a state observer.

Observability of a System

- Consider a system defined by

$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{G}\mathbf{x}(k) + \mathbf{H}u(k) & (25) \\ \mathbf{y}(k) &= \mathbf{C}\mathbf{x}(k) & (26) \end{aligned}$$

where $\mathbf{G}: n \times n$, $\mathbf{H}: n \times 1$, $\mathbf{C}: m \times n$, $\mathbf{x}: n \times 1$, $\mathbf{u}: 1 \times 1$, $\mathbf{y}: m \times 1$, and $m \leq n$.

- The solution of the state equation for a discrete-time, linear time-invariant system is

$$\mathbf{x}(n) = \mathbf{G}^n\mathbf{x}(0) + \sum_{j=0}^{n-1} \mathbf{G}^{n-j-1}\mathbf{H}\mathbf{u}(j) \tag{27}$$

- The system output is therefore given by

$$\mathbf{y}(n) = \mathbf{C}\mathbf{G}^n\mathbf{x}(0) + \sum_{j=0}^{n-1} \mathbf{C}\mathbf{G}^{n-j-1}\mathbf{H}\mathbf{u}(j) \tag{28}$$

Observability of a System

- In equation 28, the matrices $\mathbf{G}$, $\mathbf{H}$, and $\mathbf{C}$, as well as the input sequence $\mathbf{u}(0), \ldots, \mathbf{u}(n-1)$, are known. Therefore, the second term on the right-hand side of the expression is completely known.
- Since this term is always known, the problem of determining system observability can be simplified by considering only the *unforced system*, that is, the system with zero control input.
- In other words, if we can determine the initial state of the unforced system from the observation of its outputs, then we will also be able to determine it for the system subjected to a sequence of control inputs.

Definition

A system is said to be **completely observable** if every initial state $\mathbf{x}(0)$ can be determined from the observation of the system output $\mathbf{y}(k)$ over a finite number of sampling intervals.

- Considering the unforced system, its equations reduce to

$$\begin{aligned} \mathbf{x}(n) &= \mathbf{G}\mathbf{x}(n-1) \\ \mathbf{y}(n-1) &= \mathbf{C}\mathbf{x}(n-1) \end{aligned} \tag{29}$$

- Applying the state equation recursively, the system output can be written as

$$\mathbf{y}(n-1) = \mathbf{C}\mathbf{G}^{n-1}\mathbf{x}(0) \tag{30}$$

Complete State Observability

- According to the definition of observability, given the output sequence $\mathbf{y}(0), \mathbf{y}(1), \ldots, \mathbf{y}(n-1)$, a system is observable if it is possible to determine the initial state $\mathbf{x}(0)$ from these outputs.
- Expressing the outputs as functions of the initial state:

$$\begin{aligned} \mathbf{y}(0) &= \mathbf{C}\mathbf{x}(0) \\ \mathbf{y}(1) &= \mathbf{C}\mathbf{G}\mathbf{x}(0) \\ &\vdots \\ \mathbf{y}(n-1) &= \mathbf{C}\mathbf{G}^{n-1}\mathbf{x}(0) \end{aligned} \tag{31}$$

- In matrix form:

$$\begin{bmatrix} \mathbf{y}(0) \\ \mathbf{y}(1) \\ \vdots \\ \mathbf{y}(n-1) \end{bmatrix} = \begin{bmatrix} \mathbf{C} \\ \mathbf{C}\mathbf{G} \\ \vdots \\ \mathbf{C}\mathbf{G}^{n-1} \end{bmatrix} \mathbf{x}(0) \tag{32}$$

Complete State Observability

- The matrix

$$\begin{bmatrix} \mathbf{C} & \mathbf{CG} & \dots & \mathbf{CG}^{n-1} \end{bmatrix}^T$$

is called the **observability matrix**, denoted by $\mathbf{M}_O$.

- Since $\mathbf{G}$ is an $n \times n$ matrix and $\mathbf{C}$ is an $m \times n$ matrix, the observability matrix

$$\mathbf{M}_O = \begin{bmatrix} \mathbf{C} & \mathbf{CG} & \dots & \mathbf{CG}^{n-1} \end{bmatrix}^T$$

has dimensions $mn \times n$.

- Therefore, for the system of equations to have a unique solution, the matrix $\mathbf{M}_O$ must have rank n.

Definition (Observability Condition)

The necessary and sufficient condition for a system to be observable is that the rank of the observability matrix be equal to n:

$$\mathrm{rank}(\mathbf{M}_O) = n$$

Complete State Observability

- A typical situation in which a system is not completely observable or controllable occurs when pole-zero cancellations are present.
- In the case of a single-input single-output (SISO) system, the system will be both controllable and observable if no pole-zero cancellations can be performed in its transfer function.

- Consider the system defined by

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 0.5 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

- Determine whether the system is observable.
- **Solution:** The observability matrix is

$$\mathbf{M}_O = \begin{bmatrix} \mathbf{C} \\ \mathbf{CG} \end{bmatrix} = \begin{bmatrix} 0.5 & 1 \\ -0.5 & -2 \end{bmatrix} \tag{33}$$

- Since the rank of $\mathbf{M}_O$ is 2, the system is completely observable.

- A system admits multiple state-space representations. A natural question is whether controllability and observability are preserved under a change of basis in the state representation.
- If a change of basis is performed using a nonsingular transformation matrix $\mathbf{T}$, the state-space representation

$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{G}\mathbf{x}(k) + \mathbf{H}\mathbf{u}(k) \\ \mathbf{y}(k) &= \mathbf{C}\mathbf{x}(k) \\ \mathbf{x}(k) &= \mathbf{T}\mathbf{x}'(k) \end{aligned} \tag{34}$$

is transformed into the new representation

$$\begin{aligned} \mathbf{x}'(k+1) &= \mathbf{T}^{-1}\mathbf{G}\mathbf{T}\mathbf{x}'(k) + \mathbf{T}^{-1}\mathbf{H}\mathbf{u}(k) \\ \mathbf{y}(k) &= \mathbf{C}\mathbf{T}\mathbf{x}'(k) \end{aligned} \tag{35}$$

Invariance of Controllability and Observability

- The transformed matrices $\mathbf{G}'$, $\mathbf{H}'$, and $\mathbf{C}'$ are therefore given by

$$\begin{aligned} \mathbf{G}' &= \mathbf{T}^{-1}\mathbf{G}\mathbf{T} \\ \mathbf{H}' &= \mathbf{T}^{-1}\mathbf{H} \\ \mathbf{C}' &= \mathbf{C}\mathbf{T} \end{aligned} \tag{36}$$

Invariance of Controllability

- For this new state-space representation, the controllability matrix becomes

$$\begin{aligned}
\mathbf{M}'_C &= \left[\mathbf{T}^{-1}\mathbf{H} \quad \mathbf{T}^{-1}\mathbf{G}\mathbf{T}\mathbf{T}^{-1}\mathbf{H} \quad \dots \quad \mathbf{T}^{-1}\mathbf{G}^{n-1}\mathbf{T}\mathbf{T}^{-1}\mathbf{H}\right] \\
&= \left[\mathbf{T}^{-1}\mathbf{H} \quad \mathbf{T}^{-1}\mathbf{G}\mathbf{H} \quad \dots \quad \mathbf{T}^{-1}\mathbf{G}^{n-1}\mathbf{H}\right] \\
&= \mathbf{T}^{-1}\left[\mathbf{H} \quad \mathbf{G}\mathbf{H} \quad \dots \quad \mathbf{G}^{n-1}\mathbf{H}\right] \\
&= \mathbf{T}^{-1}\mathbf{M}_C
\end{aligned} \tag{37}$$

- Since **T** is nonsingular, the rank of the transformed controllability matrix $\mathbf{M}'_C$ is equal to the rank of the original controllability matrix $\mathbf{M}_C$.
- Therefore, if the system in 34 is controllable, then the transformed system in 35 is also controllable.

Invariance of Observability

- If a similar analysis is carried out for observability, it follows that the observability matrix of system 35 is

$$
\begin{aligned}
\mathbf{M'}_O &= \begin{bmatrix} \mathbf{CT} \\ \mathbf{CTT}^{-1}\mathbf{GT} \\ \vdots \\ \mathbf{CTT}^{-1}\mathbf{G}^{n-1}\mathbf{T} \end{bmatrix} = \begin{bmatrix} \mathbf{CT} \\ \mathbf{CGT} \\ \vdots \\ \mathbf{CG}^{n-1}\mathbf{T} \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{C} \\ \mathbf{CG} \\ \vdots \\ \mathbf{CG}^{n-1} \end{bmatrix} \mathbf{T} = \mathbf{M}_O\mathbf{T}
\end{aligned} \tag{38}
$$

- Since the transformation matrix $\mathbf{T}$ is nonsingular, and similarly to the controllability case, the ranks of the observability matrices $\mathbf{M'}_O$ and $\mathbf{M}_O$ are equal.
- Therefore, although the system changes basis, if the original system was observable, it will remain observable.

Duality Principle (Kalman)

- Let $\mathbf{S}_1$ be a linear time-invariant system defined by

$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{G}\mathbf{x}(k) + \mathbf{H}\mathbf{u}(k) \\ \mathbf{y}(k) &= \mathbf{C}\mathbf{x}(k) \end{aligned} \tag{39}$$

  where the matrix dimensions are: $\mathbf{x} : n \times 1$, $\mathbf{u} : r \times 1$, $\mathbf{y} : m \times 1$, $\mathbf{G} : n \times n$, $\mathbf{H} : n \times r$, and $\mathbf{C} : m \times n$.

- Let $\mathbf{S}_2$, called the **dual system** of $\mathbf{S}_1$, be defined by

$$\begin{aligned} \mathbf{x}'(k+1) &= \mathbf{G}^T\mathbf{x}'(k) + \mathbf{C}^T\mathbf{u}'(k) \\ \mathbf{y}'(k) &= \mathbf{H}^T\mathbf{x}'(k) \end{aligned} \tag{40}$$

  where $\mathbf{x}' : n \times 1$, $\mathbf{u}' : m \times 1$, and $\mathbf{y}' : r \times 1$.

Duality Principle (Kalman)

The system $\mathbf{S}_1$ is completely state controllable if and only if the dual system $\mathbf{S}_2$ is completely observable. Likewise, $\mathbf{S}_1$ is completely observable if and only if $\mathbf{S}_2$ is completely state controllable.

Duality Principle (Kalman)

- The controllability and observability matrices of system $\mathbf{S}_1$ are

$$\begin{aligned} \mathbf{M}_C &= \begin{bmatrix} \mathbf{H} & \mathbf{GH} & \ldots & \mathbf{G}^{n-1}\mathbf{H} \end{bmatrix} \\ \mathbf{M}_O &= \begin{bmatrix} \mathbf{C} \\ \mathbf{CG} \\ \vdots \\ \mathbf{CG}^{n-1} \end{bmatrix} \end{aligned} \tag{41}$$

- The controllability matrix of the dual system $\mathbf{S}_2$ is

$$\begin{aligned} \mathbf{M}'_C &= \begin{bmatrix} \mathbf{C}^T & \mathbf{G}^T\mathbf{C}^T & \ldots & (\mathbf{G}^T)^{n-1}\mathbf{C}^T \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{C} \\ \mathbf{CG} \\ \vdots \\ \mathbf{CG}^{n-1} \end{bmatrix} = \mathbf{M}_O \end{aligned} \tag{42}$$

Duality Principle (Kalman)

- The observability matrix of the dual system $\mathbf{S}_2$ is

$$\begin{aligned} \mathbf{M'}_O &= \begin{bmatrix} \mathbf{H}^T \\ \mathbf{H}^T\mathbf{G}^T \\ \vdots \\ \mathbf{H}^T(\mathbf{G}^{n-1})^T \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{H} & \mathbf{GH} & \dots & \mathbf{G}^{n-1}\mathbf{H} \end{bmatrix} = \mathbf{M}_C \end{aligned} \tag{43}$$

- As can be seen,

$$\mathbf{M'}_O = \mathbf{M}_C \qquad \text{and} \qquad \mathbf{M'}_C = \mathbf{M}_O,$$

  which confirms the duality principle.
- This duality principle was already used when deriving canonical state-space realizations from a transfer function, where the controllable canonical form and the observable canonical form are dual systems.

State Feedback Control

- We wish to control a system in which all state variables are available (measurable or accessible) and can therefore be used for feedback.

Theorem

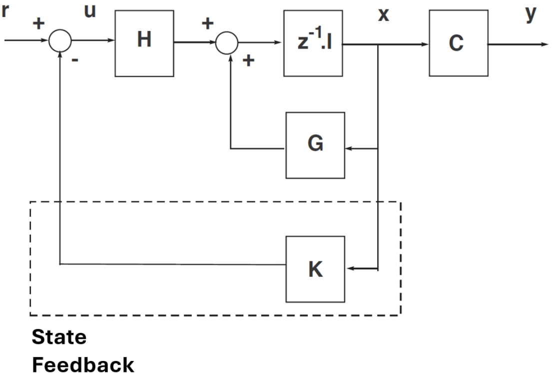
If a system is completely state controllable, then the closed-loop poles of the system can be freely placed at any desired locations $p_1, p_2, \ldots, p_n$ by means of state feedback with a suitable gain matrix $\mathbf{K}$.

State Feedback Control

- The objective of this control scheme is to make the characteristic equation of the closed-loop system take the form

$$(z - p_1)(z - p_2) \cdots (z - p_n) = 0 \tag{44}$$

- The desired pole locations are selected so that the required design specifications are satisfied.



Single-Input Single-Output Systems

- Consider the system shown in the previous figure, whose state-space equations are

$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{G}\mathbf{x}(k) + \mathbf{H}u(k) \\ \mathbf{y}(k) &= \mathbf{C}\mathbf{x}(k) \end{aligned} \tag{45}$$

where $\mathbf{G} : n \times n$, $\mathbf{H} : n \times 1$, $\mathbf{C} : 1 \times n$, $\mathbf{x} : n \times 1$, $u : 1 \times 1$, and $y : 1 \times 1$.

- The control signal applied to the system is given by

$$u(k) = r(k) - \mathbf{K}\mathbf{x}(k) \tag{46}$$

- From equations 45 and 46, with $\mathbf{K} : 1 \times n$, it follows that

$$\mathbf{x}(k+1) = \mathbf{G}\mathbf{x}(k) + \mathbf{H}\left[-\mathbf{K}\mathbf{x}(k) + r(k)\right] \tag{47}$$

$$= \left[\mathbf{G} - \mathbf{H}\mathbf{K}\right]\mathbf{x}(k) + \mathbf{H}r(k) \tag{48}$$

Single-Input Single-Output Systems

- Taking the z-transform of the closed-loop state equation, we obtain

$$z\mathbf{X}(z) = (\mathbf{G} - \mathbf{HK})\mathbf{X}(z) + \mathbf{H}R(z) \tag{49}$$

therefore,

$$(z\mathbf{I} - \mathbf{G} + \mathbf{HK})\mathbf{X}(z) = \mathbf{H}R(z) \tag{50}$$

$$\mathbf{X}(z) = (z\mathbf{I} - \mathbf{G} + \mathbf{HK})^{-1}\mathbf{H}R(z) \tag{51}$$

- The output is then given by

$$Y(z) = \mathbf{C}[z\mathbf{I} - \mathbf{G} + \mathbf{HK}]^{-1}\mathbf{H}R(z) \tag{52}$$

from which we obtain

$$F(z) = \frac{Y(z)}{R(z)} = \mathbf{C}[z\mathbf{I} - \mathbf{G} + \mathbf{HK}]^{-1}\mathbf{H} \tag{53}$$

- From 53, it can be seen that the denominator of the transfer function is $|z\mathbf{I} - (\mathbf{G} - \mathbf{HK})|$.

Single-Input Single-Output Systems

- If a certain dynamic behavior of the system is desired, it is sufficient to adjust the positions of the closed-loop poles.
- To do so, the desired pole locations $p_1, p_2, \ldots, p_n$ (or the desired characteristic equation) are specified, and thus

$$|z\mathbf{I} - \mathbf{G} + \mathbf{HK}| = (z - p_1)(z - p_2)\cdots(z - p_n) \tag{54}$$

- The coefficients of the feedback gain matrix **K** are obtained from equation 54.
- The necessary and sufficient condition for solving this system is that the rank of the state controllability matrix be equal to n, i.e., the system must be completely controllable.
- If the system is completely controllable, then the matrix $(z\mathbf{I} - \mathbf{G} + \mathbf{HK})$ is nonsingular.

- Given a system

$$\mathbf{x}(k+1) = \mathbf{G}\mathbf{x}(k) + \mathbf{H}u(k) \tag{55}$$

that is completely state controllable, if state feedback is applied according to $u(k) = r(k) - \mathbf{K}\mathbf{x}(k)$, then the closed-loop poles can be placed at $z = p_1, z = p_2, \dots, z = p_n$ by matching the characteristic equations:

$$|z\mathbf{I} - \mathbf{G} + \mathbf{H}\mathbf{K}| = (z - p_1)(z - p_2)\cdots(z - p_n) = 0 \tag{56}$$

- The left-hand side is a polynomial in z whose coefficients depend on the entries of the state-feedback matrix $\mathbf{K}$.
- By equating these coefficients to those of the desired characteristic polynomial, a system of n equations with n unknowns is obtained. Solving it yields the feedback gains.

- Consider the system described by

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -0.5 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \tag{57}$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \tag{58}$$

- Determine the values of the state-feedback gain matrix **K** such that the closed-loop poles are located at

$$p_{1,2} = 0.5 \pm j0.5.$$

- **Solution:** First, verify that the system is completely state controllable. The controllability matrix is

$$\mathbf{M}_C = \begin{bmatrix} \mathbf{H} & \mathbf{GH} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -0.5 \end{bmatrix} \tag{59}$$

  which has rank 2. Therefore, the system is completely state controllable.

- Since the system is controllable, it is possible to find a state-feedback law that satisfies the design requirements. According to equation 54,

$$|z\mathbf{I} - \mathbf{G} + \mathbf{HK}| = (z - p_1)(z - p_2)\cdots(z - p_n) = 0 \tag{60}$$

which, for this particular case, becomes

$$|z\mathbf{I} - \mathbf{G} + \mathbf{HK}| = z^2 - z + 0.5 = 0 \tag{61}$$

- Substituting the system matrices and carrying out the calculations gives

$$z^2 + z(0.5 + k_2) + (1 + k_1) = z^2 - z + 0.5 = 0 \tag{62}$$

- Equating the coefficients of both polynomials yields

$$\begin{aligned} k_1 &= -0.5 \\ k_2 &= -1.5 \end{aligned}$$

- Therefore, the state-feedback gain matrix is

$$\mathbf{K} = [-0.5 \quad -1.5].$$

- The response of the open-loop system and the response of the system with the designed state feedback are shown in Figure 4.

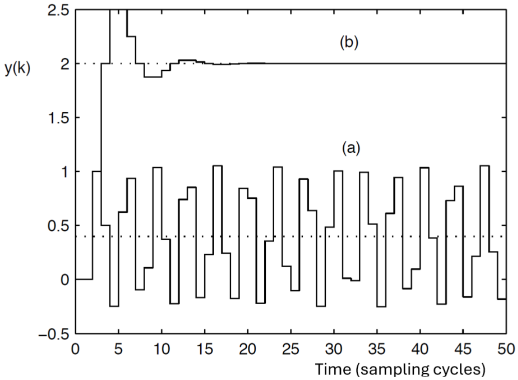


Figure 4: System response for the example: a) without feedback and b) with state feedback.

- If the system is not already in controllable canonical form, the coefficient-comparison method may lead to cumbersome calculations.
- An alternative is to transform the system into controllable canonical form, design the controller in that representation, and then convert the controller back to the original state-space representation.
- Consider a system described by

$$\mathbf{x}(k+1) = \mathbf{G}\mathbf{x}(k) + \mathbf{H}u(k) \tag{63}$$

$$y(k) = \mathbf{C}\mathbf{x}(k) \tag{64}$$

whose controllability matrix is

$$\mathbf{M}_C = \begin{bmatrix} \mathbf{H} & \mathbf{G}\mathbf{H} & \dots & \mathbf{G}^{n-1}\mathbf{H} \end{bmatrix} \tag{65}$$

- Suppose that a transformation $\mathbf{T}$ is applied that converts the system into controllable canonical form:

$$\mathbf{x}(k) = \mathbf{T}\mathbf{x}'(k) \tag{66}$$

Pole Placement by Transformation

- The transformed system is then described by

$$\mathbf{x}'(k+1) = \mathbf{T}^{-1}\mathbf{G}\mathbf{T}\mathbf{x}'(k) + \mathbf{T}^{-1}\mathbf{H}u(k) \tag{67}$$

$$y(k) = \mathbf{C}\mathbf{T}\mathbf{x}'(k) \tag{68}$$

- As shown previously in 37, the controllability matrix of the transformed system is

$$\mathbf{M}_{C'} = \mathbf{T}^{-1}\begin{bmatrix}\mathbf{H} & \mathbf{G}\mathbf{H} & \dots & \mathbf{G}^{n-1}\mathbf{H}\end{bmatrix} = \mathbf{T}^{-1}\mathbf{M}_C \tag{69}$$

and therefore the transformation matrix can be obtained as

$$\mathbf{T} = \mathbf{M}_C\mathbf{M}_{C'}^{-1} \tag{70}$$

- Once the system has been transformed into controllable canonical form and satisfies the controllability condition, the state-feedback controller can be designed. In this representation:

$$|z\mathbf{I} - \mathbf{G}' + \mathbf{H}'\mathbf{K}'| = (z-p_1)(z-p_2)\cdots(z-p_n) = 0 \tag{71}$$

Pole Placement by Transformation

- By equating coefficients, the gain matrix $\mathbf{K}'$ of the controller is determined.
- The closed-loop system in controllable canonical form is described by

$$\begin{aligned} \mathbf{x}'(k+1) &= \mathbf{T}^{-1}\mathbf{G}\mathbf{T}\mathbf{x}'(k) + \mathbf{T}^{-1}\mathbf{H}u(k) && (72) \\ u(k) &= -\mathbf{K}'\mathbf{x}'(k) + r(k) && (73) \\ y(k) &= \mathbf{C}\mathbf{T}\mathbf{x}'(k) && (74) \end{aligned}$$

and therefore

$$\begin{aligned} \mathbf{x}'(k+1) &= \mathbf{T}^{-1}\mathbf{G}\mathbf{T}\mathbf{x}'(k) - \mathbf{T}^{-1}\mathbf{H}\mathbf{K}'\mathbf{x}'(k) + \mathbf{T}^{-1}\mathbf{H}r(k) && (75) \\ &= \left(\mathbf{T}^{-1}\mathbf{G}\mathbf{T} - \mathbf{T}^{-1}\mathbf{H}\mathbf{K}'\right)\mathbf{x}'(k) + \mathbf{T}^{-1}\mathbf{H}r(k) && (76) \\ y(k) &= \mathbf{C}\mathbf{T}\mathbf{x}'(k) && (77) \end{aligned}$$

- We now reverse the transformation and return to the original representation by applying

$$\mathbf{x}' = \mathbf{T}^{-1}\mathbf{x}.$$

Pole Placement by Transformation

- Thus,

$$\mathbf{x}(k+1) = \mathbf{G}\mathbf{x}(k) - \mathbf{H}\mathbf{K}'\mathbf{T}^{-1}\mathbf{x}(k) + \mathbf{H}r(k) \tag{78}$$

$$= \left(\mathbf{G} - \mathbf{H}\mathbf{K}'\mathbf{T}^{-1}\right)\mathbf{x}(k) + \mathbf{H}r(k) \tag{79}$$

$$y(k) = \mathbf{C}\mathbf{x}(k) \tag{80}$$

- Comparing this result with the characteristic equation of the closed-loop system in the original representation,

$$|z\mathbf{I} - \mathbf{G} + \mathbf{H}\mathbf{K}| = 0 \tag{81}$$

  we obtain

$$\mathbf{K} = \mathbf{K}'\mathbf{T}^{-1} \tag{82}$$

- If the **system is uncontrollable**, it can still be transformed into controllable canonical form and a gain matrix $\mathbf{K}'$ can be computed. However, in that case $\mathbf{T}$ is singular, so it is **not possible to compute the inverse transformation**, and consequently $\mathbf{K}$ cannot be obtained.

Pole Placement by Transformation

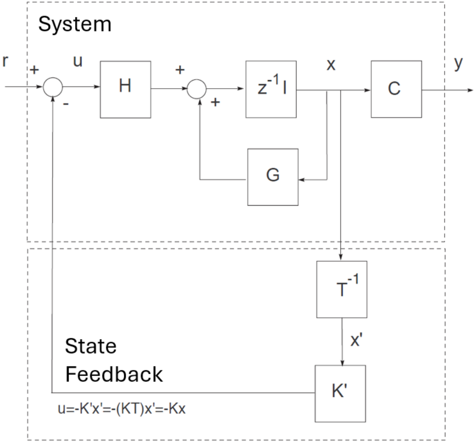


Figure 5: Transformation of a system into another state-space representation for control design.

- Consider the system described by the following equations:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} -0.5 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \tag{83}$$

$$y(k) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \tag{84}$$

  and suppose that the state-feedback gain matrix $\mathbf{K}$ is to be designed so that the closed-loop poles are located at

$$p_{1,2} = 0.5 \pm j0.5.$$

- **Solution:** First, let us verify whether the system is completely state controllable. The controllability matrix is

$$\mathbf{M}_C = \begin{bmatrix} \mathbf{H} & \mathbf{GH} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \tag{85}$$

  whose rank is 1. Therefore, the system is **not completely state controllable**.

- The system is therefore **not state controllable**. If control design is desired, it is necessary to move to a state-space representation that is controllable. To do this, we first obtain the transfer function:

$$\mathbf{F}(z) = \mathbf{C}[z\mathbf{I} - \mathbf{G}]^{-1}\mathbf{H} = \frac{z + 0.5}{z^2 + 1.5z + 0.5}$$

and then convert it directly into controllable canonical form:

$$\begin{bmatrix} x_1'(k+1) \\ x_2'(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.5 & -1.5 \end{bmatrix} \begin{bmatrix} x_1'(k) \\ x_2'(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \tag{86}$$

$$y(k) = \begin{bmatrix} 0.5 & 1 \end{bmatrix} \begin{bmatrix} x_1'(k) \\ x_2'(k) \end{bmatrix} \tag{87}$$

- In this new representation, the controllability matrix is

$$\mathbf{M'}_C = \begin{bmatrix} \mathbf{H'} & \mathbf{G'H'} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1.5 \end{bmatrix} \tag{88}$$

which has rank 2, and therefore the transformed system is controllable.

In controllable canonical form, the state-feedback gain matrix $\mathbf{K'}$ can be computed to place the closed-loop poles at the desired locations.

$$|z\mathbf{I} - \mathbf{G'} + \mathbf{H'K'}| = (z - p_1)(z - p_2)\cdots(z - p_n) = 0 \tag{89}$$

- For this particular case:

$$|z\mathbf{I} - \mathbf{G}' + \mathbf{H}'\mathbf{K}'| = z^2 - z + 0.5 = 0 \tag{90}$$

Substituting the matrices and simplifying gives

$$z^2 + z(1.5 + k_2') + (0.5 + k_1') = z^2 - z + 0.5 = 0 \tag{91}$$

Equating coefficients yields

$$\begin{aligned} k_1' &= 0 \\ k_2' &= -2.5 \end{aligned}$$

and therefore the state-feedback gain matrix is

$$\mathbf{K}' = [0 \quad -2.5]$$

In this case it is not possible to apply the inverse transformation $\mathbf{T}^{-1}$ to obtain the gain matrix in the original state-space representation.

- Let us verify this result.
  The transformation matrix relating both representations is

$$\mathbf{T} = \mathbf{M}_C \mathbf{M}'^{-1}_C \quad = \quad \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1.5 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0.5 & 1 \end{bmatrix} \tag{92}$$

  and since the rank of $\mathbf{T}$ is 1, the matrix is singular and therefore noninvertible.
  Consequently, the original system cannot be transformed back from the controllable canonical representation using $\mathbf{T}^{-1}$.

- Consider the system shown in the figure, where an additional gain k_0 has been introduced in order to adjust the static gain of the system.
  The purpose of this gain is to provide the possibility of adjusting the steady-state error of the closed-loop system.

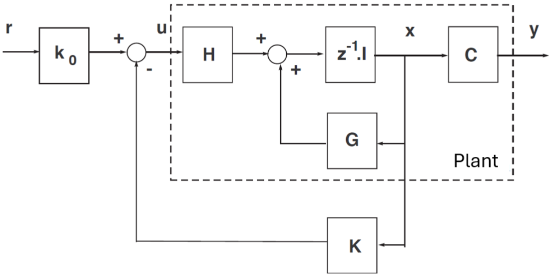


Figure 6: State-feedback control with gain adjustment

- The system equations in this case are

$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{G}\mathbf{x}(k) + \mathbf{H}u(k) \\ y(k) &= \mathbf{C}\mathbf{x}(k) \end{aligned} \qquad (93)$$

where $\mathbf{G} : n \times n$, $\mathbf{H} : n \times 1$, $\mathbf{C} : 1 \times n$, $\mathbf{x} : n \times 1$, $u : 1 \times 1$, and $y : 1 \times 1$.

- The control signal $u(k)$ applied to the system is given by

$$u(k) = k_0 r(k) - \mathbf{K}\mathbf{x}(k) \qquad (94)$$

- From equations 93 and 94 we obtain

$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{G}\mathbf{x}(k) + \mathbf{H}[-\mathbf{K}\mathbf{x}(k) + k_0 r(k)] \\ &= [\mathbf{G} - \mathbf{H}K]\mathbf{x}(k) + \mathbf{H}k_0 r(k) \end{aligned} \qquad (95)$$

- The transfer function of the system has the form

$$F(z) = \frac{Y(z)}{R(z)} = \mathbf{C}[z\mathbf{I} - \mathbf{G} + \mathbf{H}\mathbf{K}]^{-1}\mathbf{H}k_0 \qquad (96)$$

- Equation 96 shows that the denominator of the transfer function is the determinant of the matrix $[z\mathbf{I} - (\mathbf{G} - \mathbf{HK})]$, while the gain k_0 appears in the numerator, having been introduced in the forward path.
- If a specific behavior is desired:
  - To adjust the system dynamics, the closed-loop pole locations are first assigned.
  - To adjust the steady-state error, once the poles have been fixed, the value of k_0 is adjusted.
- Assuming that the system input is $R(z)$, the steady-state response is given by

$$\lim_{k\to\infty} y(k) = \lim_{z\to 1}(1 - z^{-1})Y(z) = \lim_{z\to 1}(1 - z^{-1})R(z)F(z) \qquad (97)$$

- In equation 97, if the desired steady-state value of the output for the input $R(z)$ is known, it is possible to determine the value of the gain k_0 from the right-hand side of the expression. In this way, the steady-state response can also be adjusted.

- Let us consider the same system as in the previous example. The values of the state-feedback gain have already been determined in order to place the poles at specified locations.

  We now seek to achieve zero steady-state error for a step input.
- **Solution:** Let us first verify whether the system satisfies the steady-state error specification. The closed-loop transfer function is

$$F(z) = \frac{Y(z)}{R(z)} = \mathbf{C}[z\mathbf{I} - \mathbf{G} + \mathbf{HK}]^{-1}\mathbf{H} \tag{98}$$

  Expanding this expression, we obtain

$$F(z) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} z & -1 \\ 0.5 & z-1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{z^2 - z + 0.5} \tag{99}$$

If a unit-step input is applied to the system, the steady-state output value will be

$$\begin{aligned}\lim_{k\to\infty} y(k) &= \lim_{z\to 1}(1-z^{-1})R(z)F(z)\\ &= \lim_{z\to 1}(1-z^{-1})\frac{z}{z-1}\frac{1}{z^2-z+0.5}\\ &= 2 \end{aligned} \qquad (100)$$

For a unit-step input, the steady-state output of the system is equal to 2.

If we want the steady-state output of the system to be equal to 1, we introduce a gain k_0 into the system, so that the closed-loop transfer function becomes

$$F(z) = \frac{Y(z)}{R(z)} = \mathbf{C}[z\mathbf{I} - \mathbf{G} + \mathbf{H}\mathbf{K}]^{-1}\mathbf{H}k_0 \qquad (101)$$

The steady-state response will be

$$\begin{aligned}\lim_{k\to\infty} y(k) &= \lim_{z\to 1}(1-z^{-1})R(z)F(z) \\ &= \lim_{z\to 1}(1-z^{-1})\frac{z}{z-1}\frac{k_0}{z^2-z+0.5} \\ &= \frac{k_0}{0.5} \qquad (102)\end{aligned}$$

Therefore, in order for the steady-state value of the system response to be equal to 1, it is required that $k_0 = 0.5$.

System Type Modification

- It is not always possible to satisfactorily adjust both the dynamic response and the steady-state response of a system using the two techniques discussed so far.
- This is particularly true for low-order systems, where the degrees of freedom for selecting the closed-loop pole locations are limited.
- If a second-order dynamic response and zero steady-state error are desired, the system must be of third order so that both requirements can be adjusted. If the order is lower, either the dynamics or the steady-state response can be adjusted, but not both simultaneously.
- If the system order does not allow both design requirements to be satisfied simultaneously, it is possible to increase the system type by adding an integrator in the forward path of the system (this increases both the order and the type of the system).

System Type Modification

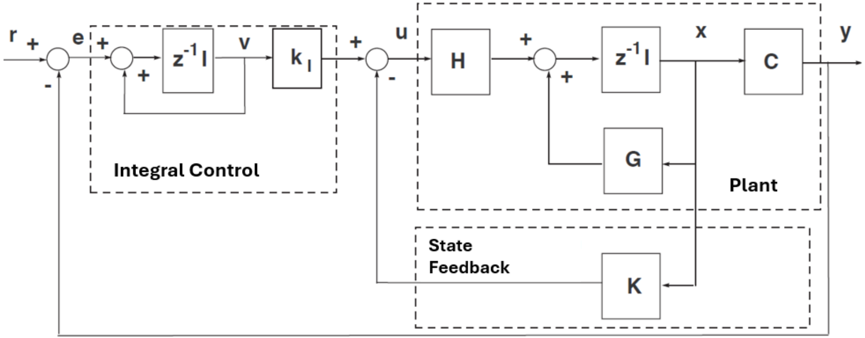


Figure 7: State feedback with an additional integrator.

System Type Modification

- The system shown in Figure 7 is described by the following equations:

$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{G}\mathbf{x}(k) - \mathbf{H}\mathbf{K}\mathbf{x}(k) + \mathbf{H}k_i v(k) \\ v(k+1) &= v(k) + e(k) \\ e(k) &= r(k) - y(k) \\ y(k) &= \mathbf{C}\mathbf{x}(k) \end{aligned} \tag{103}$$

- In this expression, $v(k+1)$ can be rewritten as

$$v(k+1) = v(k) + r(k) - \mathbf{C}\mathbf{x}(k) \tag{104}$$

- If we now consider a new augmented state vector including $v(k)$, we obtain the following expression

$$\begin{aligned} \begin{bmatrix} \mathbf{x}(k+1) \\ v(k+1) \end{bmatrix} &= \begin{bmatrix} \mathbf{G}-\mathbf{H}\mathbf{K} & \mathbf{H}k_i \\ -\mathbf{C} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ v(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r(k) \\ y(k) &= \begin{bmatrix} \mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ v(k) \end{bmatrix} \end{aligned} \tag{105}$$

System Type Modification

- The figure shows that adding an additional state variable that accumulates the error between the system input and output is essentially a state-feedback configuration.

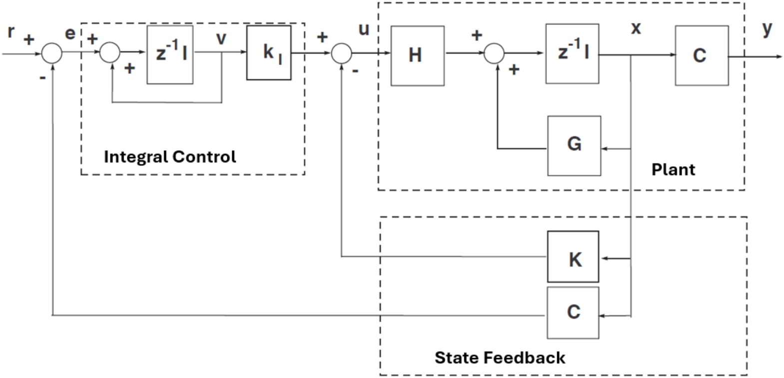


Figure 8: State feedback with an additional integrator (2).

Example

- Let us consider the same system as in the previous example, for which the state-feedback gain values have already been determined, together with the gain required to achieve zero steady-state error for a step input.

  Let us now attempt to achieve zero steady-state error for a ramp input as well.
- **Solution:** Let us evaluate the steady-state error produced by a ramp input for the system:

$$e(k) = r(k) - y(k) \tag{106}$$

and taking the z-transform, we obtain

$$E(z) = R(z) - Y(z) = R(z) - F(z)R(z) = (1 - F(z))R(z) \tag{107}$$

Example (Cont.)

To determine the steady-state error, we compute the limit of this error expression as $k \to \infty$, obtaining

$$\begin{aligned}
\lim_{k\to\infty} e(k) &= \lim_{z\to 1}(1 - z^{-1})R(z)(1 - F(z)) \\
&= \lim_{z\to 1}(1 - z^{-1})\frac{Tz^{-1}}{(1 - z^{-1})^2}\left(1 - \frac{k_0}{z^2 - z + 0.5}\right) \\
&= \frac{T(0.5 - k_0)}{0}
\end{aligned} \tag{108}$$

Except for the case $k_0 = 0.5$, where the error becomes indeterminate, the steady-state error tends to infinity in all other cases (in practice, the possibility of adjusting k_0 exactly to 0.5 is very small, so even slight deviations will cause the error to diverge).

Example (Cont.)

If an integrator is introduced while maintaining the previously computed values of the state-feedback gain matrix **K**, the system matrices with integral control become

$$\mathbf{G}' = \begin{bmatrix} \mathbf{G} - \mathbf{HK} & \mathbf{H}k_i \\ -\mathbf{C} & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -0.5 & 1 & k_i \\ -1 & 0 & 1 \end{bmatrix}$$

$$\mathbf{H}' = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{C}' = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

Example (Cont.)

The transfer function of the system becomes

$$\begin{aligned}
F'(z) &= \mathbf{C}'[z\mathbf{I} - \mathbf{G}']^{-1}\mathbf{H}' \\
&= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} z & -1 & 0 \\ 1 & z+0.5 & -k_i \\ 1 & 0 & z-1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
&= \frac{k_i}{(z-1)(z^2 - z + 0.5) + k_i}
\end{aligned}$$

Example (Cont.)

Taking the limit as $k \to \infty$ in the steady-state error expression, we obtain

$$\begin{aligned}
\lim_{k\to\infty} e(k) &= \lim_{z\to 1}(1-z^{-1})R(z)(1-F'(z)) \\
&= \lim_{z\to 1}(1-z^{-1})\frac{Tz^{-1}}{(1-z^{-1})^2}\left(1-\frac{k_i}{(z-1)(z^2-z+0.5)+k_i}\right) \\
&= \frac{0.5T}{k_i} \qquad (109)
\end{aligned}$$

The error is not completely eliminated, but it can be made very small by sampling the system with a small sampling period T and choosing a large value of k_i.

Systems with Vector Input

- We have considered the problem of designing a state-feedback controller for systems with a scalar control input.
- When the control input to the system is a vector, we have much greater freedom in selecting the control signals used to control the system.
- The controllability matrix had dimensions $n \times n$ for systems with a scalar input, whereas for vector inputs its dimensions are $n \times nr$ (assuming that the state vector has dimensions $n \times 1$ and the control input has dimensions $r \times 1$).

Example

- Consider the system defined by the following equations

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.5 & -0.5 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0.25 \end{bmatrix} \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix} \quad (110)$$

  and suppose that we wish to determine the values of the gains in the state-feedback matrix **K** (of dimension 2×2) so that the closed-loop poles are located at $p_1 = 0.5 \pm j0.5$.
- **Solution:** Let us check the state controllability matrix

$$\mathbf{M}_C = [\mathbf{H}|\mathbf{GH}] = \begin{bmatrix} 0 & 0 & 1 & 0.25 \\ 1 & 0.25 & -0.5 & -0.125 \end{bmatrix} \quad (111)$$

  which has rank 2, and therefore the system is completely state controllable.

Example (Cont.)

We want the closed-loop poles to be located at $p_1 = 0.5 - j0.5$ and $p_2 = 0.5 + j0.5$, therefore

$$|z\mathbf{I} - \mathbf{G} + \mathbf{H}\mathbf{K}| = z^2 - z + 0.5 = 0 \tag{112}$$

Substituting the matrix values, we obtain

$$z^2 + z(0.5 + k_{12} + 0.25k_{22}) + 0.5 + k_{11} + 0.25k_{21} = z^2 - z + 0.5 = 0 \tag{113}$$

and equating the coefficients of both polynomials yields

$$\begin{aligned} k_{12} + 0.25k_{22} &= -1.5 \\ k_{11} + 0.25k_{21} &= 0 \end{aligned} \tag{114}$$

We have two equations and four unknowns, which means that there are two degrees of freedom in the coefficients of the gain matrix.

- If we choose $k_{12} = k_{21} = 1$, then $k_{22} = -10$ and $k_{11} = -0.25$, giving

$$\mathbf{K} = \begin{bmatrix} -0.25 & 1 \\ 1 & -10 \end{bmatrix}$$