Control Engineering II

Topic 10: State Observers

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Problem 1

Suppose the system defined by the following equations:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} -0.8 & 1 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$[y_1(k)] = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Design a state observer for this system so that the error dynamics is as fast as possible (observer at the origin).

Solution

First, it is checked that the system is totally observable:

$$M_O = \begin{bmatrix} 1 & 2 \\ -0.8 & 2 \end{bmatrix}$$

The rank is 2, so the system is completely observable.
For the observer dynamics to be as fast as possible, we must place the poles at the origin (minimum time response or dead-beat):

$$|zI - G + K_eC| = z^2$$

$$\left| \begin{bmatrix} z+0.8 & -1 \\ 0 & z-0.5 \end{bmatrix} + \begin{bmatrix} K_{e1} \\ K_{e2} \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} \right| = z^2$$

$$\left| \begin{bmatrix} z+0.8+K_{e1} & -1+2K_{e1} \\ K_{e2} & z-0.5+2K_{e2} \end{bmatrix} \right| = z^2$$

$$(z+0.8+K_{e1})(z-0.5+2K_{e2}) - K_{e2}(-1+2K_{e1}) = z^2$$

$$z^2 + z(+0.3+K_{e1}+2K_{e2}) - 0.4 - 0.5K_{e1} + 2.6K_{e2} = z^2$$

$$\left.\begin{array}{r} +0.3+K_{e1}+2K_{e2}=0 \\ -0.4-0.5K_{e1}+2.6K_{e2}=0 \end{array}\right\}$$

We solve the system of equations and arrive at the sought-after state observer:

$$K_e = \begin{bmatrix} -0.4389 \\ 0.0694 \end{bmatrix}$$

Problem 2

Suppose the system defined by the following equations:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1.2 \\ 1 & 0 & -0.8 \\ 0 & 1 & 0.3 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 2.1 \\ -0.2 \\ 1.1 \end{bmatrix} u(k)$$

$$[y_1(k)] = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

Design a state observer for this system so that the observer poles are a double one at the origin and another at z=-0.1.

Solution

First, it is checked that the system is totally observable:

$$M_O = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0.3 \\ 1 & -0.3 & -0.7 \end{bmatrix}$$

The rank is 3, so the system is completely observable.
For the observer dynamics to contain the poles in the desired positions, the following expression must be satisfied:

$$|zI - G + K_eC| = z^2(z+0.1) = z^3 + 0.1z^2$$

$$\left| \begin{bmatrix} z & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & z \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1.2 \\ 1 & 0 & -0.8 \\ 0 & 1 & 0.3 \end{bmatrix} + \begin{bmatrix} K_{e1} \\ K_{e2} \\ K_{e3} \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \right| = z^3 + 0.1z^2$$

$$z^3 + (K_{e3} - 0.3)z^2 + (K_{e2} + 0.8)z + K_{e1} - 1.2 = z^3 + 0.1z^2$$

$$\left. \begin{array}{r} K_{e3} - 0.3 = 0.1 \\ K_{e2} + 0.8 = 0 \\ K_{e1} - 1.2 = 0 \end{array} \right\}$$

We solve the system of equations and arrive at the sought-after state observer:

$$K_e = \begin{bmatrix} 1.2 \\ -0.8 \\ 0.4 \end{bmatrix}$$

Suppose the system defined by the following equations:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0.5 & 1 \\ -0.1 & 1.2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$[y_1(k)] = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Design a state observer for this system so that the error dynamics is as fast as possible (observer at the origin). First convert the system to the observable canonical form.

Solution

First, it is checked that the system is totally observable:

$$M_O = \begin{bmatrix} 1 & 2 \\ 0.3 & 3.4 \end{bmatrix}$$

The rank is 2, so the system is completely observable.
To convert it to the observable canonical form, the transfer function must be calculated from the following expression:

$$F(z) = C(zI - G)^{-1}H = \frac{2z}{z^2 - 1.7z + 0.7}$$

We write the equations of the transformed system:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & -0.7 \\ 1 & 1.7 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u(k)$$

$$[y_1(k)] = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Now we design the observer for the transformed system. For the observer dynamics to be as fast as possible, we must place the poles at the origin (minimum time response or dead-beat):

$$\left| zI - G' + K_e^{'} C' \right| = z^2$$

$$\left| \begin{bmatrix} z & 0.7 \\ -1 & z - 1.7 \end{bmatrix} + \begin{bmatrix} K'_{e1} \\ K'_{e2} \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \right| = z^2$$

$$z^2 + z\left(K'_{e2} - 1.7\right) + K'_{e1} + 0.7 = z^2$$

We solve the system of equations and arrive at the sought-after state observer:

$$K'_e = \begin{bmatrix} -0.7 \\ 1.7 \end{bmatrix}$$

Finally, we calculate the transformation matrix to return to the original system:

$$T = M_O^{-1} M'_O = \begin{bmatrix} 1 & 2 \\ 0.3 & 3.4 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ 1 & 1.7 \end{bmatrix} = \begin{bmatrix} -0.7143 & 0 \\ 0.3571 & 0.5 \end{bmatrix}$$

$$K_e = T * K'_e = \begin{bmatrix} 0.5 \\ 0.6 \end{bmatrix}$$

Suppose the system defined by the following equations:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.4 & 1.1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$[y_1(k)] = \begin{bmatrix} 1.3 & 0.2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Design a state observer for this system so that the error dynamics is as fast as possible (observer at the origin). First convert the system to the observable canonical form.

Solution

First, it is checked that the system is totally observable:

$$M_O = \begin{bmatrix} 1.3 & 0.2 \\ -0.08 & 1.52 \end{bmatrix}$$

The rank is 2, so the system is completely observable.
It can be deduced that the system is in the controllable canonical form.
To convert it to the observable canonical form, the transfer function must be calculated from the following expression:

$$F(z) = C(zI - G)^{-1} H = \frac{0.2z + 1.3}{z^2 - 1.1z + 0.4}$$

We write the equations of the transformed system:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & -0.4 \\ 1 & 1.1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1.3 \\ 0.2 \end{bmatrix} u(k)$$

$$[y_1(k)] = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

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Now we design the observer for the transformed system. For the observer dynamics to be as fast as possible, we must place the poles at the origin (minimum time response or dead-beat):

$$\left| zI - G' + K_e'C' \right| = z^2$$

$$\left| \begin{bmatrix} z & 0.4 \\ -1 & z - 1.1 \end{bmatrix} + \begin{bmatrix} K_{e1}' \\ K_{e2}' \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \right| = z^2$$

$$z^2 + z\left(K_{e2}' - 1.1\right) + K_{e1}' + 0.4 = z^2$$

We solve the system of equations and arrive at the sought-after state observer:

$$K_e' = \begin{bmatrix} -0.4 \\ 1.1 \end{bmatrix}$$

Finally, we calculate the transformation matrix to return to the original system:

$$T = M_O'^{-1} M_O = \begin{bmatrix} 1.3 & 0.2 \\ -0.08 & 1.52 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ 1 & 1.1 \end{bmatrix} = \begin{bmatrix} -0.1004 & 0.6526 \\ 0.6526 & 0.758 \end{bmatrix}$$

$$K_e = T * K_e' = \begin{bmatrix} 0.758 \\ 0.5728 \end{bmatrix}$$

Problem 5

Suppose the system defined by the following equations:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0.8 & -0.5 \\ 1 & 0.5 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.6 \\ 0.1 \end{bmatrix} u(k)$$

$$[y_1(k)] = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Design a reduced-order state observer for this system so that the error dynamics is as fast as possible (observer at the origin). Assume for this that only variable 1 is observable.

Solution

First, we separate between measurable and non-measurable variables:

$$G_{mm} = 0.8 \quad G_{mn} = -0.5 \quad G_{nn} = 0.5 \quad G_{nm} = 1$$

We calculate the observability matrix:

$$M_O = \begin{bmatrix} 1 & 0 \\ 0.8 & -0.5 \end{bmatrix}$$

The system is completely observable. To design the reduced-order observer, we apply the following expression:

$$|zI - (G_{nn} - K_e G_{mn})| = |zI - (0.5 + K_e 0.5)| = z$$

$$z - (0.5 + K_e 0.5) = z$$

The following condition must be met:

$$0.5 + K_e 0.5 = 0$$

A possible solution is:

$$K_e = -1$$

Problem 6

Suppose the system defined by the following equations:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & -0.2 \\ 0 & 0.5 & 0.1 \\ 0 & 0.5 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 0.01 \\ 0.1 \\ 0.2 \end{bmatrix} u(k)$$

$$\begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

Design a reduced-order state observer for this system so that the error dynamics is as fast as possible (observer at the origin). Assume for this that only variables 1 and 2 are observable.

Solution

First, we separate between measurable and non-measurable variables:

$$G_{mm} = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} \quad G_{mn} = \begin{bmatrix} -0.2 \\ 0.1 \end{bmatrix} \quad G_{nn} = 1 \quad G_{nm} = \begin{bmatrix} 0 & 0.5 \end{bmatrix}$$

We calculate the observability matrix:

$$M_O = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -0.2 \\ 0 & 0.5 & 0.1 \\ 1 & -0.1 & -0.4 \\ 0 & 0.3 & 0.15 \end{bmatrix}$$

The system is completely observable. To design the reduced-order observer, we apply the following expression:

$$|zI - (G_{nn} - K_e G_{mn})| = \left| zI - \left(1 - \begin{bmatrix} k_{e1} & k_{e2} \end{bmatrix} \begin{bmatrix} -0.2 \\ 0.1 \end{bmatrix}\right) \right| = z$$

$$z - (1 + 0.2k_{e1} - 0.1k_{e2}) = z$$

The following condition must be met:

$$1 + 0.2k_{e1} - 0.1k_{e2} = 0$$

A possible solution is:

$$K_e = \begin{bmatrix} -4 & 2 \end{bmatrix}$$