# Control Engineering II

Solved Exam

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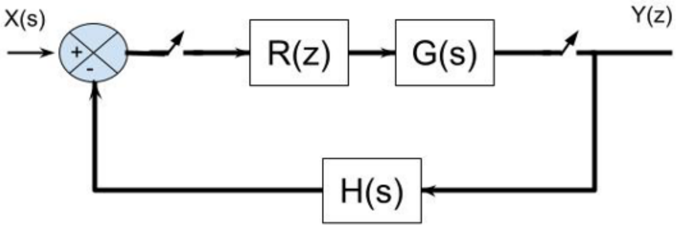
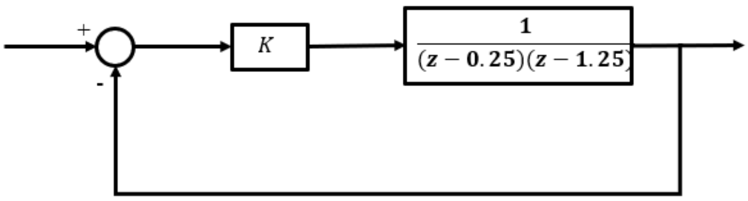
- **Exercise 1**
  Find the equivalent discrete-time transfer function of the following block diagram using the residue method:

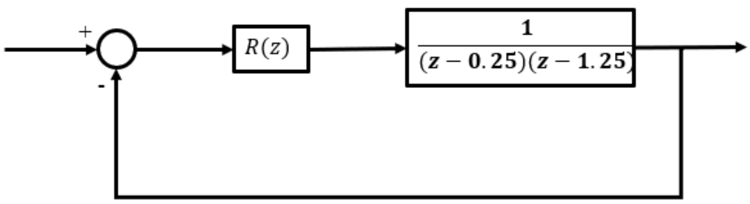



where:
$R(z) = \frac{z-1}{z}$
$G(s) = \frac{1-e^{-Ts}}{s}\frac{5}{s+1}$
$H(s) = \frac{1}{s}$

- **Exercise 2**
  Consider the following system:




  - Using the Jury test, determine the values of $K$ that make the system stable.
  - Now consider:




    Determine the simplest possible controller, using the root-locus design method, such that the closed-loop system has dominant poles at $z_d = 0.25 \pm i0.25$.
  - For this controller, determine the position error.

*Notes:*
- *Draw and derive the root locus in detail.*
- *Explain and justify the entire procedure.*

- **Exercise 3**
  Consider the following discrete-time system:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(k)$$

$$\begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

  - Determine the output controllability matrix. Is this system completely output controllable?
  - If the initial state is $x(0) = (0\ 0)^t$, what input sequence $u(0), u(1)$ is required for the system to reach the output $y(2) = (3\ 9)^t$?
  - The system now has a single output, so the output equation for the same system becomes:

$$y(k) = [3\ 1] \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

    This system is controlled using state feedback. Determine the state-feedback matrix $K$ such that the closed-loop system poles are $p_{1,2} = 0,5 \pm j0,5$.

- **Exercise 4**
  Given the system:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0.1 & 0.5 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k) \quad (1)$$

$$y(k) = \begin{bmatrix} 0.5 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \quad (2)$$

  - Determine the state observability of the system.
  - Given the output sequence $y(0) = 1$, $y(1) = 2$, for inputs $u(0) = 0$, $u(1) = 0$, determine the initial state $x(0)$ of the system.
  - Given an observer gain $K_e = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ for a full-order state observer, determine the observer poles and the system poles. Is the observer gain matrix suitable for estimating the system state?

# Solution

**Solution to Exercise 1:**

- System equations:

$$U(s) = X(s) - H(s)Y^*(s) \rightarrow U(z) = X^*(z) - H^*(s)Y^*(s)$$

$$Y(s) = G(s)R(z)U^*(s) \rightarrow Y(z) = G^*(s)R(z)U^*(s)$$

$$Y(z) = \frac{G^*(s)R(z)X^*(z) - H^*(s)Y^*(s)}{1 + H^*(s)G^*(s)R(z)}$$

- We calculate $G(z)$:

$$G^*(s) = Z\left[\frac{1-e^{-Ts}}{s}\frac{5}{s+1}\right] = (1 - z^{-1})Z\left[\frac{5}{s(s+1)}\right]$$

$$R_{-1} = \lim_{p\rightarrow 1}\left[\frac{5}{p(1-e^{-T(s-p)})}\right] = \frac{-5}{1-e^{-Ts}e^{-T}} = \frac{-5z}{z-e^{-T}}$$

$$R_0 = \lim_{p\rightarrow 0}\left[\frac{5}{(p+1)(1-e^{-T(s-p)})}\right] = \frac{5}{1-e^{-Ts}} = \frac{5z}{z-1}$$

$$G(z) = (1 - z^{-1})\left[\frac{-5z}{z-e^{-T}} + \frac{5z}{z-1}\right] = 5\frac{1-e^{-T}}{(z-e^{-T})}$$

- We calculate $H^*(s)$:

$$H^*(s) = Z\left[\frac{1}{s}\right]$$

$$R_0 = \lim_{p\rightarrow -1}\left[\frac{1}{1-e^{-T(s-p)}}\right] = \frac{1}{1-e^{-Ts}} = \frac{z}{z-1}$$

- System transfer function:

$$\frac{Y(z)}{X(z)} = \frac{\frac{z^{-1}5(1-e^{-T})}{z(z-e^{-T})}}{1+\frac{z^{-1}5(1-e^{-T})}{z(z-e^{-T})}} = \frac{5z^{-1}(1-e^{-T})}{(z-e^{-T})}\left[\frac{1}{1+5\frac{1-e^{-T}}{(z-e^{-T})}}\right]$$

# Solution

**Solution to Exercise 2:**

- $G(z) = \frac{1}{(z-0{,}25)(z-1{,}25)}$
- Critical value of $k$

  $P(z) = 1 + k\frac{1}{(z-0{,}25)(z-1{,}25)} = 0$

  $P(z) = z^2 - 1,5z + (0,3125 + k) = 0$
- Jury test:

  1. $|a_0| < a_n \Rightarrow 0.3125 + k < 1$

     $0,3125 + k < 1 \Rightarrow k < 0,8125$
     $0,3125 + k > -1 \Rightarrow k < -1,3125$

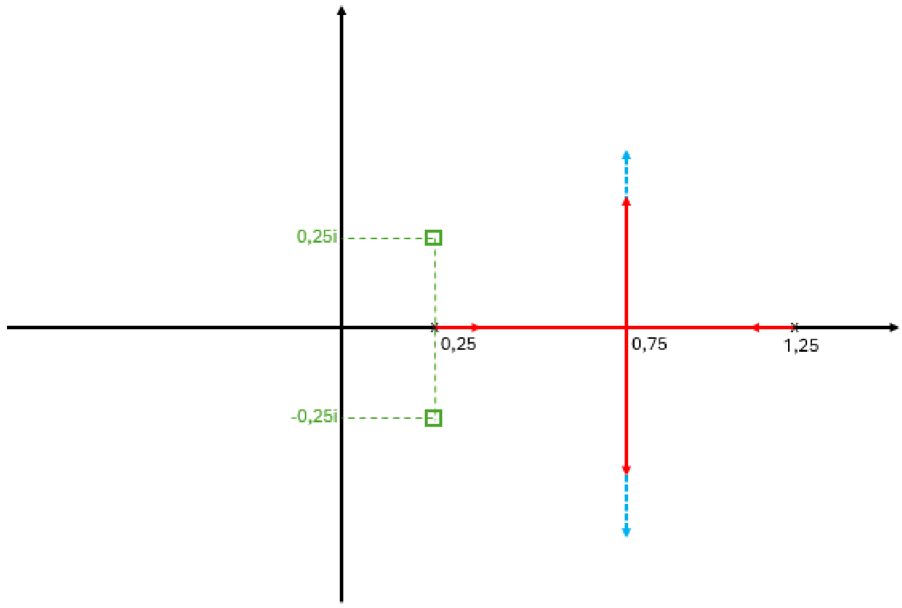
     $-1.3125 < k < 0.8125$
  2. $P(1) > 0 \Rightarrow P(1) = 1 - 1,5 + 0,3125 + k > 0$
     $-0.1875 + k > 0 \Rightarrow \quad k > 0.1875$
  3. $(-1)^2 P(-1) > 0 \Rightarrow P(-1) = 1 + 1,5 + 0,3125 + k > 0$
     $2,81 + k > 0 \Rightarrow k > -2,81$

  Combining the three conditions gives:

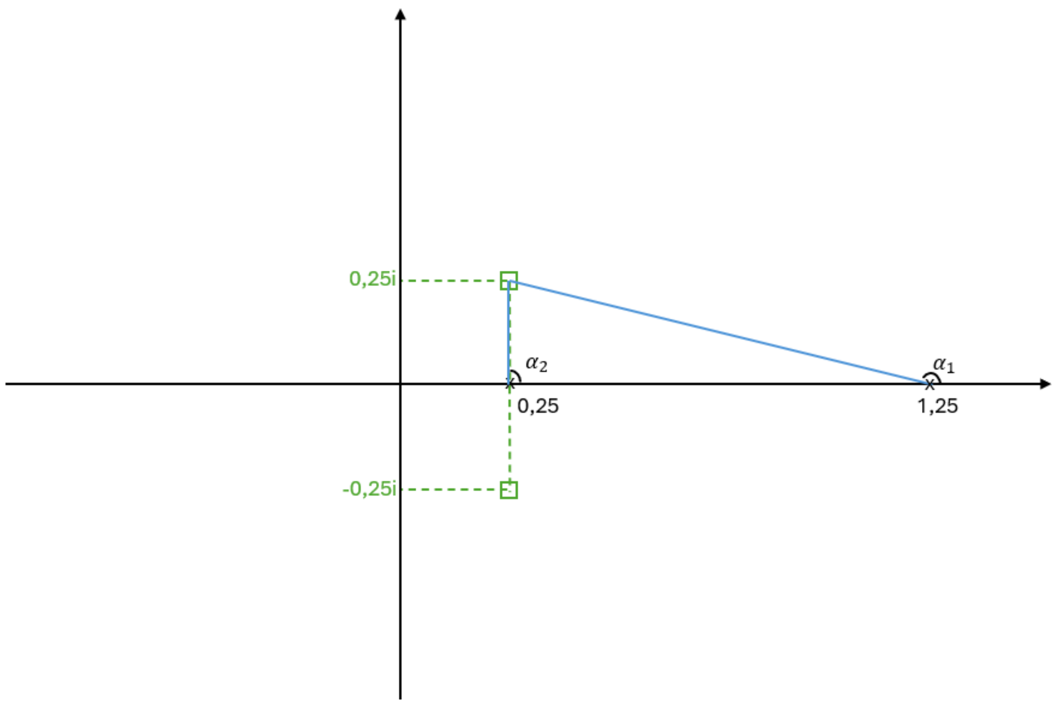
  $0,1875 \quad < \quad k < 0,6875 \Rightarrow$ Stable.
- P-type controller with $\quad z_d = 0.25 \pm i0.25$
  - number of poles $n = 2$
  - number of zeros $m = 0$
  - number of branches $n = 2$
  - number of asymptotes $n - m = 2$
  - Asymptote angles
    $\theta = \frac{(2q+1)\pi}{n-m} = \pm\frac{\pi}{2}$
  - Centroid
    $\sigma = \frac{\sum p - \sum z}{n-m} = \frac{0{,}25+1{,}25}{2} = 0.75$
  - Breakaway points
    $\frac{dk}{dz} = 0; \quad -k = z^2 - 1,5z + 0,3215$
    $\frac{dk}{dz} = 2z - 1,5 = 0 \quad \Rightarrow z = 0,75$

**Solution to Exercise 2 (Cont.):**

- Root Locus




- Angle criterion



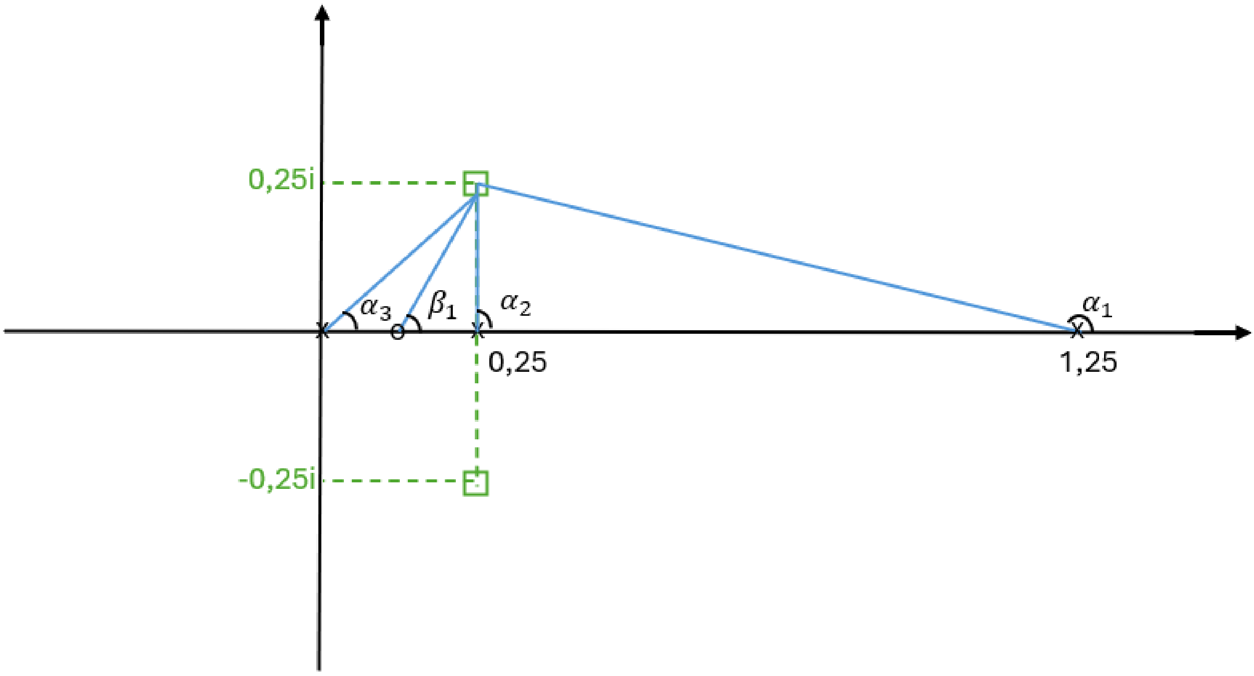

$\sum \arg p - \sum \arg z = \pi \Rightarrow \alpha_1 + \alpha_2 = \pi$

$\alpha_1 + \alpha_1 = 180$ ?

$(\pi - tg^{-1}\frac{0{,}25}{1}) + \frac{pi}{2}) = 165{,}963^\circ + 90^\circ = 255{,}963^\circ \neq \pi$

The desired dominant poles do not belong to the root locus, so we try a PD controller. We add the derivative action (a new zero and a pole at the origin) $\Rightarrow R(z) = \frac{z-c}{z}$.

**Solution to Exercise 2 (Cont.):**




- Angle criterion
  $\alpha_1 + \alpha_2 + \alpha_3 - \beta_1 = \pi$

  $255,963^\circ + tg^{-1}\frac{0{,}25}{0{,}25} - tg^{-1}\frac{0{,}25}{0{,}25-c} = 180^\circ$

  $300,96^\circ - 180^\circ = tg^{-1}\frac{0{,}25}{0{,}25-c} = 120,96^\circ$

  $\frac{0{,}25}{0{,}25-c} = -1,667 \Rightarrow c = 0,4$

  $R(z) = \frac{z-0{,}4}{z}$

- Magnitude criterion
  $|k| = \frac{|z-0{,}25||z-1{,}25||z|}{|z-0{,}4|}$

  $|k| = 0,3125$

  $R(z) = 0,3125\frac{z-0{,}4}{z}$
- We calculate the error.
  $ep = \frac{1}{1+k_p}$; $k_p = \lim_{z\to 1} 0,3125\frac{z-0{,}4}{z(z-0{,}25)(z-1{,}25)} \Rightarrow k_p = \frac{0{,}1875}{0{,}1875} = 1$

  $ep = \frac{1}{2} = 0,5 \Rightarrow ep = 50\%$

# Solution

**Solution to Exercise 3:**

- Output controllability matrix:

$$M_{cs} = [CH \quad CGH]$$

where

$$CH = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix},$$

and

$$CGH = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 5 \\ 6 & -1 & 5 \end{bmatrix}.$$

Therefore,

$$M_{cs} = \begin{bmatrix} 5 & 2 & -2 & 5 \\ 5 & 6 & -1 & 5 \end{bmatrix}$$

which indicates that the system is completely output controllable.

- Input sequence:

$$y(2) - CG^2x(0) = M_{cs} \begin{bmatrix} u(1) \\ u(0) \end{bmatrix}$$

and using the given initial values,

$$\begin{bmatrix} 3 \\ 9 \end{bmatrix} - 0 = \begin{bmatrix} 5 & -2 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} u(1) \\ u(0) \end{bmatrix},$$

we obtain the following system of equations:

$$5u(1) - 2u(0) = 3,$$
$$5u(1) + 4u(0) = 9.$$

Solving for $u(0)$ and $u(1)$, we obtain:

$$u(0) = 1, \quad u(1) = 1.$$

- Feedback gain matrix:

$$[zI - G + HK] = z^2 - z + 0.5$$

**Solution to Exercise 3 (Cont.):**

where

$$G = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}, \quad H = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}.$$

Expanding the expression,

$$\begin{bmatrix} z-2 & 0 \\ 0 & z+1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} z-2+k_1 & k_2 \\ 1k_1 & z+1+1k_2 \end{bmatrix} = z^2 - z + 0.5.$$

Matching coefficients and solving the equations, we obtain:

$$\begin{aligned} z^2 + z(-1 + 2k_2 + k_1) - 2 - 4k_2 + k_1 &= z^2 - z + 0.5, \\ -1 + 2k_2 + k_1 &= -1, \\ -2 - 4k_2 + k_1 &= 0.5. \end{aligned}$$

Solving for $k_1$ and $k_2$, we obtain:

$$K = \begin{bmatrix} \frac{5}{6} & -\frac{5}{12} \end{bmatrix}.$$

**Solution to Exercise 4:**

- The observability matrix $M_o$ is defined as:

$$M_o = \begin{bmatrix} C \\ CG \end{bmatrix} = \begin{bmatrix} 0,5 & 1 \\ 0,6 & 0,5 \end{bmatrix}$$

We calculate the determinant of $M_o$:

$$\det(M_o) = 0,5 * 0,5 - 0,6 = 0,25 - 0,6 = -0,35$$

Since $\det(M_o) \neq 0$, the system is completely observable.

- Determination of the initial state. Given the outputs and inputs, the system can be expressed as:

$$\begin{bmatrix} y(0) \\ y(1) \end{bmatrix} = \begin{bmatrix} C \\ CG \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

Substituting the given values:

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0.5 & 1 \\ 0.6 & 0.5 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

Solving the system of equations, we obtain:

$$x_1(0) = 1.76, \quad x_2(0) = 0.118$$

- The system observer equation is:

$$\dot{x}(k+1) = Gx(k) + Ke[y(k) - \hat{y}(k)] + Hu(k)$$

where $G$, $H$, $C$, and $Ke$ are the system, input, output, and observer-gain matrices, respectively.

- Characteristic Equation

The characteristic equation with observer gain $Ke = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is:

$$|zI - G + KeC| = \left| \begin{bmatrix} z-1 & 0 \\ -0.1 & z-0.5 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0,5 & 1 \end{bmatrix} \right| = \left| \begin{bmatrix} z-1 & 0 \\ 0,4 & z+0,5 \end{bmatrix} \right|$$
$$= (z-1)(z+0,5) = 0$$

The observer poles are $z = 1$ and $z = -0,5$.
The system poles are calculated from the original matrix $G$:

$$|zI - G| = \left| \begin{bmatrix} z-1 & 0 \\ -0,1 & z-0,5 \end{bmatrix} \right| = (z-1)(z-0,5) = 0$$

The system poles are $z = 1$ and $z = 0,5$.