Control Engineering II

Topic 9:
State Feedback Control

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uc3m | Universidad Carlos III de Madrid
OpenCourseWare

2026

Given the system:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} u(k)$$

determine the sequence u(0), u(1), u(2) so that the system passes from the state $(0,1,0)^T$ at instant 0 to the state $(1,1,2)^T$ at instant 3.

Solution

First, the controllability matrix must be calculated to see if the system is completely controllable in state.

$$M_c = \begin{bmatrix} H & GH & G^2H \end{bmatrix}$$

$$H = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad GH = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

$$G^2H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

$$M_C = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -2 & 2 \\ 1 & 2 & 4 \end{bmatrix} \quad det(M_C) = -12$$

Since the rank of the controllability matrix is 3, the system is completely controllable in state.

Problem 1 (2/2)

We proceed to calculate the requested solution using the following expression:

$$x\left(n\right)-G^{n}x\left(0\right)=M_{C}\begin{bmatrix}u(n-1)\\u(n-2)\\\dots\\u(0)\end{bmatrix}$$

$$\begin{bmatrix}x_1\left(3\right)\\x_2\left(3\right)\\x_3\left(3\right)\end{bmatrix}-G^{3}\begin{bmatrix}x_1\left(0\right)\\x_2\left(0\right)\\x_3\left(0\right)\end{bmatrix}=M_{C}\begin{bmatrix}u(2)\\u(1)\\u(0)\end{bmatrix}$$

$$\begin{bmatrix}1\\1\\2\end{bmatrix}-\begin{bmatrix}1&0&0\\0&-1&0\\0&0&8\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}1&1&1\\2&-2&2\\1&-2&4\end{bmatrix}\begin{bmatrix}u(2)\\u(1)\\u(0)\end{bmatrix}$$

$$\left.\begin{array}{c}u\left(2\right)+u\left(1\right)+u\left(0\right)=1\\2u\left(2\right)-2u\left(1\right)+2u\left(0\right)=2\\u\left(2\right)+2u\left(1\right)+4u\left(0\right)=2\end{array}\right\}$$

We solve the system to obtain the sought input sequence:

$$u\left(0\right)=0.3333\quad u\left(1\right)=0\quad u\left(2\right)=0.6667$$

Problem 2 (1/2)

Given the system:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} u(k)$$

determine the sequence u(o), u(1), u(2) so that the system passes from the state $(0,0,0)^T$ at instant 0 to the state $(3,2,2)^T$ at instant 3.

Solution

First, the controllability matrix must be calculated to see if the system is completely controllable in state.

$$M_c = \begin{bmatrix} H & GH & G^2H \end{bmatrix}$$

$$H = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \quad GH = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}$$

$$G^2H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$$

$$M_C = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix} \quad det(M_C) = 0$$

Since the rank of the controllability matrix is 2, the system is not completely controllable in state.

We proceed to calculate the requested solution using the following expression:

Problem 2 (2/2)

$$x(n) - G^n x(0) = M_C \begin{bmatrix} u(n-1) \\ u(n-2) \\ \cdots \\ u(0) \end{bmatrix}$$

$$\begin{bmatrix} x_1(3) \\ x_2(3) \\ x_3(3) \end{bmatrix} - G^3 \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = M_C \begin{bmatrix} u(2) \\ u(1) \\ u(0) \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & -8 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u(2) \\ u(1) \\ u(0) \end{bmatrix}$$

$$\left.\begin{array}{c} 2u(2) - 2u(1) + 2u(0) = 3 \\ u(2) + 2u(1) + 4u(0) = 2 \\ 2 = 0 \end{array}\right\}$$

It is observed that the third equation is not satisfied, so the third state is not reachable. What will be possible is to reach the desired values for the first two states.

We have a system with two equations and three unknowns, whose solution as a function of u(0) is:

$$u(1) = \tfrac{1}{6} - u(0) \quad u(2) = \tfrac{5}{3} - 2u(0)$$

If we set u(0)=1:

$$u(0) = 1 \quad u(1) = \tfrac{-5}{6} \quad u(2) = \tfrac{-1}{3}$$

Given the system:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u(k)$$

$$\begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

determine if it is completely output controllable, and calculate the sequence u(o), u(1), u(2) so that the system with initial state $(0,0,0)^T$ **has an output** $y(k) = (2,2)^T$ **at instant 3.**
What would happen if the value of C changes and becomes the following?

$$C = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

Solution

First, the output controllability matrix must be calculated to see if the system is completely output controllable.

$$M_c = \begin{bmatrix} CH & CGH & CG^2H \end{bmatrix}$$

$$M_{CS} = \begin{bmatrix} 4 & -3 & 3 \\ -1 & -2 & 6 \end{bmatrix} \quad rango(M_{CS}) = 2$$

Since the rank of the output controllability matrix is 2, the system is completely output controllable.

Problem 3 (2/3)

We proceed to calculate the requested solution using the following expression:

$$y(n) - CG^n x(0) = M_{CS} \begin{bmatrix} u(n-1) \\ u(n-2) \\ \dots \\ u(0) \end{bmatrix}$$

$$\begin{bmatrix} y_1(3) \\ y_2(3) \end{bmatrix} - CG^3 \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = M_{CS} \begin{bmatrix} u(2) \\ u(1) \\ u(0) \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 5 & -8 & -2 \\ -6 & -8 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 & -3 & 3 \\ -1 & -2 & 6 \end{bmatrix} \begin{bmatrix} u(2) \\ u(1) \\ u(0) \end{bmatrix}$$

$$\left.\begin{aligned} 4u(2) - 3u(1) + 3u(0) = 2 \\ -u(2) - 2u(1) + 6u(0) = 2 \end{aligned}\right\}$$

We have a system with two equations and three unknowns, whose solution as a function of u(0) is:

$$u(1) = \tfrac{-10}{11} + \tfrac{27}{11}u(0) \quad u(2) = \tfrac{-2}{11} + \tfrac{12}{11}u(0)$$

If we set u(0)=1:

$$u(0) = 1 \quad u(1) = \tfrac{17}{11} \quad u(2) = \tfrac{10}{11}$$

If the value of C changes, we recalculate the value of the output controllability matrix:

$$M_{CS} = \begin{bmatrix} 4 & -3 & 3 \\ 8 & -6 & 6 \end{bmatrix} \quad rango(M_{CS}) = 1$$

Problem 3 (3/3)

Since the rank of the output controllability matrix is 1, the system is not completely output controllable.
We proceed to calculate the requested solution:

$$\begin{bmatrix} y_1(3) \\ y_2(3) \end{bmatrix} - CG^3 \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = M_{CS} \begin{bmatrix} u(2) \\ u(1) \\ u(0) \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 & -3 & 3 \\ 8 & -6 & 6 \end{bmatrix} \begin{bmatrix} u(2) \\ u(1) \\ u(0) \end{bmatrix}$$

$$\left.\begin{array}{l} 4u(2) - 3u(1) + 3u(0) = 2 \\ 8u(2) - 6u(1) + 6u(0) = 2 \end{array}\right\}$$

We see that we have a system that has no solution. If we put it in terms of a generic output:

$$\begin{bmatrix} y_1(3) \\ y_2(3) \end{bmatrix} \begin{bmatrix} 4 & -3 & 3 \\ 8 & -6 & 6 \end{bmatrix} \begin{bmatrix} u(2) \\ u(1) \\ u(0) \end{bmatrix}$$

It is observed that one row is linearly dependent on the other, such that when we fix one of the two outputs, the other depends on it: $y_2(3) = 2y_1(3)$.

Problem 4

Given the system:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u(k)$$

$$\begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

determine if it is observable.

Solution

First, the observability matrix must be calculated:

$$M_O = \begin{bmatrix} C & CG & CG^2 \end{bmatrix}^T$$

$$C = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -2 \end{bmatrix} \quad CG = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -2 \\ -2 & -2 & 2 \end{bmatrix}$$

$$CG^2 = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ -2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 4 & 2 \\ 4 & 4 & 2 \end{bmatrix}$$

$$M_O = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -2 \\ 1 & -2 & -2 \\ -2 & -2 & 2 \\ -3 & 4 & 2 \\ 4 & 4 & 2 \end{bmatrix} \quad rango(M_O) = 3$$

Since the rank is three, the system is completely observable.

Problem 5 (1/2)

Given the system:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0.5 \\ -0.5 & 2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k)$$

$$[y_1(k)] = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

determine the values of the gains of the state feedback matrix K so that the closed-loop poles of the system are at 0.2 and 0.5.

Solution

First, it is checked if the system is completely controllable in state, for which the controllability matrix is calculated:

$$M_c = \begin{bmatrix} H & GH \end{bmatrix}$$

$$M_C = \begin{bmatrix} 1 & 1 \\ 0 & -0.5 \end{bmatrix} \quad det\left(M_C\right) = 0.5$$

The system is completely controllable in state. Therefore, it is possible to find a state feedback that meets the specifications through the following expression:

$$\left|zI - G + HK\right| = (z - p_1)(z - p_2)\dots(z - p_n)$$

$$\left| \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 1 & 0.5 \\ -0.5 & 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} \right| = (z - 0.2)(z - 0.5)$$

If this system is solved, the desired values of the gain matrix K are reached:

$$\left| \begin{bmatrix} z - 1 + k_1 & -0.5 + k_2 \\ 0.5 & z - 2 \end{bmatrix} \right| = z^2 - 0.7z + 0.1$$

$$z^2 + (k_1 - 3)\, z + (-2k_1 - 0.5k_2 + 2.25) = z^2 - 0.7z + 0.1$$

We have a system with two equations and two unknowns:

$$\left. \begin{array}{c} k_1 = 2.3 \\ -2k_1 - 0.5k_2 = -2.15 \end{array} \right\}$$

The solution will be:

$$k_1 = 2.3 \quad k_2 = -4.9$$

Thus, the sought-after gain matrix is:

$$K = [2.3 \quad -4.9]$$

Problem 6 (1/2)

Given the system:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} u(k)$$

$$[y_1(k)] = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

determine the values of the gains of the state feedback matrix K so that the closed-loop poles of the system are at -0.8, -0.6 and 0.5.

Solution

First, it is checked if the system is completely controllable in state, for which the controllability matrix is calculated:

$$M_c = \begin{bmatrix} H & GH & G^2H \end{bmatrix}$$

$$H = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \quad GH = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$$G^2H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -0.25 & 0.25 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$M_C = \begin{bmatrix} 2 & 2 & 2 \\ 1 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad det\,(M_C) = 4$$

Problem 6 (2/2)

The system is completely controllable in state. Therefore, it is possible to find a state feedback that meets the specifications through the following expression:

$$|zI - G + HK| = (z - p_1)(z - p_2)\dots(z - p_n)$$

$$\left| \begin{bmatrix} z & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & z \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0.5 & 0.5 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} \right|$$
$$= (z - 0.5)(z + 0.8)(z + 0.6)$$

If this system is solved, the desired values of the gain matrix K are reached:

$$\left| \begin{bmatrix} 2k_1 + z - 1 & 2k_2 & 2k_3 \\ k_1 & k_2 + z + 1 & k_3 \\ k_1 & k_2 - 0.5 & k_3 + z - 0.5 \end{bmatrix} \right| = z^3 + 0.9z^2 - 0.22z - 0.24$$

$$z^3 + (2k_1 + k_2 + k_3 - 0.5)\, z^2 + (k_1 - 1.5k_2 + 0.5k_3 - 1)\, z$$
$$+ (-k_1 + 0.5k_2 - 1.5k_3 + 0.5) = z^3 + 0.9z^2 - 0.22z - 0.24$$

We have a system with three equations and three unknowns:

$$\left. \begin{array}{r} 2k_1 + k_2 + k_3 = 1.4 \\ k_1 - 1.5k_2 + 0.5k_3 = 0.78 \\ -k_1 + 0.5k_2 - 1.5k_3 = -0.74 \end{array} \right\}$$

Problem 6 (3/3)

The solution will be:

$$k_1 = 0.72 \quad k_2 = -0.04 \quad k_3 = 0$$

Thus, the sought-after gain matrix is:

$$K = [0.72 \quad -0.04 \quad 0]$$

Problem 7

Determine the observability of the system defined by the following equations:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ -2 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} u(k)$$

$$[y_1(k)] = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

Solution

To estimate observability, the observability matrix must be calculated:

$$M_O = \begin{bmatrix} C \\ CG \\ CG^2 \end{bmatrix}$$

$$M_O = \begin{bmatrix} 1 & 2 & 1 \\ -3 & -2 & -2 \\ 1 & 2 & 3 \end{bmatrix}$$

Since the determinant of the matrix is 8, the rank will be equal to three and the system is completely observable.

Problem 8

Determine the observability of the system defined by the following equations:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & -0.8 & 0 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u(k)$$

$$[y_1(k)] = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

Solution

To estimate observability, the observability matrix must be calculated:

$$M_O = \begin{bmatrix} C \\ CG \\ CG^2 \end{bmatrix}$$

$$M_O = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -0.8 & 0 \\ 0.8 & 0.64 & 0 \end{bmatrix}$$

Since the determinant of the matrix is 0, the rank will be equal to two and the system is not completely observable.

Given the system defined by the following equations:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} -1 & 0.5 \\ 0.2 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k)$$
$$[y_1(k)] = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Determine the values of the state feedback matrix K so that the closed-loop poles of the system are at $p_{1,2} = 0.6 \pm j0.6$**.**

Solution

First, we check that the system is completely controllable in state via the controllability matrix:

$$M_C = \begin{bmatrix} H & GH \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 0.2 \end{bmatrix}$$

Since it has rank 2, the system is completely controllable, so it is possible to find a state feedback matrix that meets the conditions. This is done according to the following expression:

$$|zI - G + HK| = (z - p_1)(z - p_2) \ldots (z - p_n) = 0$$

The solution for the following system must be found:

$$\left| \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} -1 & 0.5 \\ 0.2 & 1 \end{bmatrix} + \begin{bmatrix} k_1 & k_2 \\ 0 & 0 \end{bmatrix} \right| = z^2 - 1.2z + 0.72$$
$$z^2 + k_1 z - 1.1 - k_1 + 0.2k_2 = z^2 - 1.2z + 0.72$$
$$\left. \begin{array}{r} k_1 = -1.2 \\ -1.1 - k_1 + 0.2k_2 = 0.72 \end{array} \right\} \rightarrow \begin{array}{r} k_1 = -1.2 \\ k_2 = 3.1 \end{array}$$

Finally, the necessary state feedback matrix will be:

$$K = \begin{bmatrix} -1.2 & 3.1 \end{bmatrix}$$

If a gain k_0 is included to adjust the steady-state error, calculate the value of said gain so that the output has a value of 1 when the input is a unit step.

The closed-loop transfer function will be:

$$F(z) = C(zI - G + HK)^{-1} H = \frac{0.2}{z^2 - 1.2z + 0.72}$$

Its steady-state response to a unit step input is:

$$y(\infty) = \lim_{z \to 1} \left(1 - z^{-1}\right) R(z) * F(z)$$

$$= \lim_{z \to 1} \left(1 - z^{-1}\right) \frac{z}{z-1} \frac{0.2}{z^2 - 1.2z + 0.72} = 0.3846$$

If an initial gain is introduced, it can be adjusted so that the system tends to the desired value:

$$F(z) = C(zI - G + HK)^{-1} Hk_0 = \frac{0.2k_0}{z^2 - 1.2z + 0.72}$$

Its steady-state response to a unit step input is:

$$y(\infty) = \lim_{z \to 1} \left(1 - z^{-1}\right) \frac{z}{z-1} \frac{0.2k_0}{z^2 - 1.2z + 0.72} = 0.3846k_0 = 1$$

Solving gives the sought-after value:

$$k_0 = 2.6$$

Given the system defined by the following equations:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} -0.3 & 0 \\ 1.1 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k)$$

$$[y_1(k)] = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Determine by transformation the values of the state feedback matrix K so that the closed-loop poles of the system are at $p_{1,2} = 0.6 \pm j0.6$**. For this, the system must first be converted to the controllable canonical form.**

Solution

First, we check that the system is completely controllable in state via the controllability matrix:

$$M_C = \begin{bmatrix} H & GH \end{bmatrix} = \begin{bmatrix} 1 & -0.3 \\ 0 & 1.1 \end{bmatrix}$$

Since it has rank two, the system is completely controllable.
To convert the system to the controllable canonical form, the transfer function must be calculated:

$$F(z) = C(zI - G)^{-1} H = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} z+0.3 & 0 \\ -1.1 & z-1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$F(z) = \frac{2z - 0.9}{z^2 - 0.7z - 0.3}$$

Problem 10 (2/3)

The system represented in the controllable canonical form will be:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0.3 & 0.7 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$[y_1(k)] = \begin{bmatrix} -0.9 & 2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

We calculate the controllability matrix for the transformed system:

$$M'_C = \begin{bmatrix} H' & G'H' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0.7 \end{bmatrix}$$

We now proceed to calculate the feedback matrix for the new representation:

$$|zI - G' + H'K'| = (z - p_1)(z - p_2)\dots(z - p_n) = 0$$

$$\left| \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0.3 & 0.7 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ k'_1 & k'_2 \end{bmatrix} \right| = z^2 - 1.2z + 0.72$$

$$z^2 + (k'_2 - 0.7)z + k'_1 - 0.3 = z^2 - 1.2z + 0.72$$

$$k'_1 = 1.02$$
$$k'_2 = -0.5$$

It is possible to calculate the transformation matrix T to obtain the gain matrix in the initial representation:

$$T = M_C \cdot M_C'^{-1} = \begin{bmatrix} -1 & 1 \\ 1.1 & 0 \end{bmatrix}$$

Problem 10 (3/3)

Since it is invertible, the feedback matrix can be calculated:

$$K = K' T^{-1} = \begin{bmatrix} 1.02 & -0.5 \end{bmatrix} \begin{bmatrix} 0 & 0.9091 \\ 1 & 0.9091 \end{bmatrix}$$

$$K = \begin{bmatrix} -0.5 & 0.4727 \end{bmatrix}$$

If a gain k_0 is included to adjust the steady-state error, calculate the value of said gain so that the output has a value of 1 when the input is a unit step.

The transfer function for the system with state feedback will be:

$$F(z) = C(zI - G + HK)^{-1} H = \frac{2z - 0.9}{z^2 - 1.2z + 0.72}$$

Its steady-state response to a unit step input is:

$$y(\infty) = \lim_{z \to 1} \left(1 - z^{-1}\right) R(z) * F(z)$$

$$= \lim_{z \to 1} \left(1 - z^{-1}\right) \frac{z}{z-1} \frac{2z - 0.9}{z^2 - 1.2z + 0.72} = 2.1154$$

If an initial gain is introduced, it can be adjusted so that the system tends to the desired value:

$$F(z) = C(zI - G + HK)^{-1} Hk_0 = \frac{(2z - 0.9)k_0}{z^2 - 1.2z + 0.72}$$

Its steady-state response to a unit step input is:

$$y(\infty) = \lim_{z \to 1} \left(1 - z^{-1}\right) \frac{z}{z-1} \frac{(2z - 0.9)k_0}{z^2 - 1.2z + 0.72} = 2.1154k_0 = 1$$

Solving gives the sought-after value:

$$k_0 = 0.4727$$

Problem 11 (1/2)

Given the system defined by the following equations:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} -1 & 0.5 \\ 0.2 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 & 0.5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix}$$

$$[y_1(k)] = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Determine the values of the state feedback matrix K so that the closed-loop poles of the system are at $p_{1,2} = 0.6 \pm j0.6$.

Solution

First, the controllability matrix is calculated:

$$M_C = \begin{bmatrix} H & GH \end{bmatrix} = \begin{bmatrix} 1 & 0.5 & -1 & -0.5 \\ 0 & 0 & 0.2 & 0.1 \end{bmatrix}$$

Since it has rank two, the system is completely controllable in state. The state feedback matrix will be:

$$K = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$$

We calculate the values of the matrix so that the poles are in the desired places

$$|zI - G + HK| = (z - p_1)(z - p_2)\ldots(z - p_n) = 0$$

The solution for the following system must be found:

$$\left| \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} -1 & 0.5 \\ 0.2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0.5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \right| = z^2 - 1.2z + 0.72$$

$$z^2 + (k_{11} + 0.5k_{21})z - k_{11} + 0.2k_{12} - 0.5k_{21} + 0.1k_{22} - 1.1$$

$$= z^2 - 1.2z + 0.72$$

$$\left. \begin{array}{r} k_{11} + 0.5k_{21} = -1.2 \\ -k_{11} + 0.2k_{12} - 0.5k_{21} + 0.1k_{22} - 1.1 = 0.72 \end{array} \right\}$$

The system has two equations and four unknowns, so it is possible to fix two of the unknowns and then calculate the other two. If we take $k_{11} = k_{12} = 1$, the necessary state feedback matrix will be:

$$K = \begin{bmatrix} 1 & 1 \\ -4.4 & 4.2 \end{bmatrix}$$