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Control Engineering II

Topic 10:
State Observers

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State Observers

- It is often the case that some or all of the state variables of a system are not measurable (or accessible), and only the system output variable can be measured. In these situations, it is necessary to estimate those state variables that cannot be measured directly by means of state observers.

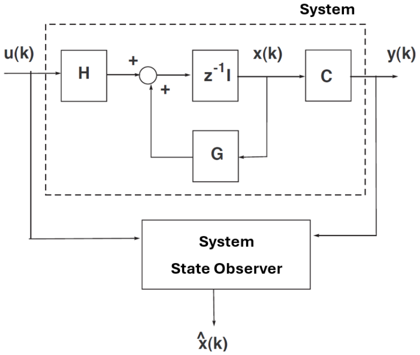

Figure 1: System with a state observer.

- If none of the state variables can be measured, the state observer must estimate all of them. This type of observer is called a *full-order observer*.
- A first idea would be to include a mathematical model of the system in the observer and feed it with the same control inputs that are applied to the real system.

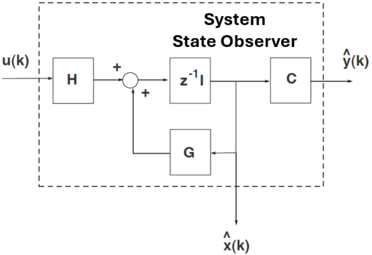

Figure 2: Full-order state observer.

- This initial scheme has three main limitations:
  1. It is necessary to know the **initial state** of the system, otherwise the model and the system will evolve differently because they do not start from the same state.
  2. The matrices **G**, **H** and **C** of the actual physical system are **not known exactly**; they are usually only estimates. As a result, the observer will accumulate deviations over time, and after a certain period, the estimated state will no longer correspond to the true state of the system.
  3. Even if the system matrices used in the observer were the exact ones, the effect of **disturbances** would cause the estimated state and the actual state to differ after some time.

- It is necessary to correct the state estimated by the observer whenever the estimated output does not match the output of the real system. To do this, a feedback of the estimation error is introduced into the observer.

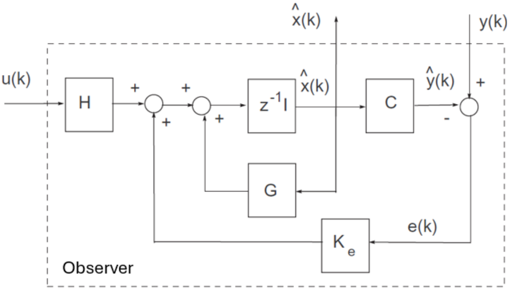

Figure 3: State observer.

Full-Order Observer

- In order to design an observer, a necessary prerequisite is that the system state can be estimated mathematically at a given instant of time. This is determined by the observability of the system.

Definition

The necessary and sufficient condition for being able to observe the state variables is that the observability condition of the system be satisfied.

Observer Transfer Function

- The observer shown in the figure has the following dynamics

$$\begin{aligned}\hat{\mathbf{x}}(k+1) &= \mathbf{G}\hat{\mathbf{x}}(k) + \mathbf{K}_e\mathbf{e}(k) + \mathbf{H}\mathbf{u}(k) \\ \hat{\mathbf{y}}(k) &= \mathbf{C}\hat{\mathbf{x}}(k) \\ \mathbf{e}(k) &= \mathbf{y}(k) - \hat{\mathbf{y}}(k)\end{aligned} \tag{1}$$

Taking the z-transform of the previous expressions, we obtain

$$z\hat{\mathbf{X}}(z) = \mathbf{G}\hat{\mathbf{X}}(z) + \mathbf{K}_e\mathbf{E}(z) + \mathbf{H}\mathbf{U}(z) \tag{2}$$

$$\hat{\mathbf{Y}}(z) = \mathbf{C}\hat{\mathbf{X}}(z) \tag{3}$$

$$\mathbf{E}(z) = \mathbf{Y}(z) - \hat{\mathbf{Y}}(z) \tag{4}$$

and therefore

$$z\hat{\mathbf{X}}(z) = (\mathbf{G} - \mathbf{K}_e\mathbf{C})\hat{\mathbf{X}}(z) + \mathbf{H}\mathbf{U}(z) + \mathbf{K}_e\mathbf{Y}(z) \tag{5}$$

or equivalently

$$\hat{\mathbf{X}}(z) = [z\mathbf{I} - \mathbf{G} + \mathbf{K}_e\mathbf{C}]^{-1}\mathbf{H}\mathbf{U}(z) + [z\mathbf{I} - \mathbf{G} + \mathbf{K}_e\mathbf{C}]^{-1}\mathbf{K}_e\mathbf{Y}(z) \tag{6}$$

Full-Order Observer

- The observer transfer function is given by

$$\hat{\mathbf{X}}(z) = \left[[z\mathbf{I} - \mathbf{G} + \mathbf{K}_e\mathbf{C}]^{-1}\mathbf{H} \quad [z\mathbf{I} - \mathbf{G} + \mathbf{K}_e\mathbf{C}]^{-1}\mathbf{K}_e \right] \begin{bmatrix} \mathbf{U}(z) \\ \mathbf{Y}(z) \end{bmatrix} \tag{7}$$

  and the observer characteristic equation is given by

$$|z\mathbf{I} - \mathbf{G} + \mathbf{K}_e\mathbf{C}| = 0 \tag{8}$$

- The observer feedback gain $\mathbf{K}_e$ must be designed so that its dynamics are appropriate for the system being observed. **The observer must have faster dynamics than the system itself.**

System with State Feedback and Observer

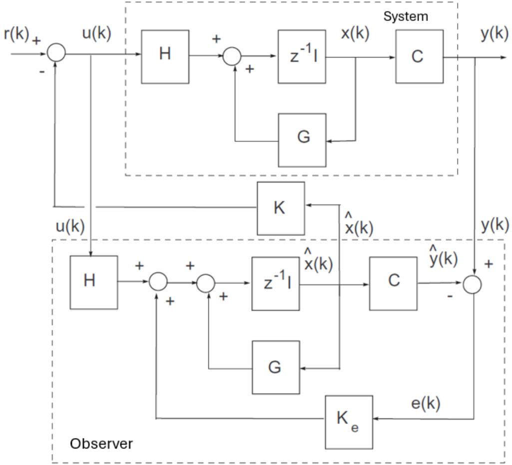

Figure 4: System with full-order observer and state feedback.

Observer Estimation Error

- We have analyzed the observer dynamics, but the quantity of greater interest is the error made by the observer, that is, the difference between the estimated state provided by the observer and the true state:

$$\tilde{\mathbf{x}}(k) = \mathbf{x}(k) - \hat{\mathbf{x}}(k) \tag{9}$$

- The state estimation error at instant $k+1$ is given by

$$\begin{aligned}
\tilde{\mathbf{x}}(k+1) &= \mathbf{x}(k+1) - \hat{\mathbf{x}}(k+1) \\
&= \mathbf{G}\mathbf{x}(k) + \mathbf{H}\mathbf{u}(k) - [\mathbf{G} - \mathbf{K}_e\mathbf{C}]\hat{\mathbf{x}}(k) - \mathbf{H}\mathbf{u}(k) - \mathbf{K}_e\mathbf{C}\mathbf{x}(k) \\
&= (\mathbf{G} - \mathbf{K}_e\mathbf{C})[\mathbf{x}(k) - \hat{\mathbf{x}}(k)] \\
&= (\mathbf{G} - \mathbf{K}_e\mathbf{C})\tilde{\mathbf{x}}(k)
\end{aligned} \tag{10}$$

- The error dynamics are determined by the eigenvalues of the matrix $\mathbf{G} - \mathbf{K}_e\mathbf{C}$.
- The eigenvalues of this matrix must lie inside the stable region, and the error dynamics should be very fast so that the estimation error vanishes as quickly as possible.

Observer Gain Design by Coefficient Matching

- The observer gain is designed in a way similar to state-feedback design. That is, the observer gain $\mathbf{K}_e$ is determined so that its characteristic equation has poles at the desired locations $p_1, p_2, \ldots, p_n$.

$$|z\mathbf{I} - \mathbf{G} + \mathbf{K}_e\mathbf{C}| = (z - p_1)(z - p_2)\ldots(z - p_n) = 0 \tag{11}$$

Consider the system defined by the following equations

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -0.5 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \quad (12)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \quad (13)$$

and suppose that we wish to design a state observer for this system, taking into account that the values of the state-feedback gain matrix **K** have already been selected so that the closed-loop poles of the system are located at $p_1 = 0.5 - j0.5$ and $p_2 = 0.5 + j0.5$.

Solution: Let us verify that the system is completely observable

$$\mathbf{M}_O = \begin{bmatrix} \mathbf{C} \\ \mathbf{CG} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (14)$$

which has rank 2, therefore the system **is observable**.
To make the error dynamics as fast as possible, we place the observer poles at the origin (**minimum-time response, or *dead-beat***). Therefore

$$|z\mathbf{I} - \mathbf{G} + \mathbf{K}_e\mathbf{C}| = z^2 = 0 \quad (15)$$

that is,

$$\left| \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & -0.5 \end{bmatrix} + \begin{bmatrix} k_{e1} \\ k_{e2} \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \right| = z^2 = 0 \quad (16)$$

from which we obtain

$$z^2 + z(k_{e1} + 0.5) + (0.5k_{e1} + k_{e2} + 1) = z^2 = 0 \quad (17)$$

and by equating coefficients we obtain $k_{e1} = -0.5$ and $k_{e2} = -0.75$.

- If the system is not in observable canonical form, the coefficient-matching method may lead to cumbersome calculations, especially for high-order systems.
- An alternative is to transform the system into observable canonical form, design the observer in this new representation, and then convert it back to the original representation.

Observer Gain Design

By Transformation

- Consider a system with the following state-space representation:

$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{G}\mathbf{x}(k) + \mathbf{H}u(k) && (18) \\ y(k) &= \mathbf{C}\mathbf{x}(k) && (19) \end{aligned}$$

and the full-order observer for this representation is given by

$$\begin{aligned} \hat{\mathbf{x}}(k+1) &= \mathbf{G}\hat{\mathbf{x}}(k) + \mathbf{H}\mathbf{u}(k) + \mathbf{K}_e(\mathbf{y}(k) - \hat{\mathbf{y}}(k)) \\ \hat{\mathbf{y}}(k) &= \mathbf{C}\hat{\mathbf{x}}(k) && (20) \end{aligned}$$

Observer Gain Design
By Transformation

- Rearranging the equation, we obtain

$$\hat{\mathbf{x}}(k+1) = (\mathbf{G} - \mathbf{K}_e\mathbf{C})\hat{\mathbf{x}}(k) + \mathbf{H}\mathbf{u}(k) + \mathbf{K}_e\mathbf{y}(k) \qquad (21)$$

  which is the expression of the full-order observer.
- From the previous expression, the relationship between $\tilde{x}(k+1)$ and $\tilde{x}(k)$, considering $\tilde{x}(k) = x(k) - \hat{x}(k)$, is

$$\tilde{\mathbf{x}}(k+1) = (\mathbf{G} - \mathbf{K}_e\mathbf{C})\tilde{\mathbf{x}}(k) \qquad (22)$$

  whose characteristic equation is

$$|z\mathbf{I} - \mathbf{G} + \mathbf{K}_e\mathbf{C}| = 0 \qquad (23)$$

- As in the case of state-feedback design, it is possible to transform the system into another canonical form, typically the observable canonical form, design the observer there, and then convert it back to the original representation.

- For the system in its original representation, the observability matrix is

$$\mathbf{M}_O = \begin{bmatrix} \mathbf{C} \\ \mathbf{CG} \\ \vdots \\ \mathbf{CG}^{n-1} \end{bmatrix} \tag{24}$$

and suppose that we apply a transformation $\mathbf{T}$ that transforms the system into observable canonical form

$$\mathbf{x}(k) = \mathbf{T}\mathbf{x}'(k) \tag{25}$$

so that the transformed system takes the form

$$\mathbf{x}'(k+1) = \mathbf{T}^{-1}\mathbf{G}\mathbf{T}\mathbf{x}'(k) + \mathbf{T}^{-1}\mathbf{H}u(k) \tag{26}$$

$$y(k) = \mathbf{C}\mathbf{T}\mathbf{x}'(k) \tag{27}$$

Observer Gain Design

By Transformation

- The relationship between the observability matrices of both representations is given by

$$\mathbf{M}'_O = \mathbf{M}_O\mathbf{T} \tag{28}$$

and therefore

$$\mathbf{M}_O^{-1}\mathbf{M}'_O = \mathbf{T} \tag{29}$$

so that the transformation matrix $\mathbf{T}$ can be obtained from the two observability matrices.

- Once the system has been transformed into observable canonical form and satisfies the complete state observability condition, the state observer can be designed.
- In this new representation we have

$$|z\mathbf{I} - \mathbf{G}' + \mathbf{K}'_e\mathbf{C}'| = (z - p_1)(z - p_2)\dots(z - p_n) = 0 \tag{30}$$

and by matching coefficients, the observer gain matrix $\mathbf{K}'_e$ can be determined.

Observer Gain Design

By Transformation

- If we apply the inverse transformation $\mathbf{T}^{-1}$ to the system and define $\tilde{x}(k) = x(k) - \hat{x}(k)$, we obtain

$$\begin{aligned} \tilde{\mathbf{x}}'(k+1) &= (\mathbf{G}' - \mathbf{K}'_e\mathbf{C}')\tilde{\mathbf{x}}'(k) \\ \mathbf{y}(k) - \hat{\mathbf{y}}(k) &= \mathbf{C}'\tilde{\mathbf{x}}'(k) \end{aligned}$$

  To return to the original representation, we apply $\mathbf{x}' = \mathbf{T}^{-1}\mathbf{x}$, and recalling that $\mathbf{G}' = \mathbf{T}^{-1}\mathbf{G}\mathbf{T}$ and $\mathbf{C}' = \mathbf{C}\mathbf{T}$, we obtain

$$\begin{aligned} \tilde{\mathbf{x}}(k+1) &= (\mathbf{G} - \mathbf{T}\mathbf{K}'_e\mathbf{C})\tilde{\mathbf{x}}(k) \\ \mathbf{y}(k) - \hat{\mathbf{y}}(k) &= \mathbf{C}\tilde{\mathbf{x}}(k) \end{aligned}$$

- Comparing this expression with the state estimation error model in the original system, we conclude that the observer gain matrix expressed in the original coordinate system has the form

$$\mathbf{K}_e = \mathbf{T}\mathbf{K}'_e \tag{31}$$

- We are interested in understanding how the system dynamics behave when the observer is introduced into the feedback loop.
- The system equations and the feedback signal are

$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{G}\mathbf{x}(k) + \mathbf{H}\mathbf{u}(k) \\ \mathbf{y}(k) &= \mathbf{C}\mathbf{x}(k) \\ \mathbf{u}(k) &= -\mathbf{K}\hat{\mathbf{x}}(k) \end{aligned} \tag{32}$$

Substituting the feedback signal into the state equation, and adding and subtracting the term $\mathbf{HKx}(k)$, we obtain

$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{G}\mathbf{x}(k) - \mathbf{HK}\hat{\mathbf{x}}(k) \\ &= [\mathbf{G} - \mathbf{HK}]\mathbf{x}(k) + \mathbf{HK}[\mathbf{x}(k) - \hat{\mathbf{x}}(k)] \end{aligned} \tag{33}$$

Replacing $\mathbf{x}(k) - \hat{\mathbf{x}}(k)$ by $\tilde{\mathbf{x}}(k)$ gives

$$\mathbf{x}(k+1) = [\mathbf{G} - \mathbf{HK}]\mathbf{x}(k) + \mathbf{HK}\tilde{\mathbf{x}}(k) \tag{34}$$

- Taking into account that the dynamics of the state estimation error produced by the observer are given by

$$\tilde{\mathbf{x}}(k+1) = [\mathbf{G} - \mathbf{K}_e\mathbf{C}]\tilde{\mathbf{x}}(k) \tag{35}$$

- Combining the state equation of the feedback-controlled system 34 and the state equation of the estimation error 35 into a single state-space model, we obtain

$$\begin{bmatrix} \mathbf{x}(k+1) \\ \tilde{\mathbf{x}}(k+1) \end{bmatrix} = \begin{bmatrix} \mathbf{G} - \mathbf{HK} & \mathbf{HK} \\ \mathbf{0} & \mathbf{G} - \mathbf{K}_e\mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ \tilde{\mathbf{x}}(k) \end{bmatrix} \tag{36}$$

- The characteristic equation of the system with state feedback and observer is obtained from 36 and is given by the following expression

$$\begin{vmatrix} z\mathbf{I} - \mathbf{G} + \mathbf{HK} & -\mathbf{HK} \\ \mathbf{0} & z\mathbf{I} - \mathbf{G} + \mathbf{K}_e\mathbf{C} \end{vmatrix} = 0 \tag{37}$$

- It can be observed that the characteristic equation of the system can be written in the form

$$|z\mathbf{I} - \mathbf{G} + \mathbf{HK}| \,.\, |z\mathbf{I} - \mathbf{G} + \mathbf{K}_e\mathbf{C}| = 0 \tag{38}$$

- Expression 38 shows that the characteristic equation of the system with state feedback and observer is the product of the characteristic equations of the feedback-controlled system and the state observer.
- They are independent, which allows them to be designed separately and subsequently combined.
- To ensure that the observer poles do not influence the system dynamics, they must decay rapidly; that is, they should be placed close to the origin or at the origin.

Reduced-Order Observer

- In systems where some of the state variables can be measured, it is sufficient to estimate only the state variables that are not measurable. These are called **reduced-order observers**; if the observer order is the smallest possible, it is called a **minimum-order observer**.

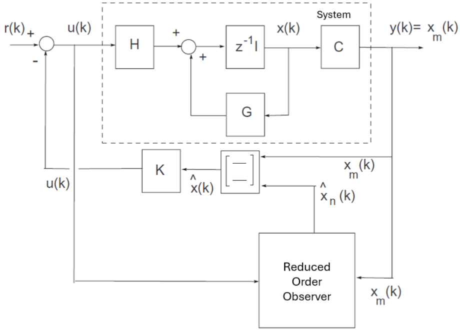

Figure 5: Reduced-order observer.

Reduced-Order Observer

- Suppose that, in the system model, some of the state variables are measurable, $\mathbf{x}_m$, while others are not, $\mathbf{x}_n$.
- We partition the state vector $\mathbf{x}$ into two subvectors:

$$\begin{aligned} \begin{bmatrix} \mathbf{x}_m(k+1) \\ \mathbf{x}_n(k+1) \end{bmatrix} &= \begin{bmatrix} \mathbf{G}_{mm} & \mathbf{G}_{mn} \\ \mathbf{G}_{nm} & \mathbf{G}_{nn} \end{bmatrix} \begin{bmatrix} \mathbf{x}_m(k) \\ \mathbf{x}_n(k) \end{bmatrix} + \begin{bmatrix} \mathbf{H}_m \\ \mathbf{H}_n \end{bmatrix} \mathbf{u}(k) \\ \mathbf{y}(k) &= \begin{bmatrix} \mathbf{I} & \mathbf{O} \end{bmatrix} \begin{bmatrix} \mathbf{x}_m(k) \\ \mathbf{x}_n(k) \end{bmatrix} \end{aligned} \tag{39}$$

and, written as separate equations, we have

$$\mathbf{x}_m(k+1) = \mathbf{G}_{mm}\mathbf{x}_m(k) + \mathbf{G}_{mn}\mathbf{x}_n(k) + \mathbf{H}_m\mathbf{u}(k) \tag{40}$$

$$\mathbf{x}_n(k+1) = \mathbf{G}_{nm}\mathbf{x}_m(k) + \mathbf{G}_{nn}\mathbf{x}_n(k) + \mathbf{H}_n\mathbf{u}(k) \tag{41}$$

Reduced-Order Observer

- From equation 40, all terms can be measured or are known except $\mathbf{x}_n(k)$. Isolating this term, the equation becomes

$$\mathbf{x}_m(k+1) - \mathbf{G}_{mm}\mathbf{x}_m(k) - \mathbf{H}_m\mathbf{u}(k) = \mathbf{G}_{mn}\mathbf{x}_n(k) \tag{42}$$

  Eq. 41 can be rewritten by grouping the measurable terms:

$$\mathbf{x}_n(k+1) = \mathbf{G}_{nn}\mathbf{x}_n(k) + [\mathbf{G}_{nm}\mathbf{x}_m(k) + \mathbf{H}_n\mathbf{u}(k)] \tag{43}$$

- If we compare equations 42 and 43 with those of a generic system

$$\mathbf{x}'(k+1) = \mathbf{G}'\mathbf{x}'(k) + \mathbf{H}'\mathbf{u}'(k) \tag{44}$$
$$\mathbf{y}'(k) = \mathbf{C}'\mathbf{x}'(k) \tag{45}$$

  such that

$$\begin{aligned}
\mathbf{x}'(k) &= \mathbf{x}_n(k) \\
\mathbf{y}'(k) &= \mathbf{x}_m(k+1) - \mathbf{G}_{mm}\mathbf{x}_m(k) - \mathbf{H}_m\mathbf{u}(k) \\
\mathbf{G}' &= \mathbf{G}_{nn} \\
\mathbf{H}'\mathbf{u}'(k) &= \mathbf{G}_{nm}\mathbf{x}_m(k) + \mathbf{H}_n\mathbf{u}(k) \\
\mathbf{C}' &= \mathbf{G}_{mn}
\end{aligned} \tag{46}$$

Reduced-Order Observer

Design

- For system 44, a state observer has the general form

$$\hat{\mathbf{x}}'(k+1) = (\mathbf{G}' - \mathbf{K}_e\mathbf{C}')\hat{\mathbf{x}}'(k) + \mathbf{H}'\mathbf{u}'(k) + \mathbf{K}_e\mathbf{y}'(k) \tag{47}$$

- Substituting the matrices and vectors with the complete expressions in 46, we obtain the observer for the non-measurable part of the system:

$$\begin{aligned} \hat{\mathbf{x}}_n(k+1) &= (\mathbf{G}_{nn} - \mathbf{K}_e\mathbf{G}_{mn})\hat{\mathbf{x}}_n(k) + \mathbf{G}_{nm}\mathbf{x}_m(k) + \mathbf{H}_n\mathbf{u}(k) \\ &\quad + \mathbf{K}_e[\mathbf{x}_m(k+1) - \mathbf{G}_{mm}\mathbf{x}_m(k) - \mathbf{H}_m\mathbf{u}(k)] \end{aligned} \tag{48}$$

- For the design of this minimum-order observer, the matrix $\mathbf{K}_e$ must be selected so that the poles of the characteristic equation are placed at the desired locations:

$$|\mathbf{z}I - (\mathbf{G}_{nn} - \mathbf{K}_e\mathbf{G}_{mn})| = 0 \tag{49}$$